Harmonic measure and harmonic analysis

Alexander Volberg

MSU

August 2020

1. Metric properties of harmonic measure

What follows is several joint works with various subset of co-authors: {J. Azzam, S. Hofmann, J. M. Martell, S. Mayboroda, M. Mourgoglou, X. Tolsa}

1a. A bit of history, the 90's

Carleson 1974: $\Omega \subset \mathbb{R}^2$ simply connected then dim $\omega \geq \frac{1}{2} + \varepsilon_0$.
Makarov 1989: $\Omega \subset \mathbb{R}^2$ simply connected then dim $\omega = 1$.
P. Jones, T. Wolff 1990: $\Omega \subset \mathbb{R}^2$ then dim $\omega \leq 1$.
J. Bourgain 1990: $\Omega \subset \mathbb{R}^{n+1}$ then dim $\omega \leq n + 1 - \varepsilon_1$.
T. Wolff, 1991: There exists $\Omega \subset \mathbb{R}^{n+1}$ such that dim $\omega \geq n + \varepsilon_2$.

2. Bishop–Jones theorem of 1991 and their question

Ω simply connected, $\Omega \subset \mathbb{R}^2$, and let Γ be some rectifiable curve. Then if $E = \Gamma \cap \partial\Omega$, then $\omega|E << \mathcal{H}^1|E$.

Their question: is the opposite true? Namely, $\omega|E << \mathcal{H}^1|E$, $\omega(E) > 0$, does this imply that $\omega|E$ is rectifiable? Meaning, in particular, the existence of rectifiable (or Lipschitz) curve Γ such that

$$\omega(\Gamma \cap E) > 0\,.$$

3. Three phase problem, joint work with X. Tolsa

In a paper from 1997 Tsirelson [Ts] proved the following result, previously conjectured by Bishop in [Bi] (see also Problem **a** in [EFS2]):

Theorem

[Ts] Let $\Omega_1, \Omega_2, \Omega_3 \subset \mathbb{R}^{n+1}$ be disjoint open connected sets, with harmonic measures $\omega_1, \omega_2, \omega_3$. Let $E \subset \partial\Omega_1 \cap \partial\Omega_2 \cap \partial\Omega_3$ so that $\omega_1, \omega_2, \omega_3$ are mutually absolutely continuous in E. Then $\omega_i(E) = 0$ for $i = 1, 2, 3$.

We remark that the planar case $n = 1$ of the preceding theorem had previously been proved by Bishop [Bi] and Eremenko, Fuglede, and Sodin [EFS1]. In the higher dimensional case, another previous partial result had been obtained by Bishop in [Bi]. Namely he had shown that if $\Omega_1, \dots, \Omega_m \subset \mathbb{R}^{n+1}$ are disjoint domains with harmonic measures $\omega_1, \dots, \omega_m$ which are mutually absolutely continuous in $E \subset \bigcap_{j=1}^m \partial\Omega_j$, then $\omega_j(E) = 0$ if $m = 5$ in $\mathbb{R}^3$, or if $m = 11$ in any dimension.

Lemma

Let $n \geq 2$ and $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain. Denote by ω^p its harmonic measure with pole at p and by G its Green function. Let $B = B(x_0, r)$ be a closed ball with $x_0 \in \partial\Omega$ and $0 < r < \mathrm{diam}(\partial\Omega)$. Then, for all $a > 0$,

$$\omega^x(aB) \gtrsim \inf_{z \in 2B \cap \Omega} \omega^z(aB)\, r^{n-1}\, G(x,y) \quad \text{for all } x \in \Omega \setminus 2B \text{ and } y \in B \cap \Omega \tag{1}$$

with the implicit constant independent of a.

