Partial regularity of harmonic maps from Alexandrov spaces into compact Riemannian manifolds

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Contents

We want to introduce some regularity result, in particular the Lipschitz regularity, for harmonic maps from Alexandrov spaces.

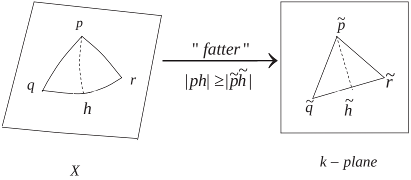
§1. A brief introduction of Alexanrdov geometry

Alexandrov spaces are of a class of singular metric spaces with curvature bounded from below in the sense of comparison triangles.

Alexandrov curvature: (A. Wald (40'), A. D. Alexandrov (40'–50'))

(X, d) is said to have **curvature $\geqslant k$** (in the sense of Alexandrov), if

- d is geodesic; I.e, $\forall\ p, q \in X$, $\exists\ \gamma$ from p to q such that $L(\gamma) = d(p, q)$;
- (Toponogov comparison) any triangle $\triangle pqr$ in X is "fatter" than the triangle $\triangle \tilde{p}\tilde{q}\tilde{r}$ in k-plane with $|pq| = |\tilde{p}\tilde{q}|$, $|qr| = |\tilde{q}\tilde{r}|$ and $|rp| = |\tilde{r}\tilde{p}|$,

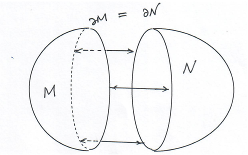


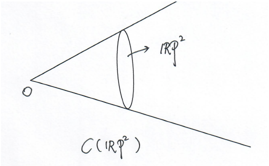
Remark: (1) If X is smooth and $d = d_g$, then $curv \geqslant k \iff sec_g \geqslant k$.

(2) $CAT(k)$ ($curv \leqslant k$ globally) can be defined by replacing"fatter" by "thiner".

Examples and constructions:

- Convex polyhedra; Riem. manifold (M, g) with $sec_g \geqslant k$;
- Curvature bounded from below (CBB) is closed under some natural geometric constructions: the gluing, cone, suspension and join:





- The condition *CBB* is stable under Gromov-Hausdorff convergence: If $X_j \xrightarrow{d_{GH}} X$ and $curv_{X_j} \geqslant k$, then $curv_X \geqslant k$.

Topology:

- $\dim(X) \in \mathbb{N} \cup \{\infty\}$, the Hausdorff dimension; (Burago–Gromov–Perelman)
- Stability theorem (Perelman) :

$$\left.\begin{array}{ll} X_j \xrightarrow{d_{GH}} X_\infty, & curv_{X_j} \geqslant k, \\ \dim X_j = \dim X_\infty, & diam(X_j) \leqslant D \end{array}\right\} \Longrightarrow X_j \overset{homeo}{\cong} X_\infty, \ as\ j\ large$$

- Topological statification : (Perelman)

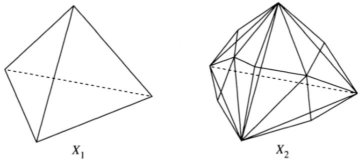
$$X = X_1 \cup X_2 \cup \cdots X_k \cup \cdots \cup X_n,$$

where the each strata X_k is an k-dim **topological manifold** or $\emptyset$.

Singularity:

Let (X, d) have *curv* $\geqslant k$, $\dim = n$.

- p is called a (metrically) singular point iff $d_{GH}\text{-}\lim_{\lambda\to\infty}(X, \lambda d, p) \neq \mathbb{R}^n$.
  (equivalently, $\lim_{r\to 0}\frac{\mu(B_r(p))}{\omega_{n-1}r^n} \neq 1$.)
- $Sing(X)$ has co-dim $\geqslant 1$ but it maybe dense. (Otsu-Shioya, 1994)
  (If $\partial X = \emptyset$, then $Sing(X)$ has co-dim $\geqslant 2$.)



