Variations on the (Eternal) Theme of Analytic Continuation

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" Between two truths of the real domain, the easiest and shortest path quite often passes through the complex domain."

P. PAINLEVÉ, 1900

I. A warm-up.

(i) **Q.** When does the series $\sum_0^\infty a_n z^n$ represent a rational function?

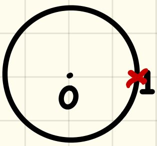
A. Iff $\exists N : \forall n \geq N$

$$\det \begin{pmatrix} a_0 & \cdots & a_n \\ a_1 & \cdots & a_{n+1} \\ \vdots & & \vdots \\ a_n & \cdots & a_{2n} \end{pmatrix} = 0$$

Ex. $\frac{1}{1-z} = \sum_0^\infty z^n$, $N = 1$.

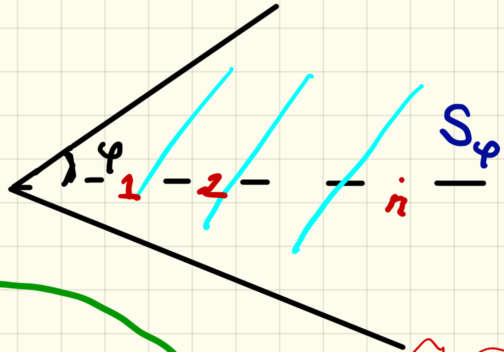
(Kronecker, 1881).

(ii) $\sum_0^\infty a_n z^n$ w/ROC $=1$ extends to $\mathbb{C} \setminus \{1\}$ iff $a_n = g(n)$, g is entire, of the minimal type. (L. Leau - 1899, S. Wigert - 1900, G. Faber - 1903.)



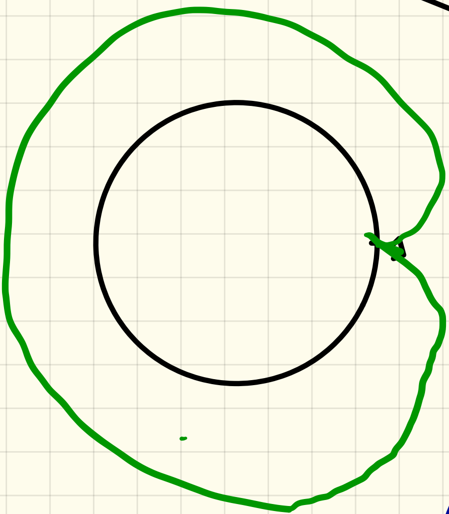
Ex. $\boxed{\sum_0^\infty \cos(\sqrt{n})\, z^n}$.

(iii)



$g \in H^\infty(S_\varphi)$

$$f(z) = \sum_0^\infty g(n) z^n$$



f extends to

$$\Omega_\varphi := \{ re^{i\theta} : 2\pi - \cot\varphi \cdot \log r > \theta > \cot\varphi \cdot \log r \}$$

(DK - '97)

(Holds also for g of "minimal type" in S_φ.)

Q. Is Ω_φ maximal possible?

A??? Most likely "Yes".

II. "More serious": ODE vs. PDE

(i) $w^{(n)} + a_{n-1}(z)w^{(n-1)} + \dots + a_0(z)w(z) = f(z)$

$w(0) = w_0, \dots, w^{(n-1)}(0) = w_{n-1}$ $\quad (*)$ (Cauchy's Problem)

Thm. If $\{a_j\}_0^{n-1}$ are analytic in Ω, $0 \in \Omega$, all solutions of $(*)$ extend to Ω.

(ii) **Example**

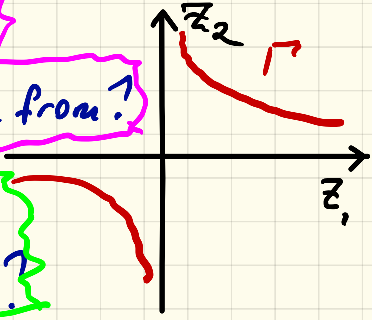
$$\frac{\partial w}{\partial z_2} = z_1^2 \frac{\partial w}{\partial z_1} \quad (**)$$

$$w(z_1, 0) = f(z_1) \text{ entire}$$

$$w(z_1, z_2) = f\left(\frac{z_1}{1 - z_1 z_2}\right)$$

– might develop singularity on

$\Gamma := \{z_1 z_2 = 1\}$



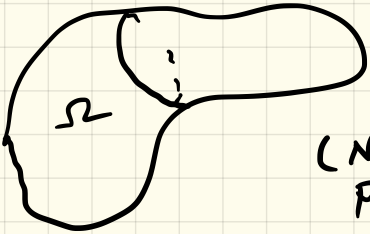
– Where do they come from?