Lemma

There is $\delta_0 > 0$ depending only on $n \geq 1$ so that the following holds for $\delta \in (0, \delta_0)$. Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded, $n-1 < s \leq n+1$, $\xi \in \partial\Omega$, $r > 0$, and $B = B(\xi, r)$. Then

$$\omega^x(B) \gtrsim_{n,s} \frac{\mathcal{H}^s_\infty(\partial\Omega \cap \delta B)}{(\delta r)^s} \quad \text{for all } x \in \delta B \cap \Omega.$$

4a. Figure illustrating G, ω

Let $\xi \in E$ and $r > 0$. Suppose that

$$\mathcal{H}^{n+1}(B(\xi, r) \cap \Omega_3) = \max_{1 \le i \le 3} \mathcal{H}^{n+1}(B(\xi, r) \cap \Omega_i). \qquad (2)$$

Then

$$\mathcal{H}^{n+1}(B(\xi, r) \cap \Omega_1) + \mathcal{H}^{n+1}(B(\xi, r) \cap \Omega_2) \le \frac{2}{3} \mathcal{H}^{n+1}(B(\xi, r)). \quad (3)$$

From Lemmas 2 and 3 we deduce that if (2) holds for $\xi \in E$ and $r > 0$, then, for $i = 1, 2$,

$$\omega_i^x(B(\xi, \delta_0^{-1} r) \gtrsim r^{n-1}\, G_i(x, y), \text{for } x \in \Omega_i \setminus B(\xi, 2r) \text{ and } y \in B(\xi, r) \cap \Omega. \quad (4)$$

Lemma

Let $a > 1$. Let $\omega_1, \omega_2, \omega_3$ and E be as in Theorem 1. There exists $b = b(a) > 1$ such that for ω_i-a.e. $\xi \in E$ there exists a sequence of ω^j-(a, b)-doubling balls $B(\xi, r_k)$ simultaneously for $j = 1, 2, 3$, with $r_k \to 0$. That is,

$$\omega_j(B(\xi, ar_k)) \leq b\,\omega_j(B(\xi, r_k)) \qquad \text{for } j = 1, 2, 3 \text{ and for all } k \geq 1.$$

It is well know that, given any Radon measure μ in $\mathbb{R}^{n+1}$, if we choose $b > a^{n+1} > 1$, then for μ-a.e. $\xi \in \mathbb{R}^{n+1}$ there exists a sequence of μ-(a, b)-doubling balls $B(\xi, r_k)$, with $r_k \to 0$. Apply this to ω_1.

We claim now for ω_1-a.e. $\xi \in E$ and $j = 2, 3$,

$$\lim_{r \to 0} \frac{\omega_1(B(\xi, ar))}{\omega_1(B(\xi, r))} \frac{\omega_j(B(\xi, r))}{\omega_j(B(\xi, ar))} = 1. \tag{5}$$

Lebesgue differentiation theorem.

7. Proof of the 3 phase theorem

We may assume the domains Ω_i to be bounded. We fix poles $p_i \in \Omega_i$ for the harmonic measures ω_i, with p_i deep inside Ω_i, and for simplicity we write $\omega_i = \omega_i^{p_i}$. We denote by $h_{i,j}$ the density function of ω_i with respect to ω_j on E. That is,

$$\omega_i|_E = h_{i,j}\,\omega_j|_E.$$

Let $\xi \in E$ be a Lebesgue point for χ_E and for all the density functions $h_{i,j}$ and so that there exists a decreasing sequence of radii $r_k \to 0$ satisfying the property described in Lemma on the previous slide, for some constant $a > 2$ big enough to be chosen below. We may assume that there exists an infinite subsequence of radii such that

$$\mathcal{H}^{n+1}(B(\xi, r_k) \cap \Omega_3) = \max_{1 \le i \le 3} \mathcal{H}^{n+1}(B(\xi, r_k) \cap \Omega_i).$$

By renaming the subsequence $\{r_k\}_k$ if necessary, we assume that this holds for all $k \ge 1$.