(Ostu-Shioya's example: the limit space of a sequence of appropriate convex surfaces in $\mathbb{R}^3$.)

Riemannian structure:

- (Perelman, Otsu-Shioya) There exists a open set (with full measure)

$$X^* \supset Reg(X) := X \backslash Sing(X)$$

is a Lipschitz manifold and a (incomplete) Riemannian metric $\{g_{ij}\}$ on X^* such that

  - $g_{ij} \in BV_{\rm loc} \cap L^\infty_{\rm loc}(X^*)$;
  - $g_{ij}(x)$ is continuous at each $x \in Reg(X)$.

Remark: (1) In general, $Reg(X)$ is not a manifold, and that $X^* \cap Sing(X)$ maybe dense.

(2) If $n = 2$, Ambrosio-Bertrand (2016'): $g_{ij} \in W^{1,p}_{\rm loc}(X^* \backslash \mathcal{S}_p)$ for each $p \in [1, 2)$, where $\mathcal{S}_p$ is a discrete set.

Analysis:

- Given a Lipschitz function f, the gradient field ∇f

$$\nabla^i f = g^{ij}\partial_j f$$

is well-defined in $L^\infty(X)$. The Sobolev norm on a open subset Ω is

$$\|f\|_{W^{1,p}(\Omega)} := \|f\|_{L^p} + \||\nabla f|\|_{L^p}$$

The Sobolev spaces $W^{1,p}(\Omega)$ (and $W_0^{1,p}(\Omega)$, $W_{\mathrm{loc}}^{1,p}(\Omega)$) is the closure of Lipschitz functions under the $W^{1,p}$-norm. (it coincides with the ones introduced by Korevaar-Schoen, Cheeger, Shanmugalingam, and Ambrosio-Gigli-Savaré).

Analysis — continue:

- Let $f \in W^{1,2}_{\rm loc}(\Omega)$. We say $f \in D(\Delta)$ if $\exists\ g \in L^2_{\rm loc}(\Omega)$ such that: it holds the divergence theorem $\int_\Omega \langle \nabla f, \nabla \phi \rangle\, d\mu = -\int_\Omega g\phi d\mu$ for all test function $\phi \in W^{1,2}_0(\Omega)$. In this case, we write $\Delta f = g$ (the Laplacian of f).

- If $\Omega = X$ and $f \in W^{1,2}(X)$, the Δf is same as the infinitesimal generator of Dirichlet form $\mathcal{E}(f,f) := \int |\nabla f|^2 d\mu$ by Gigli.

- Kuwae-Machigashira-Shioya proved that $\mathrm{Lip}_0(\Omega \cap X^*) \subset W^{1,2}_0(\Omega)$ is dense.
  That is, a $W^{1,2}$-function can be approximated by Lipschitz functions supported in almost regular part of X.

Contents

Harmonic maps (HM):

- Notations:
  - $\Omega \subset X$ —– a bounded domain, $curv_X \geqslant k$, $\Omega \cap \partial X = \emptyset$;
  - N —– a compact Riemannian manifold.
- **Energy**
  - $W^{1,2}(\Omega, N)$: Let $N \subset \mathbb{R}^\ell$ by Nash imbedding theorem. Then

$$W^{1,2}(\Omega, N) := \left\{u \in W^{1,2}(\Omega, \mathbb{R}^\ell) \middle|\ u(x) \in N \ \text{ for } \mu\text{−a.e. } x \in \Omega\right\}.$$

  - $E(u)$: let $u \in W^{1,2}(\Omega, N)$, the energy is defined by

$$E(u) := \int_\Omega e_u \mathrm{d}\mu, \qquad e_u := |\nabla u|^2.$$

- **Harmonic map (HM)** $:=$ a critical point of $E(u)$.

If the target $N = \mathbb{R}$, it is just a harmonic function.