– Why all singularities lie on the same variety?

III. Potential Theory.

(i) (G. Herglotz, 1914).

Ω = solid in $\mathbb{R}^3$, $\Gamma = \partial\Omega$ - algebraic



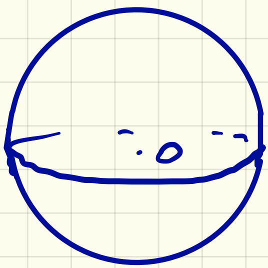
$$u_\Omega(x) = \frac{1}{4\pi}\int_\Omega \frac{dy}{|x-y|}$$

(Newtonian potential)

(a) How far can u_Ω be continued (harmonically) into Ω?

(b) What kind of singularities does it encounter inside?

Ex.



$\Gamma := \{|x| < 1\}$

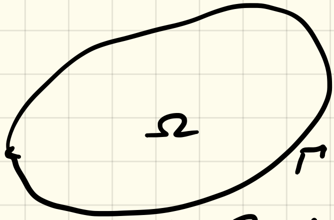
$\boxed{\text{Ans:}}$ $u_\Omega = \dfrac{c}{|x|}$.

(mean value thm).

Surprise: (a) $\mathbb{R}^3 \setminus \{0\}$ for ANY entire density $\rho(y)$ instead of 1.

(b) Algebraic when ρ is a polynomial.

"High Ground" (DK – H. S. Shapiro, '89)



$P(y)$ – polynomial, or entire

$$u_{\Omega,P}(x) = \text{const} \int_\Omega \frac{P(y)\,dy}{|x-y|}$$

Cauchy's Problem:

$$\boxed{\begin{aligned} &\Delta M = P \text{ near } \Gamma \\ &M\big|_\Gamma = \nabla M\big|_\Gamma = 0 \end{aligned} \quad (*)}$$

$$u = \begin{cases} u_{\Omega,P} & \text{outside} \\ M - u_{\Omega,P} & \text{inside} \end{cases}$$

is the desired continuation

The H. G. Question:

Are the singularities of solutions of (CP) (*) dictated by the initial variety Γ and the DO (e.g., Δ) but not by a particular data P (entire)?

J. Leray's (local) Theory (~50s-60s)

Examples : (1) $\Gamma = \{ w = z^3 \}$

(CP) $\begin{cases} \dfrac{\partial^2 u}{\partial z \partial w} = 0 \\ u|_\Gamma = zw, \ u_w|_\Gamma = z, \ u_z|_\Gamma = w \end{cases}$

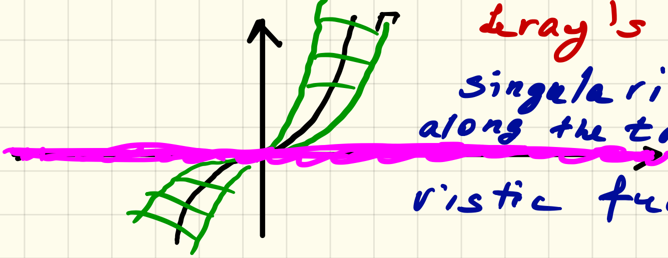
($zw - u =: M$ - modified Schwarz potential)

Caution: $(0,0)$ is a "bad" point (characteristic), Cauchy-Kovalevskaya Thm. fails).

$$u = \frac{z^4}{4} + \frac{3}{4} w^{4/3}$$

is ramified around $\{w=0\}$, a characteristic (wrt $\frac{\partial^2}{\partial z \partial w}$) plane

TANGENT to Γ at $(0,0)$.