Denote $\mathcal{B} = B(0,1)$ and consider the affine map $T_k(x) = (x - \xi)/r_k$, so that $T_k(B(\xi, r_k)) = \mathcal{B}$. For $i = 1, 2, 3$ and $k \geq 1$ take the measures

$$\omega_i^k = \frac{1}{\omega_i(B(\xi, r_k))} \, T_k \# \omega_i.$$

Notice that

$$1 = \omega_i^k(\mathcal{B}) \leq \omega_i^k(a\mathcal{B}) = \frac{\omega_i(B(\xi, ar_k))}{\omega_i(B(\xi, r_k))} \leq b,$$

where $a\mathcal{B} = B(0, a)$. Hence there is a subsequence of radii r_k so that $\omega_i^k \to \omega_i^\infty$ weakly in $\frac{a}{2}\mathcal{B}$ as $k \to \infty$, for some Borel measure ω_i^∞ such that

$$1 \leq \omega_i^\infty(\overline{\mathcal{B}}) \leq \omega_i^\infty(\tfrac{a}{2}\mathcal{B}) \leq b$$

for $i = 1, 2, 3$ and all $k \geq 1$.

For $i = 1, 2, 3$ and $k \geq 1$ consider now the functions

$$u_i^k(x) = \frac{r_k^{n-1}}{\omega_i(B(\xi, r_k))} G_i(p_i, T_k^{-1}(x)), \tag{6}$$

so that, for any C^∞ compactly supported function φ, we have

$$\int \varphi \, d\omega_i^k = \frac{\int \varphi \circ T_k \, d\omega_i}{\omega_i(B(\xi, r_k))} = \frac{1}{\omega_i(B(\xi, r_k))} \int \Delta(\varphi \circ T_k) \, G_i(p_i, x) \, dx = \tag{7}$$

$$\frac{\int \Delta\varphi(T_k x) \, G_i(p_i, x) \, dx}{r_k^2 \, \omega_i(B(\xi, r_k))} = \frac{r_k^{n-1}}{\omega_i(B(\xi, r_k))} \int \Delta\varphi(y) \, G_i(p_i, T_k^{-1} y) \, d$$

$$= \int \Delta\varphi \, u_i^k \, dy.$$

Notice also that u_i^k is a non-negative function, which is harmonic in $a\mathcal{B} \cap T_k(\Omega_i)$. Further, for $i = 1, 2$, by (4), assuming r_k small enough and choosing $a > \delta_0^{-1}$, for all $x \in \delta_0 a\mathcal{B} \cap T_k(\Omega_i)$,

$$u_i^k(x) \leq C(b). \tag{8}$$

WLOG we assume Dirichlet regularity of domains, so the functions u_i^k extend continuously by zero in $a\mathcal{B} \setminus T_k(\Omega_i)$. We continue to denote by u_i^k such extensions, which are subharmonic in $a\mathcal{B}$. By Caccioppoli's inequality and the uniform boundedness of u_i^k in $\delta_0 a\mathcal{B}$ we deduce that, for $i = 1, 2$,

$$\|\nabla u_i^k\|_{L^2(\frac{1}{4}\delta_0 a\mathcal{B})} \lesssim \|u_i^k\|_{L^2(\frac{1}{2}\delta_0 a\mathcal{B})} \lesssim_b 1.$$

In other words, this is Jensen ineq. applied to $(u_i^k)^2$.

By the Rellich-Kondrachov theorem, the unit ball of the Sobolev space $W^{1,2}(\frac{1}{4}\delta_0 a\mathcal{B})$ is relatively compact in $L^2(\frac{1}{4}\delta_0 a\mathcal{B})$, and thus there exists a subsequence of the functions u_i^k which converges *strongly* in $L^2(\frac{1}{4}\delta_0 a\mathcal{B})$ to another function $u_i \in L^2(\frac{1}{4}\delta_0 a\mathcal{B})$. Passing to a subsequence, we assume that the whole sequence of functions u_i^k converges in $L^2(\frac{1}{4}\delta_0 a\mathcal{B})$ to $u_i \in L^2(\frac{1}{4}\delta_0 a\mathcal{B})$. In particular, from (7), passing to the limit if follows that

