A new sphere theorem via the eigenmaps

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at

Geometric Measure Theory and Geometric Analysis in Moscow

§1

Main result

$$\mathbb{S}^n(r) := \{ x \in \mathbb{R}^{n+1} \mid |x|_{\mathbb{R}^{n+1}} = r \}$$

$$\begin{array}{ccccl} \iota : \mathbb{S}^n(r) & \hookrightarrow & \mathbb{R}^{n+1} & : & \text{isometric} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \\ x & \longmapsto & x & & \text{eigenmap} \end{array}$$

i.e.

$$\begin{cases} \iota^* g_{\mathbb{R}^{n+1}} = g_{\mathbb{S}^n} \\ \Delta x_i + n r^{-2} x_i \equiv 0 \end{cases}$$

Main theorem A (H.)

$\forall n \in \mathbb{Z}_{\geq 2}$, $\forall K \in \mathbb{R}$, $\forall \tau, \forall D \in (0, \infty)$

$\exists \delta := \delta(n, K, \tau, D) > 0$ s.t. if

$$\fint_{M^n} |\Phi^* g_{\mathbb{R}^{n+1}} - g| \, d\mathrm{vol}_g \leq \delta$$

for $\begin{cases} \cdot \ (M^n, g) : \text{closed Riem. with } \mathrm{Ric}^g_{M^n} \geq K g \ \& \ \mathrm{diam}(M^n, d_g) \leq D \\ \cdot \ \Phi = (\varphi_1, \cdots, \varphi_{n+1}) : M^n \to \mathbb{R}^{n+1} : \text{eigenmap with } \fint_{M^n} |\varphi_i|^2 d\mathrm{vol}_g \geq \tau, \end{cases}$

then $M^n \approx \mathbb{S}^n$: diffeomorphic.

Key

① Synthetic notion of "$\mathrm{Ric} \geq K$ & $\dim \leq N$" for metric measure space (X, d, m), so-called "$\mathrm{RCD}(K, N)$ space"

② Generalization of a theorem of Takahashi in submanifold theory to $\mathrm{RCD}(K, N)$ spaces.