Leray's principle:

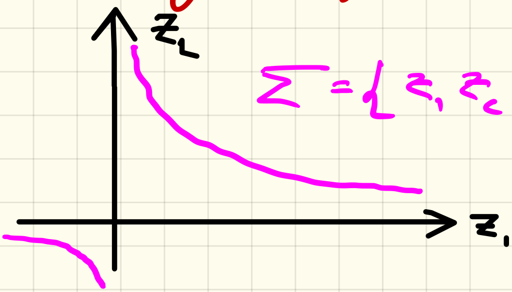
singularities propagate along the tangent characteristic from Γ (locally)

(2) Recall

$$\begin{cases} \dfrac{\partial w}{\partial z_2} = z_1^2 \dfrac{\partial w}{\partial z_1} \\ w(z_1, 0) = f(z_1) \text{ - entire} \end{cases}$$

$$w = f\left(\frac{z_1}{1 - z_1 z_2}\right).$$

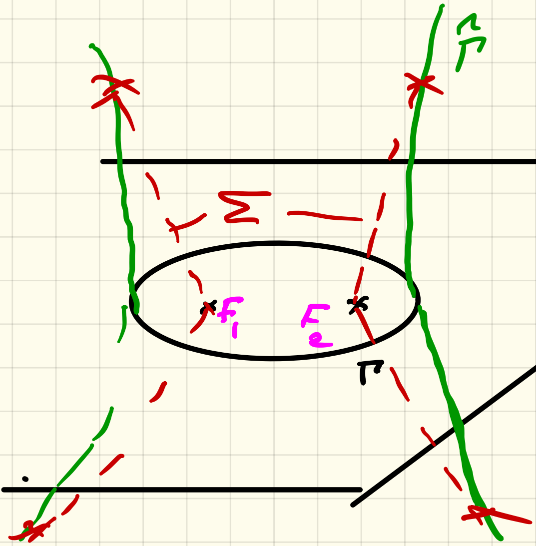
Σ = sing. set is the "Leray tangent" to $\{z_2 = 0\}$ at ∞.



(3) (a) Ellipses/Ellipsoids/Spheres

$\hat{\Gamma} = \left\{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ x, y \in \mathbb{C}\right\}$

has 4 char. points



Σ_j Leray's char. tangent

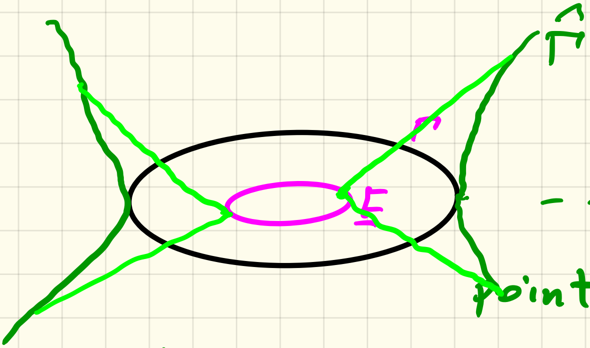
All log potentials with entire densities extend to $\mathbb{R}^2 \setminus \{\text{foci}\}$

- Generally speaking Leray's Principle is local. But in 2-dim is <u>global</u> for D.O. $\mathcal{L} := \Delta +$ lower terms. Thanks to Riemann's method of integrating hyperbolic eqs in 2-var.
- In dim ≥ 3, it is only known to be <u>global</u> for:
  - spheres, spherical cylinders (DK-H.S. Shapiro, '89, G. Johnsson - '90).
  - ALL Quadratic Surfaces (G. Johnsson, '94-'95).

<u>Q.</u> - When should we expect bounded, algebraic singularities?

- What is the reason behind unbounded "polar" singularities, e.g., for a sphere?

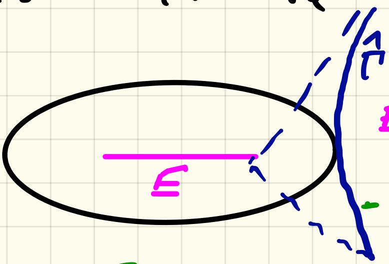
(b) Oblate spheroid: $\Gamma := \left\{ \frac{x_1^2}{a^2} + \frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} = 1 \quad a > b \right\}$.



