Ring isomorphisms of Murray–von Neumann algebras

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Outline

1. Intoduction
   - Short history
   - Prelimanaries
2. Main results
   - Continuity of ring isomorphisms in the measure topology
   - General form of ring isomorphisms

Outline

Intoduction

We shall start with the following famous result due to J. von Neumannn.

THEOREM 4.2: Let $\mathfrak{R}$, $\mathfrak{R}'$ be regular rings such that $\mathfrak{R}$ has order $n \geq 3$ and such that $\bar{R}_{\mathfrak{R}}$ and $\bar{R}_{\mathfrak{R}'}$ are lattice-isomorphic. Then there exists a ring-isomorphism of $\mathfrak{R}$ and $\mathfrak{R}'$ which generates the given lattice-isomorphism.

J. von Neumann
Continuous geometry
Foreword by Israel Halperin, Princeton Mathematical Series, No. 25 Princeton University Press, Princeton, N.J. (1960).

Operator algebras version of von Neumann Theorem

- Let $\mathcal{M}$ be a von Neumann algebra;
- let $P(\mathcal{M})$ the set of all projections in $\mathcal{M}$;
- let $S(\mathcal{M})$ be a $*$-algebra of all measurable operators with respect to $\mathcal{M}$. [a]

[a] definitions we shall give later

Theorem 1.1

Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras of type II_1 and let $\Phi : P(\mathcal{M}) \to P(\mathcal{N})$ be a lattice isomorphism. Then there exists a unique ring isomorphism $\Psi : S(\mathcal{M}) \to S(\mathcal{N})$ such that $\Phi(l(x)) = l(\Psi(x))$ for any $x \in S(\mathcal{M})$, in particular, $\Phi(p) = l(\Psi(p))$ for any $p \in P(\mathcal{M})$.

Here, $l(x)$ is the left projection of x.

Problems of homomorphisms on semifields

Естественно возникает следующий вопрос:

П р о б л е м а I. *Всякий ли алгебраический гомоморфизм* $R^{\Delta} \to R^{\Delta'}$ *является непрерывным, т. е. представляет собой гомоморфизм тихоновских полуполей?*

П р о б л е м а II. *Существует ли алгебраический гомоморфизм* $\psi\colon R^{\Delta} \to R$, *удовлетворяющий условиям* $\psi(1_q) = 0$ *для всех* $q \in \Delta$, $\psi(1_{\Delta}) = 1$?

M. Ya. Antonovskii, V. G. Boltyanskii, Tikhonov semifields and certain problems in general topology, Russian Mathematical Surveys (1970), 25(3):1-43.

Problems of homomorphisms on semifields

В 1930 г. С. Улам [20] поставил следующую проблему о существовании двузначных счетно-аддитивных мер:

Проблема III. *Существует ли нетривиальная двузначная счетно-аддитивная мера* $\mu: D^{\Delta} \to R$, *удовлетворяющая условию:* $\mu(1_q) = 0$ *для любого* $q \in \Delta$?

Предыдущие результаты позволяют обобщить известную теорему Макки (см. [2]).

Теорема 6. *Пусть* K — *сепарабельное нетривиально нормированное тело,* Δ *фиксировано. Тогда следующие проблемы эквивалентны:*

1) *Существует ли разрывный* K*-гомоморфизм* $\varphi: K^{\Delta} \to K$?

2) *Существует ли нетривиальная счетно-аддитивная двузначная мера* μ *на* D^{Δ} *такая, что* $\mu(1_q) = 0$, *для всех* $q \in \Delta$, $\mu(1_{\Delta}) = 1$? (Улам [3].)

Sh. A. Ayupov, Homomorphisms of a class of rings and two-valued measures on Boolean algebras, Funct. Anal. Appl., 11:3 (1977), 217–219.

Question of M. Mori and his conjecture

Let $LS(\mathcal{M})$ be a $*$-algebra of all locally measurable operators with respect to $\mathcal{M}$.

Question 1.2

Let $\mathcal{M}, \mathcal{N}$ be von Neumann algebras. What is the general form of ring isomorphisms from $LS(\mathcal{M})$ onto $LS(\mathcal{N})$?[a]

[a] M. Mori, Lattice isomorphisms between projection lattices of von Neumann algebras, arXiv:2006.08959 (2020).

- Mori himself gave an answer to the above Question in the case of von Neumann algebras of type I_∞ and III.

