Monge-Ampère Operator for Plurisubharmonic Functions with Analytic Singularities

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Defining Nonlinear Operators for Nonsmooth Functions

Real Monge-Ampère Operator

For convex $u \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^n$

$$MA(u) = \det D^2 u = Jac(\nabla u)$$

and

$$\int_E \det D^2 u \, d\lambda = \lambda(\nabla u(E)), \qquad E \subset \Omega.$$

If u is just convex one can define the *gradient image*

$$\nabla u(x) = \{y \in \mathbb{R}^n \colon u(x) + \langle \cdot - x, y \rangle \leq u\},$$

$$\nabla u(E) = \bigcup_{x \in E} \nabla u(x).$$

This defines a Radon measure $MA(u)$ such that $MA(u_j) \to MA(u)$ weakly if $u_j \to u$ in L^1_{loc}. We also have the following solution of the Dirichlet problem:

Theorem Assume that Ω is a bounded strictly convex domain in $\mathbb{R}^n$. Let φ be continuous on $\partial\Omega$ and μ a positive Radon measure in Ω such that $\mu(\Omega) < \infty$. Then there exists a unique solution to the following Dirichlet problem:

$$\begin{cases} u \in CVX(\Omega) \cap C(\bar{\Omega}), \\ MA(u) = \mu, \\ u = \varphi \text{ on } \partial\Omega. \end{cases}$$

Real Hessian Operator

For $m = 1, \ldots, n$ and smooth u we set

$$H_m(u) := S_m(D^2 u) = \sum_{1 \le i_1 < \cdots < i_m \le n} \lambda_{i_1} \ldots \lambda_{i_m},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $D^2 u$.
Then $H_1 = \Delta$ and $H_n = MA$.

A C^2 function u is called *m-convex* if $H_m(u + a|x|^2) \ge 0$ for $a \ge 0$.
Equivalently, $H_j(u) \ge 0$ for $j \le m$.

A nonsmooth u is *m-convex* if locally it is a limit of a decreasing sequence of smooth m-convex functions.

Trudinger, Wang, 1999 For every m-convex u one can uniquely define a positive measure $H_m(u)$ so that $H_m(u_j) \to H_m(u)$ weakly if $u_j \to u$ in L^1_{loc}.

Complex Monge-Ampère Operator

For $u \in C^2(\Omega)$, $\Omega \subset \mathbb{C}^n$, we set

$$CMA(u) = \det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right).$$

We have

$$(dd^c u)^n = dd^c u \wedge \cdots \wedge dd^c u = 4^n n! CMA(u) d\lambda,$$

where $d = \partial + \bar{\partial}$, $d^c = i(\bar{\partial} - \partial)$, $dd^c = 2i\partial\bar{\partial}$.

Example (Shiffman-Taylor, 1974, Kiselman, 1982)

$$u(z) = (-\log|z_1|)^{1/n}(|z_2|^2 + \cdots + |z_n|^2 - 1)$$

Then u is plurisubharmonic (psh) near the origin in $\mathbb{C}^n$, u is smooth away from $\{z_1 = 0\}$ but $(dd^c u)^n$ has unbounded mass near $\{z_1 = 0\}$.

Bedford-Taylor, 1982 For $u \in PSH \cap L^\infty_{loc}$ one can well define $(dd^c u)^n$ as a positive Radon measure so that $(dd^c u_j)^n \to (dd^c u)^n$ weakly if u_j decreases to u.

Recursive definition: for $k = 1, \dots, n$

$$(dd^c u)^k := dd^c \left(u (dd^c u)^{k-1} \right)$$

is a closed positive current.

Demailly, 1985 The same for $u \in PSH(\Omega) \cap L^\infty_{loc}(\Omega \setminus U)$, $U \Subset \Omega$.

Monotone convergence is essential:

Example (Cegrell, 1984) There exists a sequence $u_j \in PSH \cap C^\infty$, $0 \le u_j \le 1$, converging in L^1_{loc} (and thus in L^p_{loc} for every $p < \infty$) to $u \in PSH \cap C^\infty$ but $(dd^c u_j)^n$ does not converge weakly to $(dd^c u)^n$.