$E := \{ x_3 = 0,\ x_1^2 + x_2^2 = a^2 - b^2 \}$

- the caustic, each point is the meeting point of 2 char. $\mathbb{C}$-lines tangent to $\hat{\Gamma}$

Prolate spheroid $\Gamma := \left\{ \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{b^2} = 1 \right\}$



$E = \{ |x_1| \le \sqrt{a^2 - b^2},\ x_2 = x_3 = 0 \}$

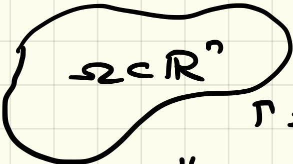
- each point on the caustic E is the meeting point of 2 tangent at ∞ characteristics.

Conjecture: This behaviour is true in general.

- True, (sort of), in 2D.

IV. Singularities of Solutions to BVP

1. Dirichlet's Problem



$\Gamma = \partial\Omega$ – algebraic

"Data" f – entire, polyn.

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \Gamma \end{cases} \Rightarrow u \text{ is real-anal. in } \overline{\Omega}.$$

Q. (i) Where do singularities of u might occur outside Ω?

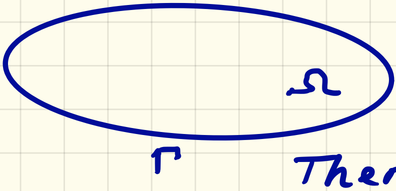
(ii) – How do singularities relate to the geometry of Γ?

– Is there the "worst" data responsible for ALL possible singularities that might occur?

Few facts:

- Classical (E. Heine, G. Lamé, M. Ferrers, ...)

Prop. $\Omega := \left\{ x : \sum_1^n x_j^2/a_j^2 < 1,\ a_1 > \dots > a_n > 0 \right\}$



$$\begin{cases} \Delta u = 0 \text{ in } \Omega \\ u = f, \text{ pol. of deg } N \end{cases} \quad (*)$$

Then, u is a harmonic polynomial of degree $\leq N$.

Thm. (DK & H.S. Shapiro, '92) For any entire data f, the solution u of $(*)$ is entire.

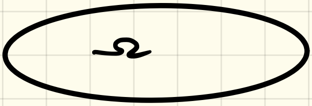
(D. Armitage, '04 - refinement).

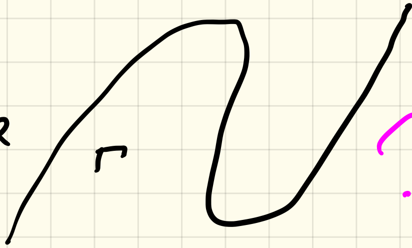
Conjecture (DK-HSS) Ellipsoids are the only domains for which the THM holds.

H. Render ('05) TRUE... provided $\Gamma = \{P(x) = 0\}$, P - polynomial, $P = \underbrace{P_m + \dots + P_0}_{\text{homogeneous}}$, $P_m \geq 0$ in $\mathbb{R}^n$ (i.e., elliptic).

discussion

$n=2$, $\Gamma=\{(x,y): x^2+y^2-1+\varepsilon P_{2n+1}(x,y)=0\}$





a "quintic" Γ

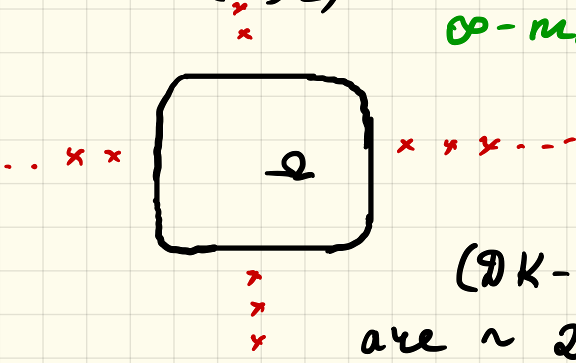
???

Ad hoc true for cubics (E. Lundberg & H. Render)

$\deg \Gamma \geq 5$, odd ??? "intuition"

- Data "sits" on 2 pieces of Γ, so DP becomes overdetermined (due to analytic continuation in $\mathbb{C}^2$!) (Goursat's problem).