③ Topological stability result

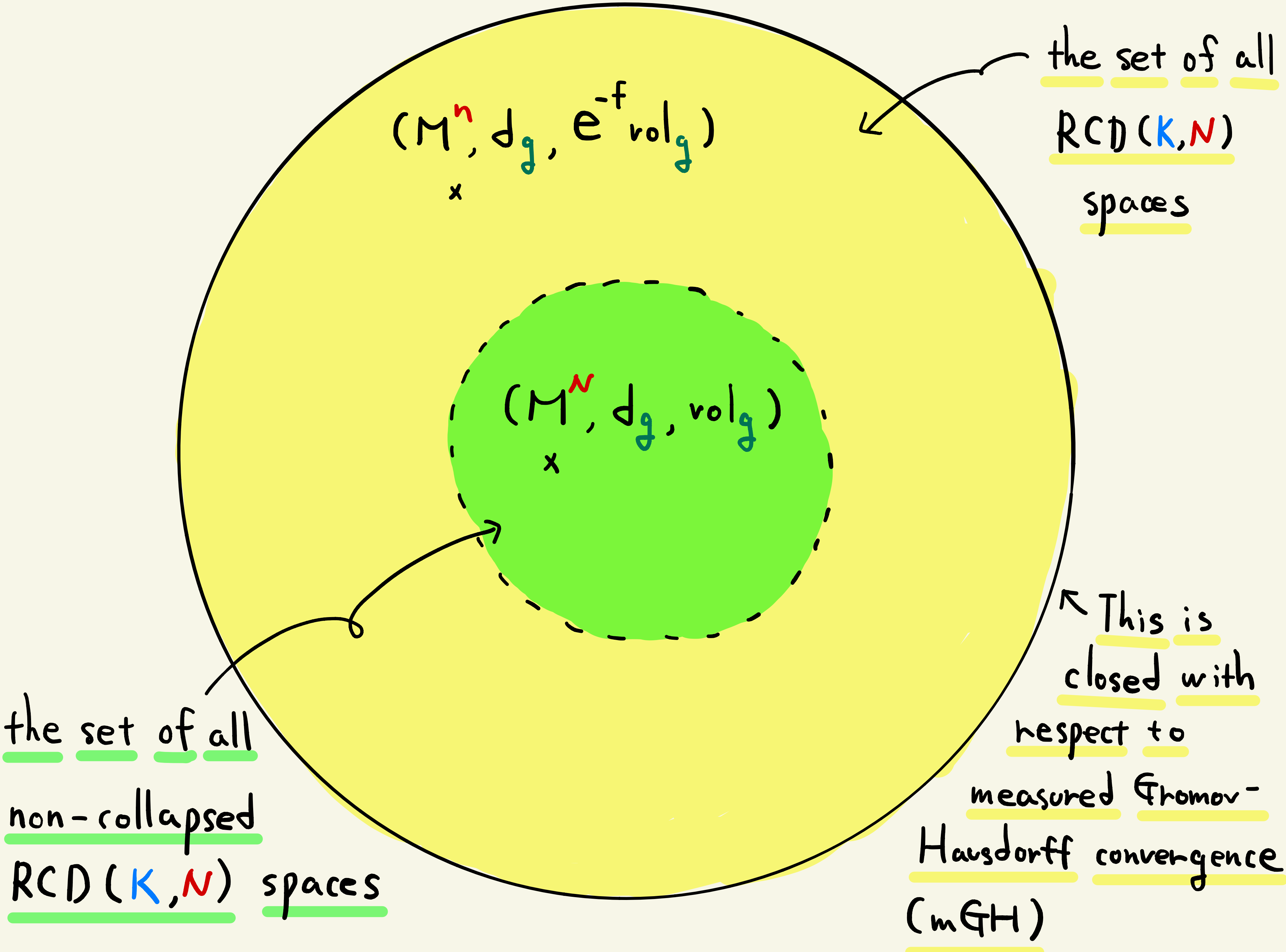
§ 2

$\mathrm{RCD}(K, N)$ space

$(M^n_x, d_g, e^{-f}\mathrm{vol}_g)$

the set of all $RCD(K,N)$ spaces

This is closed with respect to measured Gromov-Hausdorff convergence (mGH)


$(M^n_x, d_g, e^{-f}\mathrm{vol}_g)$
$(M^N_x, d_g, \mathrm{vol}_g)$
the set of all RCD(K,N) spaces
This is closed with respect to measured Gromov-Hausdorff convergence (mGH)
the set of all non-collapsed RCD(K,N) spaces

Definition - Theorem (Lott-Villani, Sturm, Ambrosio-Gigli-Savaré, Gigli, Ambrosio-Gigli-Mondino-Rajala, Ambrosio-Mondino-Savaré, Erbar-Kuwada-Sturm, De Philippis-Gigli etc.)

- $N \in [1, \infty]$, $K \in \mathbb{R}$, (X, d, m) : metric measure space

① (X, d, m) : $RCD(K, N)$ space

$\overset{\text{def}}{\Longleftrightarrow}$

(1.1) $H^{1,2}(X, d, m)$: Hilbert space

$\rightsquigarrow$ $^{\exists 1} g$ s.t "$d = d_g$", $^{\exists 1}\Delta$: Laplacian

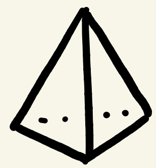
(1.2) $\frac{1}{2}\Delta|\nabla f|^2 \geq \frac{(\Delta f)^2}{N} + g(\nabla \Delta f, \nabla f) + K|\nabla f|^2$

(1.3) Sobolev-to-Lipschitz property & volume growth condition

② (X, d, m) : non-collapsed $RCD(K, N)$ space

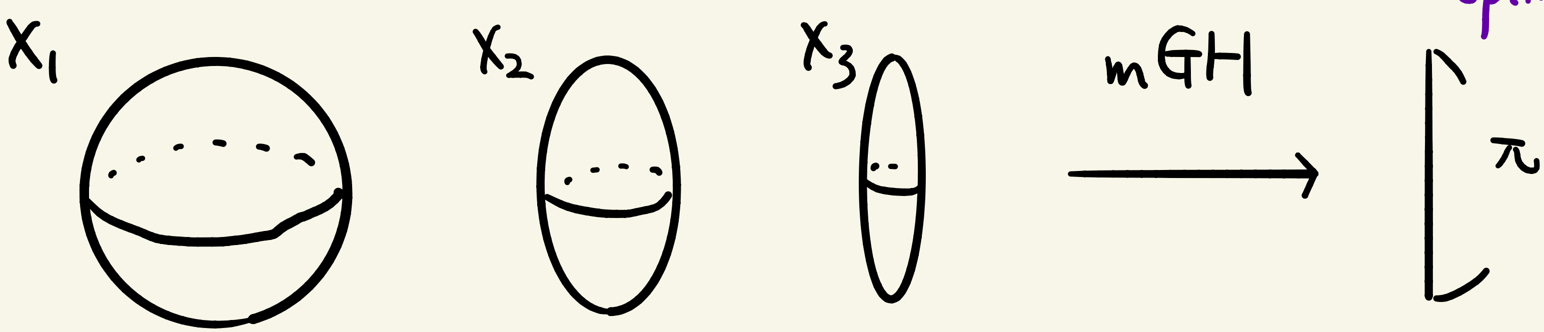
$\overset{\text{def}}{\Longleftrightarrow}$ (X, d, m) : $RCD(K, N)$ space with $m = \mathcal{H}^N$.