$$\int \varphi \, d\omega_i^\infty = \int \Delta\varphi \, u_i \, dx, \tag{9}$$

for any C^∞ function φ compactly supported in $\frac{1}{4}\delta_0 a\mathcal{B}$. So

$$\omega_i^\infty = \Delta u_i \tag{10}$$

Consider the function

$$u(x) = u_1(x) - u_2(x).$$

Note that, by the previous slide

$$\Delta u_1 - \Delta u_2 = \omega_1^\infty - \omega_2^\infty = 0.$$

We claim that

$$\omega_1^\infty = \omega_2^\infty \quad \text{in } \tfrac{1}{2}a\mathcal{B}. \tag{11}$$

Explanation....
Assuming (11) for the moment, we deduce that $\Delta u = 0$ in $\frac{1}{4}\delta_0 a\mathcal{B}$ in the sense of distributions.
So u is harmonic in the ball. It is also $= 0$ on $\operatorname{supp}\omega_1^\infty = \operatorname{supp}\omega_2^\infty$.
So this support lies in a real analytic variety if $u \neq 0$ identically.

Let us check that u does not vanish identically in $\frac{1}{4}\delta_0 a\mathcal{B}$. Since the domains Ω_1 and Ω_2 are disjoint, it follows that

$$u_1^k \cdot u_2^k = 0,$$

hence

$$u_1 \cdot u_2 = 0.$$

If it were true that $u_1 = u_2$ identically, then we would conclude that separatetely $u_1 = 0$ and $u_2 = 0$ identically. But $\Delta u_i = \omega_i^\infty \neq 0$ Contradiction.
Next we intend to get a contradiction by showing that u vanishes in a set of positive Lebesgue measure in $\mathcal{B} \subset \frac{1}{4}\delta_0 a\mathcal{B}$, which is impossible because the zero set of any harmonic non-vanishing function is a real analytic variety.

Recall that, by choosing $\Omega_3 \cap B(\xi, r_k)$ having the largest volume, we conclude that there exists a set $F_k \subset B(\xi, r_k) \setminus (\Omega_1 \cup \Omega_2)$ such that $\mathcal{H}^{n+1}(F_k) \gtrsim r_k^{n+1}$. Hence, denoting $G_k = T_k(F_k)$, we infer that

$$\int_{G_k} |u_1^k - u_2^k| \, dx = 0.$$

We may assume that χ_{G_k} converges weakly in $L^2(\mathcal{B})$ to some non-negative function $g \in L^2(\mathcal{B})$. Clearly, $\|g\|_{L^2(\mathcal{B})} \lesssim 1$ and

$$\int_{\mathcal{B}} g \, dx = \langle \chi_{\mathcal{B}}, g \rangle = \lim_k \langle \chi_{\mathcal{B}}, \chi_{G_k} \rangle \gtrsim 1.$$

Also, by the strong convergence of $|u_1^k - u_2^k|$ and the weak convergence of χ_{G_k},

$$\int |u_1 - u_2| \, g \, dx = \lim_k \int |u_1^k - u_2^k| \, \chi_{G_k} \, dx = 0,$$

which implies that $u_1 - u_2$ vanishes on a set of positive Lebesgue measure in $\mathcal{B}$ and provides the aforementioned contradiction.

15. Two phase problem, joint work with J. Azzam, M. Mourgoglou and X. Tolsa

Set

$$\Lambda_1 = \left\{ \xi \in E^*\colon 0 < h(\xi) := \frac{d\omega_2}{d\omega_1}(\xi) = \lim_{r\to 0} \frac{\omega_2(B(\xi,r))}{\omega_1(B(\xi,r))} = \lim_{r\to 0} \frac{\omega_2(E\cap B(\xi,r))}{\omega_1(E\cap B(\xi,r))} < \infty \right\}$$

and

$$\Gamma = \{\xi \in \Lambda_1 : \xi \text{ is a Lebesgue point for } h \text{ with respect to } \omega_1\}.$$

Again, by Lebesgue differentiation for measures Γ has full measure in E^* and hence in E.