  - **minimizing HM** = a (local) minimizer of $E(u)$.
  - **(weakly) HM**, if we perturb the embedding $N \subset \mathbb{R}^\ell$, we get

$$\frac{d}{dt}E(\pi(u+tv))\Big|_{t=0} = 0,$$

Euler-Lagrange equation:

$$\Delta u + A(\nabla u, \nabla u) = 0,$$

where A is the second fundamental form of $N \subset \mathbb{R}^\ell$;

  - minimizing HM $\subset$ (weakly) HM.

Recall the regularity for HM between smooth manifolds:

- $\dim(\Omega) = 2$, weakly HM is smooth (Morrey, Schoen, Helein (91'));
- $\dim(\Omega) \geq 3$, Energy miniming map is smooth off a closed set $Z_u \subset \Omega$ with $\dim(Z_u) \leq n - 3$.

  If $sec_N \leq 0$, the exceptional set $Z_u = \emptyset$. (Schoen-Uhlenbeck (83'), Giaquinata-Ginsti);
- There is a weakly HM $u : \mathbb{B}^3 \to \mathbb{S}^2$, everywhere discontinuous (Riviere (95')).

The study of HM in the singular setting was initiated by M. Gromov & R. Schoen. They proved the following regularity result:

THEOREM (GROMOV-SCHOEN, 92', KOREVAAR-SCHOEN, 93')

Let B_1 be the unit ball in $\mathbb{R}^n$ and let Y be of CAT(0). Suppose $u : (B_1, g) \to Y$ is a minimizing map, where the Riemannian matric g is of C^1. Then u is Lipschitz continuous in $B_{1/2}$.

After this, the regularity problem for the HM in the singular settings can be described as the following three directions:

- from a singular domain into a smooth target;
- from a smooth domain into a singular target;
- Both the domain and the target are singular.

Remark:

- In the smooth setting: the basic tools are
  - E-L equation $\Delta u = -A(\nabla u, \nabla u)$,
  - Monotonicity: the normalized energy $r^{2-n} \int_{B_r(x)} e_u d\mu$ is non-increasing.
- Main different in the three directions:

domain	target	E-L equation	Monotonicity
singular	smooth	Yes	No
smooth	singular	No	Yes
singular	singular	No	No

Now, we will focus on: **from a singular domain into a smooth target.**

Partial Hölder continuity (for HM from a singular domain):

THEOREM

- *(Y. G. Shi, 1996) Let B_1 be the unit ball in $\mathbb{R}^n$, and let g be a L^∞-metric on B_1, with $\Lambda^{-1}I \leqslant g \leqslant \Lambda I$, where Λ is a positive constant, I is the unit matrix. Assume $u : (B_1, g) \to N$ is an energy minimizing map. Then u is Hölder continuous off a closed set Z_u with $\mathcal{H}^{n-2}(Z_u) = 0$.*

- *(F. H. Lin, 1997) Let Ω be a open domain in an Alexandrov space X of CBB. Suppose $u : \Omega \to N$ is an energy minimizing map, then u is Hölder continuous off a closed exceptional set Z_u with $\dim(Z_u) \leqslant n - 3$.*

Remark: If we assume only $g \in L^\infty$ (in the first item), the Hölder continuity is sharp.

More generally, the **Hölder continuity** also holds for energy minimizing maps $u : \Omega \subset X \to Y$ in the following settings:

X	Y	
simplicial complex	$CAT(0)$-complex	J. Chen ('95), Daskalopoulos-Mese ('08 quantitative Hölder index
Riem. polyhedron	$CAT(0)$-space	Eells-Fuglede ('01)
Riem. polyhedron	$CAT(1)$-space, image in regular ball	Fuglede ('03)
manifold with L^∞-Riem. metric	smooth	Y. G. Shi ('96) Ishizuka-C. Y. Wang ('08) (for (weak) HM)

- Recall that any Hölder continuous HM between two smooth manifolds can be improved to be smooth, by the elliptic regularity theory.
- From the above Gromov-Schoen's Lipschitz regularity theorem, a natural question is how to improve the Hölder continuity (in the above settings) to a "higher regularity".
- (Jost-Lin conjecture: From Hölder to Lipschitz)
  F. H. Lin (1997) conjectured that the Hölder continuity for minimizing maps, from an Alexandrov space with CBB into a compact Riemannian manifold, can be improved to the Lipschitz continuity. A similar problem about Lipschitz regularity was given by Jost (1998).

