Mixed Type Hermite–Padé Approximants for Nikishin System

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Plan of the Talk

1. Padé approximants and Markov theorem

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2. Hermite–Padé approximants for Nikishin system

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2. Hermite–Padé approximants for Nikishin system
3. Mixed type approximants and applications

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1. Padé approximants and Markov theorem
2. Hermite–Padé approximants for Nikishin system
3. Mixed type approximants and applications
4. Our results for Mixed type approximants

Padé approximants, orthogonal polynomials and Markov theorem

Let s be a finite positive Borel measure supported on $[-1,1]$ and $\widehat{s}(z) := \int_{-1}^{1} \frac{s(x)}{z-x}$ be the Cauchy transform of s.

For any $n \in \mathbb{Z}_+$ there exist polynomials $P_{n,0}$ and $P_{n,1}$ s.t. $\deg P_{n,0} \le n$, $P_{n,0} \not\equiv 0$ and $\left(P_{n,0}\widehat{s} + P_{n,1}\right)(z) = O(z^{-n-1})$, $z \to \infty$.

$\pi_n = -P_{n,1}/P_{n,0}$ is n-th Padé approximant to $\widehat{s}$ at ∞.

Theorem (Markov)

$\lim_{n\to\infty} \pi_n(z) = \widehat{s}(z), \quad z \in \mathbb{C} \setminus [-1,1].$

$P_{n,0}$ is orthogonal: $\int_{-1}^{1} P_{n,0}(x) x^k ds(x) = 0, \quad k < n.$

Consider a zero counting measure: $\mu(P_{n,0}) := \sum_{\{x:\, P_{n,0}(x)=0\}} \delta_x.$

If $s' > 0$ a.e. on $[-1,1]$, then $*$-weak $\lim_{n\to\infty} \frac{1}{n}\mu(P_{n,0}) = \frac{dx}{\pi\sqrt{1-x^2}}$.

Hermite–Padé approximants, multiple orthogonal polynomials and normality

Let $s_1, \dots, s_d$ be a collection of finite positive Borel measures on $\mathbb{R}$.

For any $\vec{n} \in \mathbb{Z}_+^d$ there exist polynomials $P_{\vec{n},0}, P_{\vec{n},1}, \dots P_{\vec{n},d}$ s.t. $\deg P_{\vec{n},0} \le |\vec{n}| := n_1 + \cdots n_d$, $P_{\vec{n},0} \not\equiv 0$ and as $z \to \infty$ we have $\left(P_{\vec{n},0}\widehat{s}_j + P_{\vec{n},j}\right)(z) = O(z^{-n_j-1})$.

The set of rational functions $\pi_{\vec{n},j} = -P_{\vec{n},j}/P_{\vec{n},0}$ is called Hermite–Padé approximants to $\widehat{s}_1, \dots, \widehat{s}_d$ at ∞.

$P_{\vec{n},0}$ is a multiple orthogonal polynomial:
$\int P_{\vec{n},0}(x)x^k ds_j(x) = 0, \quad k < n_j, \quad j = 1, \dots, d.$

If $\deg P_{\vec{n},0} = |\vec{n}|$, then $\pi_{\vec{n},j}$ are unique and the index $\vec{n}$ is called normal.
If all $\vec{n} \in \mathbb{Z}_+^d$ are normal, then the system $s_1, \dots, s_d$ is perfect.

Nikishin system

We need the following notation. Let σ_1, σ_2 be two measures with non-intersecting supports, then we define a measure $\langle \sigma_1, \sigma_2 \rangle$:

$$d\langle \sigma_1, \sigma_2 \rangle (x) := \widehat{\sigma}_2(x) d\sigma_1(x), \qquad \widehat{\sigma}(x) := \int \frac{d\sigma(t)}{x-t}.$$

Take a collection $\{\Delta_j\}_{j=1}^d$ of intervals s.t. $\Delta_j \cap \Delta_{j+1} = \varnothing$. Let $\{\sigma_j\}_{j=1}^d$ be a system of measures s.t. $\mathrm{supp}\sigma_j \subset \Delta_j$ and

$$\langle \sigma_1 \rangle := \sigma_1, \qquad \left\langle \sigma_1, \sigma_2, \dots, \sigma_j \right\rangle := \left\langle \sigma_1, \left\langle \sigma_2, \dots, \sigma_j \right\rangle \right\rangle.$$

