The quadruple bubble in the plane

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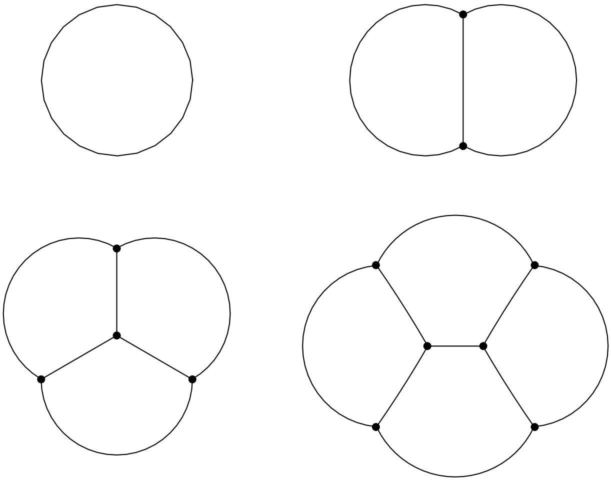
Joint works with: Tamagnini (2018) and Tortorelli (2020)



Minimal clusters

Planar N-cluster:
$\boldsymbol{E} = (E_1, \ldots, E_N)$ with $E_j \subset \mathbb{R}^2$, $|E_j \cap E_k| = 0$ for $j \neq k$

External region: $E_0 = \mathbb{R}^2 \setminus \bigcup_{j=1}^{N} E_j$

Area: $\boldsymbol{m}(\boldsymbol{E}) = (|E_1|, \ldots, |E_N|) \in \mathbb{R}^N$

Perimeter: $P(\boldsymbol{E}) = \frac{1}{2} \sum_{j=0}^{N} P(E_j) \in \mathbb{R}$

see Maggi: *sets of finite perimeter...*

Existence of optimal clusters:

We say that $\boldsymbol{E}$ is a *minimal cluster* of prescribed areas $\boldsymbol{a} \in \mathbb{R}_+^N$ if $P(\boldsymbol{E})$ is minimum among all clusters with $\boldsymbol{m}(\boldsymbol{E}) = \boldsymbol{a}$.

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Such minimal cluster exist for any $\boldsymbol{a} \in \mathbb{R}_+^N$ (Almgren, Morgan)

Explicit solutions:

$N = 1$: isoperimetric problem (Steiner 1890)
$N = 2$: double bubble (Foisy et al. 1993)
$N = \infty$: honeycomb (Hales 1999)
$N = 3$: triple bubble (Wichiramala 2004, Lawlor 2019)

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Soap bubble conjecture: the regions of a minimal cluster are connected (Morgan)
Why do we consider non-connected regions?

Regularity of minimal clusters

(Almgren, Taylor, Morgan. . .)

- regions are bounded by finitely many circular arcs (or straight segments);
- the arcs meet in triples with equal angles of 120 degrees;
- the sum of the signed curvatures of the three arcs meeting in a triple point is zero;
- to each region E_i a pressure p_i can be assigned, so that the arc between regions E_i and E_j has signed curvature $p_i - p_j$ (and $p_0 = 0$ can be assumed).

Cluster which satisfy these conditions will be called *stationary clusters*.

Pressure formula

Pressures $\boldsymbol{p} = (p_1, \dots, p_n)$ are Lagrange multipliers associated to the area constraint:

$$\left[\frac{d}{dt} P(\boldsymbol{E}(t))\right]_{t=t_0} = \boldsymbol{p} \cdot \left[\frac{d}{dt} \boldsymbol{m}(\boldsymbol{E}(t))\right]_{t=t_0}$$

Applying the previous formula to a rescaling of a stationary cluster $\boldsymbol{E}$ we obtain

$$P(\boldsymbol{E}) = 2\,\boldsymbol{p} \cdot \boldsymbol{m}(\boldsymbol{E})$$

The double bubble

Main difficulty: prove that minimizers are connected.