The obstructions from Hölder to Lipschitz:

1. The elliptic regularity theory does not work :

$$\text{Elliptic Regularity}: \quad g \in C^{\alpha}, \quad h \in C^{1,\alpha}, \quad u \in C^{\alpha} \Rightarrow u \in C^{1,\alpha}.$$

But, it is not true for $\alpha = 0$. Indeed, $g \in C^0$, $h \in C^\infty$, $u \in C^0 \not\Rightarrow u \in \mathrm{Lip}$.

- **Example**:(T. Jin, V. Maz'ya & J. Schaftingen (2009))
  Let $\alpha(r) = \frac{-n}{(n-1)\log\frac{r_0}{r}}$ for some r_0 enough large, and let
  $$\sqrt{g}g^{ij}(x) := \delta_{ij} + \alpha(|x|)(\delta_{ij} - \frac{x_i x_j}{|x|^2}) \in C(B_1, \mathbb{R}^{n\times n}).$$
  Then: $u = x_1 \cdot \log\frac{r_0}{|x|}$ solves $\partial_i(\sqrt{g}g^{ij}\partial_j u) = 0$ on B_1, but $Du \notin L^\infty(B_{1/2})$.
- Recall that $\{g_{ij}\}$ on $X^* \subset X$,
  - $\{g_{ij}\} \in BV_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(X^*)$, and is continuous at $x \in X^* \backslash Sing(X)$;
  - but, $\{g_{ij}\}$ may be NOT continuous on a *dense* subset of X^*.

The obstructions — continue:

2. Gromov-Schoen's and Korevaar-Schoen's method:

THEOREM (GROMOV-SCHOEN, 92', KOREVAAR-SCHOEN, 93')

From smooth domain to singular target:
Let B_1 be the unit ball in $\mathbb{R}^n$ and let Y be of $CAT(0)$. Let $u : (B_1, g) \to Y$ be a minimizing map, where the g is of C^1. Then u is Lipschitz continuous in $B_{1/2}$.

- The Lipschitz norm of u on $B_{1/2}$ depends on the C^1-norm of g.
- Daskalopoulos-Mese (08'): It depends on the Lip norm of g.
- Serbinowski (95'), Breiner-Fraser-Huang-Mese-Sargent-Zhang (18') extends the regularity to HM into a regular ball of a $CAT(1)$-space. The Lipschitz norm of u on $B_{1/2}$ depends on the Lip norm of g.

The obstructions — continue:

3. Bochner method:

Theorem (Z-Zhu, 18')

Both domain and target are singular:
Let $B_R(x)$ be a ball with radius R in an n-dimensional Alexandrov space with curv $\geqslant k$. Y is a $CAT(0)$-space. Then any minimizing map $u : B_R(x) \to Y$ is Lipschitz continuous in $B_{R/2}(x)$.

- In fact, we deduced a weak Bochner inequality:

$$\Delta e_u \geqslant 2nke_u$$

  in the sense of distributions. This implies $e_u \in L^\infty_{\mathrm{loc}}$.

- Recently, Gigli-Tyulenev consider how to extend a Bochner-type formula for HM from an RCD-space into a $CAT(0)$-space.

The obstructions — continue:

3. Bochner method:

- It is essential to assume that u is $CAT(0)$-valued.
- Indeed, recall the classical Bochner formula for $u : M \to N$

$$\Delta e_u \geqslant 2Ke_u - 2\kappa e_u^2,$$

provided $Ric_M \geqslant K$ and $sec_N \leqslant \kappa$.