The Nikishin system is defined as follows (E.M. Nikishin, 1980):

$$s_{1,j} := \left\langle \sigma_1, \dots, \sigma_j \right\rangle.$$

Normality of a Nikishin system

E. Nikishin, 1980, diagonal case.
Z. Bustamante, G. Lopes Lagomasino, 1992, $d = 2$.
K. Driver, H. Stahl, 1995.
A. Gonchar, E. Rakhmanov, V. Sorokin, 1997, $\mathbb{Z}_+^d(\searrow)$.
U. Fidalgo Prieto, G. Lopes Lagomasino, 2011, perfectness.

Let $\mathbb{Z}_+^d(\searrow)$ be a set of "decreasing" multi-indexes $\vec{n} \in \mathbb{Z}_+^d$ s.t.
$j < k \;\Rightarrow\; n_k \leq n_j + 1$.

Theorem (GRS, 1997)

If $\vec{n} \in \mathbb{Z}_+^d(\searrow)$, then $\vec{n}$ is normal.

Proof of normality in case $d = 2$, $n_1 \geq n_2 - 1$

$$\int P_{\vec{n},0} x^k ds_{1,1} = 0,\ k < n_1, \quad \int P_{\vec{n},0} x^k ds_{1,2} = 0,\ k < n_2 \quad \Rightarrow$$

$$\int P_{\vec{n},0} x^k d\sigma_1 = 0,\ k < n_1, \quad \int \widehat{P_{\vec{n},0}\sigma_1} x^k d\sigma_2 = 0,\ k < n_2 \quad \Rightarrow$$

$\widehat{P_{\vec{n},0}\sigma_1} = O(z^{-n_1-1}),\ z \to \infty$ and has $\geq n_2$ zeros on $\Delta_2 \quad \Rightarrow$

$$H(\mathbb{C} \setminus \Delta_1) \ni \frac{\widehat{P_{\vec{n},0}\sigma_1}}{p_{\vec{n}}} = O(z^{-|\vec{n}|-1}),\ z \to \infty \quad \Rightarrow$$

$\int P_{\vec{n},0} x^k \frac{d\sigma_1}{p_{\vec{n}}} = 0,\ k < |\vec{n}| \quad \Rightarrow \quad P_{\vec{n},0}$ has $\geq |\vec{n}|$ zeros on Δ_1

Vector equilibrium potential, $d = 2$

Let $0 \le q \le \frac{1}{2}$. Consider a class of vector measures (μ_1, μ_2) s.t.

$$S(\mu_j) \subset \Delta_j, \quad |\mu_1| = 1, \quad |\mu_2| = q$$

In this class there exists a unique measure (λ_1, λ_2) s.t.

$$2V^{\lambda_1}(x) - V^{\lambda_2}(x) \begin{cases} = & w_1, \quad x \in S(\lambda_1), \\ \ge & w_1, \quad x \in \Delta_1, \end{cases}$$

$$2V^{\lambda_2}(x) - V^{\lambda_1}(x) \begin{cases} = & w_2, \quad x \in S(\lambda_2), \\ \ge & w_2, \quad x \in \Delta_2, \end{cases}$$

where $V^{\lambda_j}(x) := -\int \log|x-t|\,d\lambda_j(t)$ is a log potential of λ_j.

Limiting zero distribution and convergent, $d = 2$

Consider a ray sequence Λ of multi-indexes $\vec{n} \in \mathbb{Z}_+^d(\searrow)$ s.t. $\frac{n_2}{|\vec{n}|} \to q \in \left[0, \frac{1}{2}\right]$ as $|\vec{n}| \to \infty$.

Theorem (GRS, 1997)

If $\sigma_j' > 0$ a.e. on Δ_j ($1 \le j \le d$) then

$$\lim_{\vec{n}\in\Lambda} \frac{1}{|\vec{n}|}\mu(P_{\vec{n},0}) = \lambda_1, \quad \lim_{\vec{n}\in\Lambda} \frac{1}{|\vec{n}|}\mu(p_{\vec{n}}) = \lambda_2.$$

The proof is based on the Gonchar–Rakhmanov theorem about lim zero distribution of orthogonal polynomials with varying weight, 1984.