Result of M. Mori

Theorem B. *Let M, N be von Neumann algebras of type I_∞ or III. If $\Psi\colon LS(M) \to LS(N)$ is a ring isomorphism, then there exist a real *-isomorphism $\psi\colon M \to N$ (which extends to a real *-isomorphism from $LS(M)$ onto $LS(N)$) and an invertible element $y \in LS(N)$ such that $\Psi(x) = y\psi(x)y^{-1}$, $x \in LS(M)$.*

- Mori conjectured that the representation of ring isomorphisms, mentioned above for type I_∞ and III cases holds also for type II von Neumann algebras.[a]

[a] M. Mori, Lattice isomorphisms between projection lattices of von Neumann algebras, arXiv:2006.08959 (2020).

Isomorphisms of C^*-algebras

4.1.20. Theorem. *Let $\mathscr{A}$ and $\mathfrak{B}$ be two C^*-algebras and suppose that $\mathscr{A}$ and $\mathfrak{B}$ are isomorphic. Then they are $*$-isomorphic.*

4.1.22. Corollary. *Let $\mathscr{A}$ (resp. $\mathfrak{B}$) be a C^*-algebra on a Hilbert space $\mathscr{H}$ (resp. $\mathscr{K}$) such that its weak closure $\bar{\mathscr{A}}$ (resp. $\bar{\mathfrak{B}}$) contains $1_{\mathscr{H}}$ (resp. $1_{\mathscr{K}}$). Let Φ be an isomorphism of $\mathscr{A}$ onto $\mathfrak{B}$. Then there exists an invertible positive element h in $\bar{\mathscr{A}}$ such that $\Phi(a)=\Phi_1(hah^{-1})$ $(a\in\mathscr{A})$, where Φ_1 is a $*$-isomorphism of $\mathscr{A}$ onto $\mathfrak{B}$. In particular if $\mathscr{A}$ is a W^*-algebra or a simple C^*-algebra with identity, then h belongs to $\mathscr{A}$.*

S. Sakai
C^*-algebras and W^*-algebras
Reprint of the 1971 edition. Classics in Mathematics.
Springer-Verlag, Berlin, 1998. xii+256 pp.

Abelian case

- If the von Neumann algebra $\mathcal{M}$ is abelian then it is $*$-isomorphic to $L_\infty(\Omega, \Sigma, \mu)$;
- $S(\mathcal{M}) \cong S(\Omega, \Sigma, \mu)$ is the algebra of all measurable complex functions on (Ω, Σ, μ);
- A.G. Kusraev by means of Boolean-valued analysis establishes necessary and sufficient conditions for existence of band-preserving non trivial algebra automorphisms on $S(\Omega, \Sigma, \mu)$;
- In particular, he has proved that $S[0, 1]$ admits discontinuous algebra automorphisms which identically act on the Boolean algebra $P(L_\infty[0, 1])$.[a]

[a] A. G. Kusraev, *Automorphisms and derivations in an extended complex f-algebra*, Sib. Math. J. **47** (2006) 97–107.

Type I_n case

- $\mathcal{M}$ is a vNa of type I_n, $n > 1$ with the center $Z(\mathcal{M})$;
- $\mathcal{M}$ is $*$-isomorphic to $M_n(Z(\mathcal{M}))$ of all $n \times n$ matrices over $Z(\mathcal{M})$;
- $S(\mathcal{M})$ is $*$-isomorphic to the algebra $M_n(Z(S(\mathcal{M})))$, where $Z(S(\mathcal{M})) = S(Z(\mathcal{M}))$;
- each algebra automorphisms Φ of $S(\mathcal{M})$ can be uniquely represented in the form

$$\Phi(x) = a\overline{\Psi}(x)a^{-1}, x \in S(\mathcal{M}),$$

where $a \in S(\mathcal{M})$ is an invertible element and $\overline{\Psi}$ is an extension of a $*$-automorphism Ψ of the center $S(Z(\mathcal{M}))$.[a]

[a] S. Albeverio, S. Ayupov, K. Kudaybergenov, R. Djumamuratov, *Automorphisms of central extensions of type I von Neumann algebras*, Studia Math. **207** (2011), 1-17.