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Tool: rotate a subcluster along an edge (two connected components can share at most one edge)

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Main idea: enlarge class of minimizers including clusters which enclose more area than prescribed. This enables to easily prove that E_0 is connected (no internal camera)

Weak minimizers

We say that $\boldsymbol{E}$ is a <u>*weak minimizer*</u> with prescribed areas $\boldsymbol{a} \in \mathbb{R}^N_+$ if $P(\boldsymbol{E})$ is minimum among all sets with $\boldsymbol{m}(\boldsymbol{E}) \geq \boldsymbol{a}$.
Notation $\boldsymbol{E} \in \mathcal{M}^*(\boldsymbol{a})$.

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Properties of weak minimizers:

- the external region E_0 is connected;
- no negative pressures: $\boldsymbol{p} \geq \mathbf{0}$;
- if $|E_i| > a_i$ then $p_i = 0$;
- if the total number of connected components is $M \leq 6$ then the weak minimizer is a (strong) minimizer.

Möbius transforms / monotonicity

Möbius transformations preserve stationarity of clusters!

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Passing throught Möbius transformations it is easy to *inflate*/*deflate* triangular components of a cluster.

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Monotonicity: when a triangular component is inflated (resp. deflated) both the area and the radius of each edge increases (resp. decreases).

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Uniqueness: in double-bubbles and triple-bubbles there is a one-to-one corrispondence $\boldsymbol{p} \mapsto \boldsymbol{a}$ between pressures and areas. (Montesinos)

Four equal areas

with Andrea Tamagnini (unifi): Minimal clusters of four planar regions with the same area (ESAIM COCV 2018)

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with Vincenzo M. Tortorelli (unipi): The quadruple planar bubble enclosing equal areas is symmetric (Calc. Var. 2020)

Variational tools

Variation I. Remove a component C with an edge of length ℓ and replace it by a disk of equal area. Then $\ell \leq 2\sqrt{\pi}\sqrt{|C|}$.

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Variation II. Remove a component C of the region E_i with n edges and rescale the resulting cluster to recover the prescribed areas. Then

$$|C| \geq \frac{16\pi|E_i|^2}{n^2 P^2(\boldsymbol{E})}\left(1 - \frac{16\pi|E_i|}{n^2 P^2(\boldsymbol{E})}\right).$$

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Variation III. Remove a component C with n edges and recover the measure by enlarging an external edge ℓ of the same region E_i. Then

$$p_i \geq \frac{2\sqrt{\pi}}{n\sqrt{|C|}} - \frac{2}{\ell}.$$

A priori estimate on the number of components

Let M be the number of connected components of a weak minimal N-cluster $\boldsymbol{E} \in \mathcal{M}^*(\boldsymbol{a})$ with $N \geq 3$. Then

$$M \leq \frac{9}{20} N^2 \frac{\|\boldsymbol{a}\|_{\frac{1}{2}}}{\|\boldsymbol{a}\|_{-1}}.$$

$$\|\boldsymbol{a}\|_p = \Big(\sum_{j=1}^{N} |a_j|^p \Big)^{1/p}$$

Isoperimetric inequality

$$P(\boldsymbol{E}) = \frac{1}{2}\sum_{i=0}^{N} P(E_i) \geq \sqrt{\pi}\left[\sqrt{\sum_{i=1}^{N}|E_i|} + \sum_{i=1}^{N}\sqrt{|E_i|}\right]$$

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If $E_i = E_i' \cup E_i''$ is disconnected:

$$P(E_i) \geq 2\sqrt{\pi}\left(\sqrt{|E_i'|} + \sqrt{|E_i''|}\right)$$

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big / small components...