Corollary

$$\lim_{\vec{n}\in\Lambda} \pi_{\vec{n},j}(z) = \widehat{s}_{1j}(z), \quad z \in \mathbb{C} \setminus \Delta_1.$$

Nikishin matrix of measures

The generators $\sigma_1, \dots, \sigma_d$ of a Nikishin system $s_{1,1}, \dots, s_{1,d}$ allows to define a Nikishin matrix:

$$s_{j,k} := \left\langle \sigma_j, \sigma_{j+1}, \dots, \sigma_k \right\rangle, \quad k > j.$$

$$s_{j,k} := \left\langle \sigma_j, \sigma_{j-1}, \dots \sigma_k \right\rangle, \quad k < j.$$

Mixed type approximants

G. López Lagomasino, S. Medina Peralta, J. Szmigielski in 2019 introduced the following *mixed type Hermite–Padé* problem.

Find polynomials $P_{n,0}, P_{n,1}, \dots, P_{n,d}$ s.t. $P_{n,0} \not\equiv 0$ and the following interpolation conditions at infinity hold:

$$\begin{array}{ll}
L_{n,0} := P_{n,0} & = O(z^n), \\
L_{n,1} := P_{n,0}\widehat{s}_{1,1} + P_{n,1} & = O(z^{-1}), \\
L_{n,2} := P_{n,0}\widehat{s}_{2,1} + P_{n,1}\widehat{s}_{2,2} + P_{n,2} & = O(z^{-1}), \\
\vdots & \quad\vdots \\
L_{n,d} := P_{n,0}\widehat{s}_{d,1} + P_{n,1}\widehat{s}_{d,2} + \cdots + P_{n,d-1}\widehat{s}_{d,d} + P_{n,d} & = O(z^{-n-1}).
\end{array}$$

They prove that for any integer n there exists a unique (up to normalization) solution and $P_{n,0}$ has n zeros on Δ_1 (normality).

Weak asymptotics and convergence

G. López Lagomasino, S. Medina Peralta, J. Szmigielski find weak asymptotics of $P_{n,0}$ and $L_{n,j}$ and derived Markov type result:

Theorem (LMS)

$$\lim_{n\to\infty}(-1)^j\frac{P_{n,j}}{P_{n,0}}(z)=\widehat{s}_{1,j}(z),\quad z\in\mathbb{C}\setminus\Delta_1.$$

They were inspired by the paper devoted to Digasperis–Procesi equation and the special case $d=2$ of the mixed type approximants (M. Bertola, M. Gekhtman, J. Szmigielski, 2010).

Motivation. Digasperis–Procesi equation

Dispersive shallow wave propagation equation:

$$u_t - u_{xxt} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}, \quad (x,t) \in \mathbb{R}^2, \ b \in \mathbb{R}.$$

The equation admits weak peakon solutuions:
$u(x,t) = \sum_{i=1}^{n} m_i(t) e^{|x-x_i(t)|}$ s.t.

$$\dot{x}_k = \sum_{i=1}^{n} m_i e^{|x_k - x_i|}, \quad \dot{m}_k = (b-1) \sum_{i=1}^{n} m_k m_i \operatorname{sgn}(x_k - x_i) e^{|x_k - x_i|},$$

Two cases are known to be integrable: $b = 2$ (Camassa–Holm) and $b = 3$ (Digasperis–Procesi). The evolution can be described by means of direct and inverse spectral problem. The evolution of the spectral data is described by quadratic (CH) and cubic (DP) string.
The inverse problem for CH is solved by Stieltjes contined fractions.
The inverse problem for DP is solved by mixed type approximants ($d = 2$).

Statement of the mixed type HP problem

We consider the following mixed type interpolation problem.