Bedford-Taylor, 1976 Let Ω be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$, $\varphi \in C(\partial\Omega)$ and $f \in C(\bar{\Omega})$, $f \geq 0$. Then there exists unique solution to the following Dirichlet problem

$$\begin{cases} u \in PSH(\Omega) \cap C(\bar{\Omega}), \\ (dd^c u)^n = f\, d\lambda, \\ u = \varphi \text{ on } \partial\Omega. \end{cases}$$

Kołodziej, 1998 May take nonnegative $f \in L^p(\Omega)$ where $p > 1$.

Example (Cegrell, 1982) For

$$u(z) = \log|z_1 \dots z_n| = \log|z_1| + \dots + \log|z_n|$$

consider two sequences from $PSH \cap C^\infty$ decreasing to u:

$$u_j = \frac{1}{2}\log\left(|z_1 \dots z_n|^2 + \frac{1}{j}\right),$$
$$v_j = \frac{1}{2}\log\left(|z_1|^2 + \frac{1}{j}\right) + \dots + \frac{1}{2}\log\left(|z_n|^2 + \frac{1}{j}\right).$$

Then $(dd^c u_j)^n \to 0$ and $(dd^c v_j)^n \to n!2^n\delta_0$ weakly.

Domain of Definition of CMA

Let $\mathcal{D}$ denote the subclass of the class of psh functions consisting of those psh u such that there exists a measure μ such that $(dd^c u_j)^n \to \mu$ weakly for any smooth psh u_j decreasing to u.

One can easily show that $\mathcal{D}$ is a maximal subclass of PSH where one can define CMA in such a way that it is continuous for decreasing sequences.

If $\Omega \subset \mathbb{C}^2$ and $u \in C^2(\Omega)$ then

$$\int_\Omega \varphi (dd^c u)^2 = -\int_\Omega du \wedge d^c u \wedge dd^c \varphi, \qquad \varphi \in C_0^\infty(\Omega).$$

B. (2004) If $n = 2$ then $\mathcal{D} = PSH \cap W^{1,2}_{loc}$.

B. (2006) For a negative $u \in PSH(\Omega)$, $\Omega \subset \mathbb{C}^n$, TFAE

i) $u \in \mathcal{D}$;

ii) **For every** $u_j \in PSH \cap C^\infty$ decreasing to u the sequence $(dd^c u_j)^n$ is locally weakly bounded;

iii) **For every** $u_j \in PSH \cap C^\infty$ decreasing to u the sequences

$$|u_j|^{n-p-2} du_j \wedge d^c u_j \wedge (dd^c u_j)^p \wedge \omega^{n-p-2}, \qquad p = 0, 1, \dots, n-2,$$

$(\omega = dd^c|z|^2)$ are locally weakly bounded;

iv) **There exists** $u_j \in PSH \cap C^\infty$ decreasing to u such that the sequences

$$|u_j|^{n-p-2} du_j \wedge d^c u_j \wedge (dd^c u_j)^p \wedge \omega^{n-p-2}, \qquad p = 0, 1, \dots, n-2,$$

$(\omega = dd^c|z|^2)$ are locally weakly bounded.

Complex Hessian Operator

For $u \in C^2(\Omega)$, $\Omega \subset \mathbb{C}^n$, and $m = 1, \ldots, n$

$$CH_m(u) := S_m(u_{j\bar{k}}) = \sum_{1 \le i_1 < \cdots < i_m \le n} \lambda_{i_1} \ldots \lambda_{i_m},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the complex Hessian $(u_{j\bar{k}}) = (\partial^2 u / \partial z_j \partial \bar{z}_k)$. Then $CH_1 = \Delta/4$ and $CH_n = CMA$.

We have

$$(dd^c u)^m \wedge \omega^{n-m} = 4^n m!(n-m)! CH_m(u) d\lambda,$$

where $\omega = dd^c|z|^2$.

A C^2 function u is called *m-subharmonic* if $CH_m(u + a|z|^2) \ge 0$ for $a \ge 0$. Equivalently, $CH_j(u) \ge 0$ for $j \le m$.

A nonsmooth u is *m-subharmonic* if locally it is a limit of a decreasing sequence of smooth m-subharmonic functions.