$\boldsymbol{E} \in \mathcal{M}^*(1,1,1,1)$

Explicit computation on well choosen competitor:
$P(\boldsymbol{E}) \leq 11.1962$

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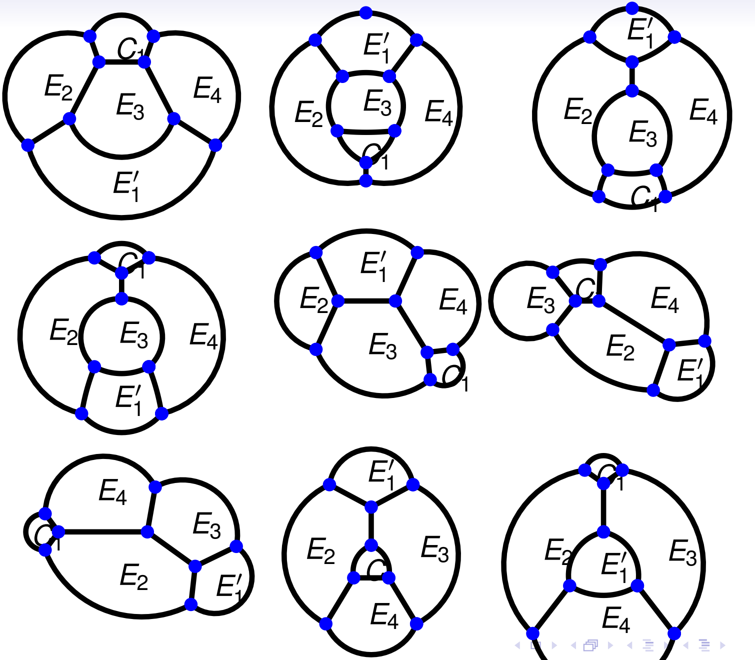
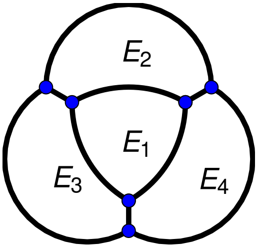
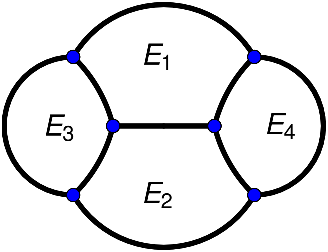
- Minimal clusters have at most $4+2$ components (big + small).
- Exclude there can be two small components.
- Reduce to 9 possible topologies for clusters with $4+1$ components.
- Exclude them: minimal clusters have 4 connected components.
- Two topologies: sandwich, flower.
- Exclude flower.

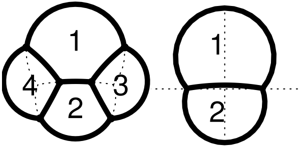

C_1
E_2
E_3
E_4
E'_1

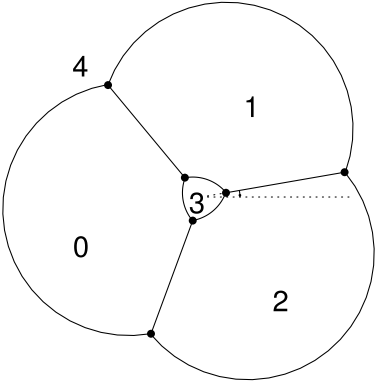
Flower / Sandwich





the Sandwich is symmetric





the Sandwich is symmetric

Let $F_k = T(E_k)$ with $T(z) = \frac{1}{\bar{z}}$, $\det DT(z) = \frac{1}{z^4}$

$$|E_1| - |E_2| = \iint_{F_1} \frac{1}{|x+iy|^4}\,dxdy - \iint_{F_2} \frac{1}{|x+iy|^4}\,dxdy$$

$$= \int_0^{\frac{3}{2}\pi} \int_{r_1(t,\theta)}^{r_2(t,\theta)} \left[\frac{1}{|1+re^{i(\theta+t)}|^4} - \frac{1}{|1+re^{i(\theta-t)}|^4} \right] r\,dr\,dt.$$

but

$$|1+re^{i\alpha}|^2 = 1 + 2r\cos\alpha + r^2$$

and for $t \in (0, \frac{2}{3}\pi]$ and $\theta \in (0, \frac{\pi}{3})$

$$\cos(\theta + t) < \cos(\theta - t).$$