Problem A. *Given a multi-index* $\vec{n} \in \mathbb{Z}_+^d$ *find polynomials* $P_{\vec{n},0}$, $P_{\vec{n},1}, \dots, P_{\vec{n},d}$ *s.t.* $P_{\vec{n},0} \not\equiv 0$ *and the following interpolation conditions at* $z = \infty$ *hold:*

$$
\begin{aligned}
L_{\vec{n},0} &:= P_{\vec{n},0} && = O(z^{|\vec{n}|}), \\
L_{\vec{n},1} &:= P_{\vec{n},0}\widehat{s}_{1,1} + P_{\vec{n},1} && = O(z^{-n_1-1}), \\
L_{\vec{n},2} &:= P_{\vec{n},0}\widehat{s}_{2,1} + P_{\vec{n},1}\widehat{s}_{2,2} + P_{\vec{n},2} && = O(z^{-n_2-1}), \\
&\vdots && \vdots \\
L_{\vec{n},d} &:= P_{\vec{n},0}\widehat{s}_{d,1} + P_{\vec{n},1}\widehat{s}_{d,2} + \cdots + P_{\vec{n},d-1}\widehat{s}_{d,d} + P_{\vec{n},d} && = O(z^{-n_d-1}).
\end{aligned}
$$

The existence of a solution is obvious.

Perfectness

Theorem

Let $P_{\vec{n},0}, P_{\vec{n},1}, \dots, P_{\vec{n},d}$ be a solution to the problem A for $\vec{n} \in \mathbb{Z}_+^d$, then $L_{\vec{n},j-1}$ has $n_j + \dots + n_d$ zeros on Δ_j, $j = 1, \dots, d$.

In particular $\deg P_{\vec{n},0} = |\vec{n}|$ (as far as $L_{\vec{n},0} := P_{\vec{n},0}$). All the indexes are normal and Nikishin systems are perfect with respect to the mixed type HP problem.

In addition to n_j zeros at ∞ $L_{\vec{n},j}$ has some finite zeros. This property is typical for Nikishin systems (remember $p_{\vec{n}}$ in GRS proof).

Connection to type II HP

Proposition

If $\vec{n} \in \mathbb{Z}_+^d(\searrow)$, then the problem A is equivalent to the HP type II approximation problem for $s_{1,1}, s_{1,2}, \ldots, s_{1,d}$:

$$P_{\vec{n},0}\widehat{s}_{1,j} - (-1)^j P_{\vec{n},j} = O(z^{-n_j-1}),\ z \to \infty, \quad j = 1, \ldots, d.$$

On the one hand for $\vec{n} = (0, \ldots, 0, n) = |\vec{n}|\vec{e}_d$ the problem A coincides with the López– Medina– Szmigielski problem. On the other hand for $\vec{n} \in \mathbb{Z}_+^d(\searrow)$ the problem A is equivalent to type II HP.

Vector equilibrium potential

Let $\vec{q} = (q_1, \ldots, q_d)$, $q_j \geq 0$, $q_1 + \cdots + q_d = 1$. Consider a class of vector measures $\vec{\mu} = (\mu_1, \ldots, \mu_d)$ s.t.

$$S(\mu_j) \subset \Delta_j, \quad |\mu_j| = q_j + \cdots + q_d.$$

In this class there exists a unique equilibrium measure $\vec{\lambda} = (\lambda_1, \ldots, \lambda_d)$ satisfying

$$-V^{\lambda_{j-1}}(x) + 2V^{\lambda_j}(x) - V^{\lambda_{j+1}}(x) \begin{cases} = & w_j, \quad x \in S(\lambda_j), \\ \geq & w_j, \quad x \in \Delta_j, \end{cases}$$

where $\lambda_0 := \lambda_{d+1} := 0$.

Limiting zero distribution and convergence

Consider a ray sequence Λ of multi-indexes $\vec{n} \in \mathbb{Z}_+^d$ s.t. $\frac{n_j}{|\vec{n}|} \to q_j$.

Theorem

If $\sigma_j' > 0$ a.e. on Δ_j ($1 \le j \le d$) then

$$\lim_{\vec{n}\in\Lambda} \frac{1}{|\vec{n}|}\mu(L_{\vec{n},j-1}) = \lambda_j, \quad j = 1, \dots, d$$

In particular $\lim_{\vec{n}\in\Lambda} \frac{1}{|\vec{n}|}\mu(P_{\vec{n},0}) = \lambda_1$.