Conjecture If u is m-subharmonic then $u \in L^p_{loc}$ for $p < nm/(n-m)$.

B. 2005 True for $p < n/(n-m)$.

Dinew-Kołodziej, 2012 True for $p < nm/(n-m)$ if the unbounded locus of u is relatively compact.

Psh Functions with Analytic Singularities

A psh u is said to have *analytic singularities* if locally it can be written in the form

$$u = c \log |F| + b,$$

where $c \geq 0$, $F = (f_1, \ldots, f_m)$ is a tuple of holomorphic functions and b is bounded.

Certainly in general $u \notin \mathcal{D}$!

Definition of CMA (Andersson-Wulcan, 2014)

For $u = \log|F| + b$, $F = (f_1, \ldots, f_m)$, and $k = 1, \ldots, n$ we want to define

$$(dd^c u)^k := dd^c\big(\mathbf{1}_{\{F\neq 0\}} u (dd^c u)^{k-1}\big).$$

For this one needs to show that $T_{k-1} := \mathbf{1}_{\{F\neq 0\}}(dd^c u)^{k-1}$ extends across $\{F = 0\}$ to a closed positive current and uT_{k-1} has a finite mass near $\{F = 0\}$.

If $m = 1$ then we say that u has *divisorial singularities*. Then $\log|F|$ is pluriharmonic, b is a bounded psh function, and thus $T_{k-1} = (dd^c b)^{k-1}$. In this case we have therefore

$$(dd^c u)^k = [F = 0] \wedge (dd^c b)^{k-1} + (dd^c b)^k,$$

where $[F = 0] = dd^c(\log|F|)$ is the current of integration along $\{F = 0\}$.

Reduction to Divisorial Singularities

For $m > 1$ we use the resolution of singularities. Assume that u is defined in open $X \subset \mathbb{C}^n$ and $Z := \{F = 0\}$. By Hironaka there exists a complex manifold X' and a proper holomorphic mapping $\pi : X' \to X$ such that the exceptional divisor $E := \pi^{-1}Z$ is a hypersurface and $\pi|_{X' \setminus E} \to X \setminus Z$ is a biholomorphism. We then have $\pi^* F = f_0 F'$, where f_0 is a holomorphic function such that $E = \{f_0 = 0\}$ and F' is a nonvanishing tuple of holomorphic functions. Then

$$\pi^* u = \log |f_0| + \log |F'| + \pi^* b = \log |f_0| + B$$

has divisorial singularities. One can then show that

$$T_{k-1} = \mathbf{1}_{\{F \neq 0\}} (dd^c u)^{k-1} = \pi_* (dd^c B)^{k-1}$$

is indeed a closed positive current and similarly one can show that $u T_{k-1}$ has a locally bounded mass near Z.

The definition of $(dd^c u)^k$ for psh u with analytic singularities is related to the intersection theory and Segre numbers in analytic geometry introduced by Tworzewski in 1995.

Theorem 1 (Andersson-B-Wulcan) Let u be a negative psh function with analytic singularities. Assume that $\chi_j(t)$ are bounded nondecreasing (in t) convex functions on $(-\infty, 0]$, decreasing to t as $j \to \infty$. Then for every $k = 1, \ldots, n$ we have weak convergence of currents

$$(dd^c(\chi_j \circ u))^k \to (dd^c u)^k.$$

Remark This result can be treated as an alternative definition of $(dd^c u)^k$.

Sketch of proof Similarly as before we may reduce it to the case of divisorial singularities: $u = \log|f| + v$, where f is holomorphic and v bounded psh. Assume that χ_j are smooth. Let γ_j be convex on $(-\infty, 0]$ such that $\gamma_j(-1) = \chi_j(-1)$ and $\gamma_j' = (\chi_j')^k$. Then they are also bounded nondecreasing (in t) and $\gamma_j(t)$ decreases to t as $j \to \infty$.