We blow-up at such ξ's. The last stage of the previous proof does not work. We have no extra domain to prove that blow-up functions u_1, u_2 are such that $u := u_1 - u_2$ is zero on a set of positive Lebesgue measure. But we still conclude that Z_u is a real analytic variety. Let's see.

$\mathcal{F} = \{c\mathcal{H}^n \,|\, V : c > 0, V$ is n-dimensional hyperplane containing the origin$\}$

Lemma

Let $\xi \in \partial\Omega_1 \cap \partial\Omega_2$ be such that $h(\xi) \in (0, \infty)$ and $c_j > 0$ are such that $c_j T_{\xi,r_j}[\omega_1] \to \omega_1^\infty \in Tan(\omega_1, \xi)$, then $c_j T_{\xi,r_j}[\omega_2] \to h(\xi)\omega_1^\infty$ (ω_i-a.e. $\xi \in \Gamma$, for example). So $Tan(\omega_1, \xi) \cap \mathcal{F} \neq \emptyset$.

Here $\{r_j\}$ is the doubling sequence for, say ω_1 at ξ and also Ω_1 has **thick complement** at those scales. Then we have that Ω_2 has **thick complement** at those scales too! As a result $u^j_{1,2} := c_j r_j^{n-1} G_{1,2}(T^{-1}_{\xi,r_j} x, p_{1,2})$ will converge to u_1, u_2, and $u = h(\xi)u_1 - u_2$ will have real analytic Z_u (as before), which will imply $\mathrm{Tan}(\omega_1, \xi) \cap \mathcal{F} \neq \emptyset$.

16b.

Why if Ω_1 has **thick complement** at some scales, then we have that Ω_2 has **thick complement** at the same scales too! Here is why.

Lemma

Let B be a ball of radius r centered at $\partial\Omega$, and $Vol(B \setminus \Omega) \geq cr^{n+1} = cVol(B)$. Let also $\omega(4B) \leq C\omega(\delta_0 B)$. Then $Vol(2\delta_0 B \cap \Omega) \geq c' r^{n+1}$.

Proof.

Let φ be bell function supported on $2\delta_0 B$, and $\varphi = 1$ on $\delta_0 B$. Then

$\omega(\delta_0 B) \leq \int \varphi d\omega = \int u\Delta\varphi \leq r^{-2}\left(\max_{2\delta_0 B} u\right) \cdot Vol(2\delta_0 B \cap \Omega) \leq Cr^{-2} Vol(2\delta_0 B \cap \Omega)\omega(4B) \leq C' r^{-n-1} Vol(2\delta_0 B \cap \Omega)\omega(\delta_0 B)$.

And we get $Vol(2\delta_0 B \cap \Omega) \geq c' r^{n+1}$. □

17. Amazing lemma

Lemma

Let Ω_1 *and* Ω_2 *be as above and let* $\xi \in \Gamma$*. If* $Tan(\omega_1, \xi) \cap \mathcal{F} \neq \varnothing$*, then* $Tan(\omega_1, \xi) \subset \mathcal{F}$*.*

It is a rather difficult lemma of Kenig–Preiss–Toro, that uses seriously the technique of tangent measures of Preiss paper ann. of Math., 1987, and also uses again the fact that $\omega_1|E$, $\omega_2|E$ are mutually absolutely continuous.

But then, by another theorem of Preiss from the same article, ω_i **are doubling in all scales for every** $\xi \in \Gamma$**.**

In particular, both Ω_1, Ω_2 have **thick complement** at **EVERY** scale for any $\xi \in \Gamma$. (The constants of course are not uniform, they depend on ξ.)