Corollary

$$\lim_{|\vec{n}|\to\infty} (-1)^j \frac{P_{\vec{n},j}}{P_{\vec{n},0}}(z) = \widehat{s}_{1,j}(z), \quad z \in \mathbb{C} \setminus \Delta_1.$$

Matrix Riemann–Hilbert problem, preliminaries

Denote by $M_{d+1}(\mathbb{C})$ the space of square matrices of size $d+1$ over $\mathbb{C}$. Let E_{jk} be the standard basis in $M_{d+1}(\mathbb{C})$:

$$\left[E_{jk}\right]_{lm} = \delta_{jl}\delta_{km}.$$

For $m = 1, \ldots, d$ define the operator $T_m : \ M_2(\mathbb{C}) \to M_{d+1}(\mathbb{C})$:

$$T_m(B) := \sum_{j \neq m, m+1} E_{j\,j} + \\ [B]_{11} E_{m\,m} + [B]_{12} E_{m\,m+1} + [B]_{21} E_{m+1\,m} + [B]_{22} E_{m+1\,m+1}.$$

Assume that the generators $(\sigma_1, \ldots, \sigma_d)$ are absolutely continuous measures and σ_j' are Hölder continuous.

Matrix Riemann–Hilbert problem

1) Y is an analytic matrix valued function $\mathbb{C} \setminus \left(\bigcup_{j=1}^{d} \Delta_j\right) \to M_{d+1}(\mathbb{C})$.
2) Y has continuous limits when approaching $\bigcup_{j=1}^{d} \Delta_j$ from above and below, and these limits are related by the jump matrix:

$$Y_+(x) = Y_-(x) T_m \begin{pmatrix} 1 & -2\pi i \sigma_m'(x) \\ 0 & 1 \end{pmatrix}, \quad x \in \Delta_m.$$

3) Y has the following behavior near ∞:

$$Y(z) = \left(\mathrm{Id} + O\left(z^{-1}\right)\right) \mathrm{diag}(z^{|\vec{n}|}, z^{-n_1}, \ldots, z^{-n_d}), \quad z \to \infty.$$

Equivalence between mixed type HP and RH problem

Consider a matrix Y:

$$Y := \begin{pmatrix} P_{\vec{n},0} & L_{\vec{n},1} & \cdots & L_{\vec{n},d} \\ c_{\vec{n},1} P_{\vec{n}-\vec{e}_1,0} & c_{\vec{n},1} L_{\vec{n}-\vec{e}_1,1} & \cdots & c_{\vec{n},1} L_{\vec{n}-\vec{e}_1,d} \\ \vdots & \vdots & \ddots & \vdots \\ c_{\vec{n},d} P_{\vec{n}-\vec{e}_d,0} & c_{\vec{n},d} L_{\vec{n}-\vec{e}_d,1} & \cdots & c_{\vec{n},d} L_{\vec{n}-\vec{e}_d,d} \end{pmatrix},$$

where $\vec{e}_1, \ldots, \vec{e}_d$ is the standard basis in $\mathbb{R}^d$, $P_{\vec{n},0}$ are monic and $c_{\vec{n},j}$ are some constants s.t.

$$\lim_{z\to\infty} c_{\vec{n},j} z^{n_j} L_{\vec{n}-\vec{e}_j,j}(z) = 1.$$

Proposition

Y is a solution of the Riemann–Hilbert problem above.

It is a unique solution if we properly specify the behavior at the edge points of Δ_j.

Conclusion

1) We suggest a mixed type interpolation problem for a Nikishin system which in particular cases is reduced to the classical Hermite–Pade problem and to the problem motivated by the Degasperis–Procesi equation.
2) Nikishin systems are perfect with respect to this interpolation problem.
3) By means of Gonchar–Rakhmanov vector equilibrium potential we find weak asymptotics and prove Markov type theorem.
4) The interpolation problem allows Matrix Rieman–Hilbert formulation.

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What's next?

The solution of this problem has many more remarkable properties and this is our ongoing project:

1) interlacing of the zeros of $L_{\vec{n}}$ and $L_{\vec{n}+\vec{e}_j}$;
2) $(d+2)$ -term recurrence relations;
3) Christoffel–Darboux formula;
4) ratio asymptotics.

Thank you for your attention!