- Suppose that $\kappa = 1$ and $K = 0$, then $\Delta e_u \geqslant -2{e_u}^2$. To conclude $e_u \in L^\infty_{\rm loc}$, we must assume a monotonicity for the $r^{2-n}\int_{B_r(x)} e_u d\mu$ (or $e_u \in L^{n/2}$).

Main result:

THEOREM (H. GE, W. JIANG & Z, ARXIV:1907.09646)

From singular domain to smooth target:
Let Ω be a bounded domain of an n-dimensional Alexandrov space with CBB, and let N be a compact smooth Riemannian manifold.
Then any continuous HM (need not be energy minimizing map) $u : \Omega \to N$ is locally Lipschitz continuous in Ω.

Remark: (1) The Lipschitz constant of u depends on $n, k, \Omega, \int_\Omega e_u d\mu$ and the $\sup_N |A|$, where A is the second fundament form of $N \subset \mathbb{R}^\ell$.

(2) This solves the above Jost-Lin's conjecture.

CONTENTS

Notations:

- $\Omega \subset X$ —— a bounded domain, $curv_X \geqslant k$, $\Omega \cap \partial X = \emptyset$;
- N —— a compact Riemannian manifold.

Main ingredients:

- Euler-Langrange equation:

$$\Delta u = -A(\nabla u, \nabla u).$$

(recalling that ∇u makes sense in L^p, and Δu is a measure.)

- Alfors regularity: $c_1 \leqslant \frac{\mu(B_r(x))}{r^n} \leqslant c_2$ for all ball $B_r(x) \subset \Omega$.

- Bochner formula for functions: $\Delta|\nabla f|^2 \geqslant -c|\nabla f|^2$ for any harmonic function on Ω.

- Gradient estimate for heat kernel:

$$|\nabla p_t(\cdot, y)|(x) \leqslant \frac{c}{\sqrt{t}\mu(B_{\sqrt{t}}(x))} \exp\Big(-\frac{d^2(x,y)}{5t} + ct\Big).$$

Lemma (Decay estimate, due to T. Huang & C. Y. Wang 10')

Let Ω, N, u be as above. Then it holds: For any $\alpha \in (0,2)$, if

$$\mathrm{osc}_{B_{r_0}(x_0)} u < \epsilon, \quad \text{(depending on } \alpha, n, k, \Omega, \mathrm{inj}(N), \sup_N |A|,)$$

then for any ball $B_r(x) \subset B_{r_0}(x_0)$,

$$r^{2-n} \int_{B_r(x)} e_u(y) d\mu(y) \leqslant C r^{\alpha}, \qquad C = C_{\alpha,n,k,\Omega,r_0}.$$

Remark: By this decay estimate, Huang-Wang obtain a $W^{1,p}$-regularity property for harmonic map from the Euclidean domain to N, when the oscillation is sufficiently small.

Lemma (Ge-Jiang-Z, 19')

Let $u \in W^{1,2}(B_R) \cap L^\infty(B_R)$ *solve*

$$\Delta u = f \in L^1(B_R), \quad f \geqslant 0.$$

Then

$$|\nabla u|(x) \leqslant C\|u\|_{L^\infty} + C\int_{B_R}\Big(\frac{f(y)}{d^{n-1}(x,y)} + |\nabla u|(y)\Big)d\mu(y)$$

for almost all $x \in B_{R/4}$.

Remark: The proof is based on the gradient estimate of heat kernels.

By using the above gradient estimate and a potential-theoretic argument, one can improve the Huang-Wang's $W^{1,p}$-regularity to the following $W^{1,\infty}$-regularity:

LEMMA (GE-JIANG-Z, 19')

Let Ω, N, u be as above. Then $\exists\epsilon > 0$ (depending on $n, k, \Omega, \mathrm{inj}(N)$, and $\sup_N |A|$), such that if

$$\mathrm{osc}_{B_{r_0}(x_0)} u < \epsilon,$$

then for any $Q \in N$ near the Image(u),

$$\sup_{B_r(x)} |\nabla u_Q| \leqslant C,$$

where $u_Q := d_N^2(Q, u(x))$.

Thank you very much!