18. The Alt-Caffarelli-Friedman monotonicity formula

The following theorem contains the Alt-Caffarelli-Friedman monotonicity formula:

Theorem

Let $B(x, R) \subset \mathbb{R}^{n+1}$, and let $u_1, u_2 \in W^{1,2}(B(x, R)) \cap C(B(x, R))$ be nonnegative subharmonic functions. Suppose that $u_1(x) = u_2(x) = 0$ and that $u_1 \cdot u_2 \equiv 0$. Set

$$\gamma(x, r) = \left(\frac{1}{r^2} \int_{B(x,r)} \frac{|\nabla u_1(y)|^2}{|y - x|^{n-1}} dy \right) \cdot \left(\frac{1}{r^2} \int_{B(x,r)} \frac{|\nabla u_2(y)|^2}{|y - x|^{n-1}} dy \right). \tag{12}$$

Then $\gamma(x, r)$ is a increasing function of $r \in (0, R)$ and $\gamma(x, r) < \infty$ for all $r \in (0, R)$. That is,

$$\gamma(x, r_1) \leq \gamma(x, r_2) < \infty \quad \text{for} \quad 0 < r_1 \leq r_2 < R. \tag{13}$$

For for $0 < r < R/4$ and $\xi \in \partial\Omega_1 \cap \partial\Omega_2$, let Ω_1 and Ω_2 have **thick complement at scale r at ξ**. Then by Jensen inequality

$$\frac{\omega_i(B(\xi,r))}{r^n} \lesssim \left(\frac{1}{r^2}\int_{B(\xi,2r)} \frac{|\nabla u_i(y)|^2}{|y-\xi|^{n-1}}dy\right)^{\frac{1}{2}} \lesssim \left(\frac{1}{r^{n+3}}\int_{B(\xi,4r)} |u_i|^2\right)^{\frac{1}{2}} \tag{14}$$

But $\left(r^{-n-3}\int_{B(\xi,4r)}|u_i|^2\right)^{\frac{1}{2}} \lesssim r^{-n}\omega_i(B(\xi,r))$ **by thickness!**
Hence,

$$\frac{\omega_1(B(\xi,r))}{r^n}\,\frac{\omega_2(B(\xi,r))}{r^n} \asymp \left(\frac{1}{r^{n+3}}\int_{B(\xi,4r)} |u_1|^2\right)^{\frac{1}{2}}\left(\frac{1}{r^{n+3}}\int_{B(\xi,4r)} |u_2|^2\right)^{\frac{1}{2}} \tag{15}$$

The main part of this is **by simultaneous thickness at the same scale** r at ξ.

Again let Ω_1 and Ω_2 have **thick complement at scale** r_0 **at** ξ_0. Then if $\xi \in B(\xi_0, r_0)$

$$\left(\frac{1}{r_0^{n+3}}\int_{B(\xi,4r_0)}|u_1|^2\right)^{\frac{1}{2}}\left(\frac{1}{r_0^{n+3}}\int_{B(\xi,4r_0)}|u_2|^2\right)^{\frac{1}{2}} \le$$

$$\left(\frac{1}{r_0^{n+3}}\int_{B(\xi_0,8r_0)}|u_1|^2\right)^{\frac{1}{2}}\left(\frac{1}{r_0^{n+3}}\int_{B(\xi_0,8r_0)}|u_2|^2\right)^{\frac{1}{2}} =: C(\xi_0, r_0).$$

And so by monotonicity and the previous slide for any $\xi \in E_m \cap B(\xi_0, r_0)$

$$\frac{\omega_1(B(\xi, r))}{r^n} \cdot \frac{\omega_2(B(\xi, r))}{r^n} \le mC(\xi_0, r_0),\ r \le r_0\,.$$

Here E_m appears from Egorov theorem, we need uniform thickness, and it works for E_m such that $\omega_i(E_m \cap B(\xi_0, r_0)) \to \omega_i(B(\xi_0, r_0))$.