Ex. P. Ebenfelt's "TV screen" $\Gamma := \{x^4+y^4=1\}$ ('92)



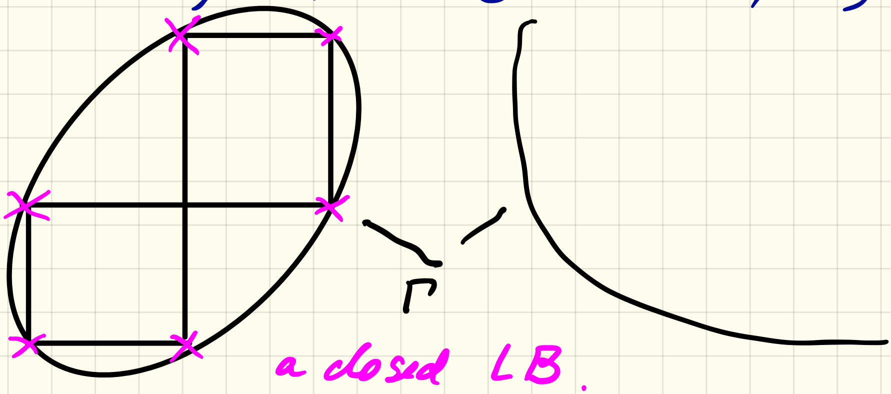
∞-many singularities for the data $f = x^2+y^2$

(DK-EL, '10) the nearest are $\sim 2^{3/4}$ from $(0,0)$

Finding Singularities

(S. Bell, P. Ebenfelt, DK, H. S. Shapiro, E. Lundberg '05 - '10, 2D!)

"Lightning Bolts" (A. Kolmogorov, V. Arnold '50s, Hilbert's 13th Problem).

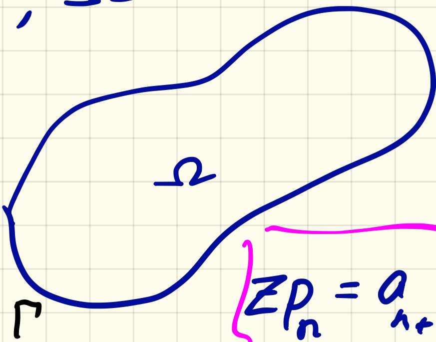


The size of LB in $\mathbb{C}^2$ (!)
~ how close to the origin the singularity appears.

3D, HIGHER DIM. - virtually nothing is known except for rather imprecise estimates of Render for elliptic surfaces.

Applications to... orthogonal polynomials

$\mathbb{R}^2$, Ω



Bergman OP: $\int_\Omega p\bar{q}\, dA = 0$ (area)

"Recurrence" (P_n - OP of deg n)

$$\boxed{zP_n = a_{n+1,n}P_{n+1} + a_{n,n}P_n + \dots a_{n-N+1,n}P_{n-N+1}}$$

$\Longleftrightarrow$ $\exists K$: $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u|_\Gamma = \text{pol. of deg } n \end{cases}$

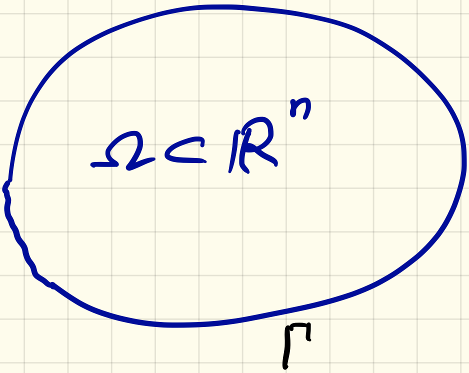
u = pol. of deg $\le n+K$

<u>Thm</u> (M. Putinar - N. Stylianopolous, '07
DK - N. Stylianopolous, '10)

Either condition $\Rightarrow$ $N=3$, Ω = ellipse, $\{P_n\}$ are Chebyshev.

Existence of SOME domain with no recurrence - L. Lempert, '78.

V. OTHER BVP – Eigenfunctions



$$(*) \quad \Delta u + \lambda u = 0$$

$$u|_\Gamma = 0$$

Γ - algebraic

Q. Where are singularities of u?

- Essentially nothing is known.

- Ω = ellipsoid, ALL eigenfunctions are entire (of exponential type) – DK - HSS (unpublished), H. Render (unpublished), well-known (with "bad" proofs)

? **Conjecture** ? If ALL eigenfunctions are entire, Ω = ellipsoid, i.e., "singularities sound out the shape."

References can be found in

DK & E. Lundberg, Linear Holomorphic PDE and Classical Potential Theory, Math. Surveys & Monographs, 232, AMS, 2018.

Большое спасибо!

Thank you! Be safe!