Example

① ( , d_{length}, H^2) : non-collapsed $RCD(0, \underline{2})$ space
optimal

(more general, Alexandrov space ⇒ non-collapsed $RCD(K,N)$ space
by Petrunin & Zhang-Zhu)

② $\left([0,\pi], d_{[0,\pi]}, \sin^{N-1}t\,dt\right)$: $RCD(N-1, \underline{N})$ space
optimal



$$\frac{\mathrm{vol}_{g_1}}{\mathrm{vol}_{g_1} X_1}, \quad \frac{\mathrm{vol}_{g_2}}{\mathrm{vol}_{g_2} X_2}, \quad \longrightarrow \quad \frac{\sin^2 t\,dt}{\int_0^\pi \sin^2 t\,dt}$$

Example

③ $\left(\mathbb{R}^n, d_{\mathbb{R}^n}, e^{-\frac{|x|^2}{2}}\mathcal{L}^n\right)$: $RCD(1, \underline{\infty})$ space

optimal

(more general, gradient Ricci soliton $\Rightarrow RCD(K, \infty)$ space)

④ Their measured Gromov-Hausdorff limit spaces

by Gigli-Mondino-Savaré.

Theorem (Mondino-Naber, Bruè-Semola, DePhilippis-Gigli, Han, H.)

(X, d, m) : $RCD(K, N)$ space with $N < \infty$

$\Rightarrow$ ① $\exists ! \, n \in \{0, 1, 2, \ldots, \lfloor N \rfloor\}$ s.t. for m-a.e. $x \in X$,

$\forall \, T_x X \simeq \mathbb{R}^n$ $\quad (\dim(X) := n)$

② (X, d, m) : $RCD(K, n)$ space $\Leftrightarrow$ $\mathrm{tr}\,\mathrm{Hess}_f = \Delta f$

③ If (X, d) : compact, then

$\mathrm{tr}\,\mathrm{Hess}_f = \Delta f \Leftrightarrow m = a\mathcal{H}^n \quad \exists \, a \in (0, \infty)$

In particular,

Theorem (Cheeger-Colding)

$$\begin{cases} \cdot \ (X_i, d_i, \mathcal{H}^n) \xrightarrow{mGH} (X, d, \mathcal{H}^n) \\ \quad : \text{compact non-collapsed } RCD(K, n) \text{ spaces} \\ \cdot \ X \text{ has no singular set} \end{cases}$$

$\Rightarrow$ $X_i \approx X$: homeomorphic for $^\forall i$: suff. large

Moreover if X_i, X are C^∞-Riem., then "homeomorphism" can be improved to "diffeomorphism".

§3

Takahashi's theorem

&

its generalization to RCD(K,N) space

Theorem (Takahashi)

(M^n, g) : closed Riem. mfd

$\Rightarrow$ ① $^\exists \Phi : M^n \longrightarrow \mathbb{R}^{k+1}$: isometric eigenmap

$$\Rightarrow \begin{cases} (1.1) & |\Phi|_{\mathbb{R}^{k+1}} \equiv \text{constant.} \\ (1.2) & \Phi \text{ is a minimal immersion into } S^k(|\Phi|_{\mathbb{R}^{k+1}}) \end{cases}$$

② Any isometric minimal immersion into $S^k(a)$

can be obtained by ①.

Recall

(X, d, m) : compact $RCD(K, N)$ space with $N < \infty$.

$\Rightarrow \quad H^{1,2}(X, d, m) \hookrightarrow L^2(X, m)$: compact operator

$\Rightarrow \quad 0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \longrightarrow \infty$

: eigenvalues of $-\Delta$.

In particular "eigenmap" makes sense.

Main theorem B (H.)

$\begin{cases} \cdot\ (X, d, m) : \text{compact } RCD(K, N) \text{ space} \\ \qquad \text{with } N < \infty \ \& \ n := \dim(X) \\ \cdot\ \Phi : X \to \mathbb{R}^{k+1} : \text{isometric eigenmap} \end{cases}$

$\Rightarrow$ · ① $n \leq k$

· ② $|\Phi|_{\mathbb{R}^{k+1}} \equiv \text{constant}$

$\Leftrightarrow m = a\mathcal{H}^n$ for $^\exists a \in (0, \infty)$ &

$(X, d, \mathcal{H}^n)$: non-collapsed $RCD(K, n)$ space

· ③ $|\Phi|_{\mathbb{R}^{k+1}} \equiv \text{constant}$

$\Rightarrow$ Φ is locally $(1 \pm \varepsilon)$-biLipschitz embedding into $\mathbb{S}^k(|\Phi|_{\mathbb{R}^{k+1}})$.