Then on $\{f \neq 0\}$

$$\begin{aligned}
\left(dd^c(\chi_j \circ u)\right)^k &= \left(\chi_j'' \circ u\, du \wedge d^c u + \chi_j' \circ u\, dd^c u\right)^k \\
&= \left(k\chi_j'' \circ u\, du \wedge d^c u + \chi_j' \circ u\, dd^c u\right) \wedge \left(\chi_j' \circ u\, dd^c u\right)^{k-1} \\
&= d\left((\chi_j' \circ u)^k d^c u\right) \wedge (dd^c u)^{k-1} \\
&= dd^c(\gamma_j \circ u) \wedge (dd^c v)^{k-1} \\
&= dd^c\left(\gamma_j \circ u\, (dd^c v)^{k-1}\right)
\end{aligned}$$

Since none of the above currents charges $\{f = 0\}$, the equality holds everywhere. One can show that

$$\gamma_j \circ u\, (dd^c v)^{k-1} \to u(dd^c v)^{k-1}$$

weakly as $j \to \infty$. $\square$

Psh Functions with Analytic Singularities on Compact Kähler Manifolds

(X, ω) Kähler manifold

We say that $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is ω-psh if locally $u := g + \varphi$ is psh, where g is a local Kähler potential, that is $\omega = dd^c g$. We say that such a φ has analytic singularities if locally u has analytic singularities.

Theorem 2 (Andersson-B-Wulcan) Let (X, ω) be a compact Kähler manifold and φ an ω-psh function on X with analytic singularities. Then

$$\int_X (\omega + dd^c\varphi)^n = \int_X \omega^n - \sum_{k=1}^{n-1} \int_Z (\omega + dd^c\varphi)^k \wedge \omega^{n-k},$$

where Z is the singular set of φ. In particular,

$$\int_X (\omega + dd^c\varphi)^n \leq \int_X \omega^n.$$

Example Let $X = \mathbb{P}^n$, $n \geq 2$, and let ω be the Fubini-Study metric.Set

$$\varphi([z_0 : z_1 : \cdots : z_n]) := \log \frac{|z_1|}{|z|}, \qquad z \in \mathbb{C}^{n+1} \setminus \{0\}.$$

Then $(\omega + dd^c\varphi)^n = 0$.

Sketch of proof of Theorem 2 For $k = 0, \ldots, n-1$

$$T_k := \mathbf{1}_{X \setminus Z}(\omega + dd^c\varphi)^k$$

is a closed positive current on X. Locally we have

$$(\omega + dd^c\varphi)^n = dd^c\big((g + \varphi)T_{n-1}\big) = \omega \wedge T_{n-1} + dd^c(\varphi T_{n-1})$$

and therefore

$$\int_X (\omega + dd^c\varphi)^n = \int_X \omega \wedge T_{n-1} = \int_X (\omega + dd^c\varphi)^{n-1} \wedge \omega - \int_Z (\omega + dd^c\varphi)^{n-1} \wedge \omega.$$

Continuing this way we will get the result. $\square$

Another Definition on Hermitian Manifolds

In the previous definition on Kähler manifold, if we approximate as in Theorem 1, we would locally consider regularizations of the form

$$\chi_j(g+\varphi) \to g+\varphi = u$$

(e.g. $\chi_j(t) = \max\{t, -j\}$). Instead, we may consider direct regularizations of the form

$$\chi_j \circ \varphi \to \varphi.$$

For this we have to generalize Theorem 1 to quasi-psh functions. (φ is called *quasi-psh* if locally it can be written as $\varphi = u + \psi$, where u is psh and ψ is smooth).

Theorem 3 Let φ be a negative quasi-psh function with analytic singularities on a complex manifold X of dimension n and assume that η is a smooth $(1,1)$-form on X. Then for $k = 1, \ldots, n$ the current $(\eta + dd^c\varphi)^k$ can be uniquely defined in such a way that if χ_j is a sequence of bounded nondecreasing convex functions on $(-\infty, 0]$ such that $\chi_j(t)$ decreases to t as j increases to ∞ then

$$(\eta + dd^c(\chi_j \circ \varphi))^k \longrightarrow (\eta + dd^c\varphi)^k$$

weakly as $j \to \infty$.

With this new definition we the obtain the right total mass:

Corollary If X is compact and ω is Kähler then

$$\int_X (\omega + dd^c\varphi)^n = \int_X \omega^n.$$

In the above example we will now get $(\omega + dd^c\varphi)^n = [z_1 = 0] \wedge \omega^{n-1}$.

Thank you!