21. Using thickness again and estimating $R_n \omega_i^*$ on E_m

Now we can prove that for any $\xi \in E_m \cap B(\xi_0, r_0))$

$$(R_n^* \omega_i)(\xi) \leq A_n m C(\xi_0, r_0),\ i = 1, 2.$$

21a. Illustration to $R^*\omega$ boundedness

22. NTV1: Non-homogeneous T1 theorem of Nazarov–Treil–Volberg, Acta Math. 2002

Theorem

Let μ in $\mathbb{R}^{n+1}$ be such that on a set E, $\mu(E) > 0$, we have $\theta^*_m(\mu, x) < \infty$ and $(R^*_m\mu)(x) < \infty$. Then there exists $E' \subset E$ such that $\mu(E') > 0$ and $R_m : L^2(E', \mu) \to L^2(E', \mu)$ is a bounded operator.

Here

$$\theta^*_m(\mu, x) = \limsup_{r\to 0} \frac{\mu(B(x, r))}{r^m}\,.$$

23. NTV2: Nazarov–Tolsa–Volberg rectifiability theorem

Theorem

Let μ *in* $\mathbb{R}^{n+1}$ *be such that on a set* E, $\mu(E) > 0$, *we have* $\theta_n^*(\mu, x) > 0$ *and* $R_n : L^2(E, \mu) \to L^2(E, \mu)$ *is a bounded operator. Then there exists* $E' \subset E$ *such that* $\mu(E') > 0$ *and such that* E' *lies in the graph of the Lipschitz map* $\mathbb{R}^n \to \mathbb{R}^{n+1}$.

24. One phase problem, J. Azzam, S. Hofmann, J. M. Martell, S. Mayboroda, M. Mourgoglou and X. Tolsa

- J. Azzam, S. Hofmann, J.M. Martell, S. Mayboroda, M. Mourgoglou, X. Tolsa, and A. Volberg. *Rectifiability of harmonic measure.* Geom. Funct. Anal. (GAFA), 26(3) (2016), 703–728.
- J. Azzam, M. Mourgoglou and X. Tolsa. *Mutual absolute continuity of interior and exterior harmonic measure implies rectifiability.* Preprint (2015). To appear in Comm. Pure Appl. Math.
- J. Azzam, M. Mourgoglou, X. Tolsa, and A. Volberg. *On a two-phase problem for harmonic measure in general domains.* Preprint arXiv:1608.04022 (2016).
- C. J. Bishop. *A characterization of Poissonian domains.* Ark. Mat. 29 (1991), no. 1, 1–24.
- J. Bourgain. *On the Hausdorff dimension of harmonic measure in higher dimension.* Invent. Math. 87 (1987), no. 3, 477–483.
- L. Caffarelli and S. Salsa. *A geometric approach to free boundary problems.* Graduate Texts in Math. 64. Amr. Math.

A. Eremenko, B. Fuglede and M. Sodin. *On the Riesz charge of the lower envelope of δ-subharmonic functions.* Potential Analysis 1 (1992), 191–204.

A. Eremenko, B. Fuglede and M. Sodin. *Harmonic measure for three disjoint domains in R^n*, in Linear and Complex Analysis Problem Book 3 (V.P. Havin, N.K. Nikolski, eds.) (1994), 323–325.

S. Hofmann, J.M. Martell, S. Mayboroda, X. Tolsa and A. Volberg. *Absolute continuity between the surface measure and harmonic measure implies rectifiability*. Preprint 2015. arXiv:1507.04409.

C. Kenig, D. Preiss, T. Toro. *Boundary structure and size in terms of interior and exterior harmonic measure in higher dimensions*. J. Amer. Math. Soc. 22 (2009), no. 3, 771–796.

C. Kenig and T. Toro. *Free boundary regularity for harmonic measures and Poisson kernels,* Ann. of Math. 150 (1999) 369–454.

X. Tolsa. *Analytic capacity, the Cauchy transform, and non-homogeneous Calderón-Zygmund theory*. Volume 307 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2014.

B. Tsirelson. *Triple points: from non-Brownian filtrations to harmonic measures.* Geom. Funct. Anal. (GAFA), Vol. 7 (1997) 1096-1142.