· ④ $|\Phi|_{\mathbb{R}^{k+1}} \equiv \text{constant}$ & $n = k$

$\Rightarrow$ Φ : isometry from X to $\mathbb{S}^k(|\Phi|_{\mathbb{R}^{k+1}})$

§4

Proof of Main theorem A under Main theorem B.

Main theorem A (H.)

$\forall n \in \mathbb{Z}_{\geq 2}$, $\forall K \in \mathbb{R}$, $\forall \tau, \forall D \in (0, \infty)$

$\exists\, \delta := \delta(n, K, \tau, D) > 0$ s.t. if

$$\fint_{M^n} |\Phi^* g_{\mathbb{R}^{n+1}} - g| \, d\mathrm{vol}_g \leq \delta$$

for $\begin{cases} \cdot\ (M^n, g) : \text{closed Riem. with } \mathrm{Ric}^g_{M^n} \geq K g \ \& \\ \qquad \mathrm{diam}(M^n, d_g) \leq D \\ \cdot\ \Phi = (\varphi_1, \cdots, \varphi_{n+1}) : M^n \to \mathbb{R}^{n+1} : \text{eigenmap with} \\ \qquad \fint_{M^n} |\varphi_i|^2 \, d\mathrm{vol}_g \geq \tau, \end{cases}$

then $M^n \approx \mathbb{S}^n$: diffeomorphic.

Proof

If not, $\exists (M_i^n, g_i) : \mathrm{Ric}_{M_i^n}^{g_i} \geq k g_i$ & $\mathrm{diam}(M, d) \leq D$

$\exists \Phi_i : M_i^n \to \mathbb{R}^{n+1}$: eigenmap s.t.

$$\begin{cases} \cdot \ M_i^n \not\approx \mathbb{S}^n \\ \cdot \ \displaystyle\int_{M_i^n} |\Phi_i^* g_{\mathbb{R}^{n+1}} - g_i| \, d\mathrm{vol}_{g_i} \to 0 \\ \cdot \ \displaystyle\int_{M_i^n} |\varphi_{i,j}|^2 \, d\mathrm{vol}_{g_i} \geq \tau, \ (\Phi_i = (\varphi_{i,1}, \cdots, \varphi_{i,n+1})) \end{cases}$$

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Step 1 ((Compactness of moduli of $RCD(K,N)$ spaces)

$$\left(M_i^n, d_{g_i}, \frac{\mathrm{vol}_{g_i}}{\mathrm{vol}_{g_i} M_i^n}\right) \xrightarrow{mGH} {}^{\exists}(X, d, m) : RCD(K, n) \text{ space}$$

Step 2 (Spectral convergence)

"Eigenvalues & eigenfunctions behave continuously"

In particular

- Φ_i "$\to$" ${}^{\exists}\Phi : X \to \mathbb{R}^{n+1}$: eigenmap
- Φ : isometric because of $\fint_{M_i^n} |\Phi_i^* g_{\mathbb{R}^{n+1}} - g_i| \, d\mathrm{vol}_{g_i} \to 0$.

Step 3 (Non-collapsed)

$$\sqrt{\dim(X)} = \fint_X |g| \, dm$$

$$= \fint_X |\Phi^* g_{\mathbb{R}^{n+1}}| \, dm$$

$$= \lim_{i\to\infty} \fint_{M_i^n} |\Phi_i^* g_{\mathbb{R}^{n+1}}| \, d\mathrm{vol}_{g_i} = \lim_{i\to\infty} \fint_{M_i^n} |g_i| \, d\mathrm{vol}_{g_i} = \sqrt{n}$$

In particular

$$\begin{cases} \cdot \quad m = a\mathcal{H}^n \text{ for } {}^{\exists} a \in (0,\infty) \\ \cdot \quad (M_i^n, d_{g_i}, \mathrm{vol}_{g_i}) \xrightarrow{\text{mGH}} (X, d, \mathcal{H}^n) \end{cases}$$

Step 4 (Generalized Takahashi's theorem)

Main theorem B $\Rightarrow$ $(X, d, \mathcal{H}^n) \simeq (\mathbb{S}^n(b), d_{\mathbb{S}^n(b)}, \mathcal{H}^n)$

for ${}^\exists b \in (0, \infty)$

Step 5 (Topological stability theorem)

$$M_i^n \approx X = \mathbb{S}^n(a) \text{ for } {}^\forall i : \text{suff. large}$$

$\Rightarrow$ Contradiction! //

Remark

① Main theorem A holds for non-collapsed RCD(k,n) spaces after replacing "diffeomorphic" by "homeomorphic".

② The proof of Main theorem B is based on

$$\begin{cases} \cdot \text{ "blow-up analysis"} \\ \cdot \ -\frac{1}{4} d\Delta|\Phi|^2_{\mathbb{R}^{k+1}} = \nabla^* \Phi^* g_{\mathbb{R}^{k+1}} \end{cases}$$

Thank you

very much!!