Outline

von Neumann algebras

- H be a Hilbert space, $B(H)$ be the $*$-algebra of all bounded linear operators on H, $\mathcal{M}$ be a von Neumann algebra in $B(H)$;
- $P(\mathcal{M})$ the set of all projections in $\mathcal{M}$;
- $e, f \in P(\mathcal{M})$ are called *equivalent* if there exists an element $u \in \mathcal{M}$ such that $u^*u = e$ and $uu^* = f$;
- $e, f \in \mathcal{M}$ notation $e \precsim f$ means that there exists a projection $q \in \mathcal{M}$ such that $e \sim q \leq f$;
- $p \in \mathcal{M}$ is said to be *finite*, if it is not equivalent to its proper sub-projection;
- $e \in P(\mathcal{M})$ is abelian, if $e\mathcal{M}e$ is an abelian algebra;
- a finite von Neumann algebra $\mathcal{M}$ without nonzero abelian projections is called of type II_1.

Murray-von Neumann algebras

- M be a von Neumann algebra and let $P(M)$ be a set of all projections in M;
- A linear operator x affiliated with M is called *measurable* with respect to M if $\chi_{(\lambda,\infty)}(|x|)$ is a finite projection for some $\lambda > 0$.
- $S(M)$ be the set of all measurable operators w.r.t. M;
- $S(M)$ equipped with the algebraic operations of the strong addition and multiplication and taking the adjoint of an operator;
- In the case, when M is a finite von Neumann algebras, the algebra $S(M)$ is referred to as the Murray-von Neumann algebra associated with M.

Regularness of $S(\mathcal{M})$

- Let $a \in S(\mathcal{M})$ and let $a = v|a|$ be the polar decomposition of a;
- $l(a) = vv^*$ and $r(a) = v^*v$ are left and right supports of the element a, respectively;
- $s(a) = l(a) \vee r(a)$ is the support of the element a;
- there is a unique element $i(a)$ in $S(\mathcal{M})$ such that $ai(a) = l(a)$, $i(a)a = r(a)$, $ai(a)a = a$, $i(a)l(a) = i(a)$ and $r(a)i(a) = i(a)$.
- the element $i(a)$ is called the *partial inverse* of the element a.
- $S(\mathcal{M})$ is a regular $*$-algebra.[a][b]

[a] K. Saitô, *On the algebra of measurable operators for a general AW^*-algebra.* II. Tohoku Math. J. **23** (1971), 525–534.

[b] S.K. Berberian, Baer $*$-rings. Die Grundlehren der mathematischen Wissenschaften, Band 195. Springer-Verlag, New York-Berlin, 1972.

Measure topology

- Let τ be a a faithful normal finite trace on M. A measure topology t_τ on $S(M)$:

$$N(\varepsilon,\delta)=\left\{x\in S(M):\ \tau\left(\chi_{(\varepsilon,\infty)}(|x|)\right)\leq\delta\right\},\ \varepsilon,\delta>0;$$

- $(S(M),t_\tau)$ is a complete metrizable topological algebra.[a]

[a] E. Nelson, J. Funct. Anal. **15** (1974) 103–116.

Example 1.3

- if $M=\ell_\infty$, then $S(M)\cong s\equiv\mathbb{R}^{\aleph_0}+i\mathbb{R}^{\aleph_0}$;
- if $M=L_\infty(0,1)$, then $S(M)\cong S(0,1)$;
- if $M=B(H)$, then $S(M)\cong B(H)$.

Various isomorphisms of $*$-algebras

For $*$-algebras $\mathcal{A}$ and $\mathcal{B}$, a (not necessarily linear) bijection $\Phi : \mathcal{A} \to \mathcal{B}$ is called

- a ring isomorphism if it is additive and multiplicative;
- a real algebra isomorphism if it is a real-linear ring isomorphism;
- an algebra isomorphism if it is a complex-linear ring isomorphism;
- a real $*$-isomorphism if it is a real algebra isomorphism and satisfies $\Phi(x^*) = \Phi(x)^*$ for all $x \in \mathcal{A}$;
- an $*$-isomorphism if it is a complex-linear real $*$-isomorphism.

Reduction of the general case to the case of a von Neumann algebra with a faithful normal finite trace

- $\mathcal{M}$ and $\mathcal{N}$ be arbitrary type II_1 von Neumann algebras with f.n.s.t. $\tau_{\mathcal{M}}$ and $\tau_{\mathcal{N}}$, respectively;
- Φ is a ring isomorphism Φ from $S(\mathcal{M})$ onto $S(\mathcal{N})$;
- there exists a family $\{z_i\}_{i\in I}$ of mutually orthogonal central projections in $\mathcal{M}$ with $\bigvee\limits_{i\in I} z_i = \mathbf{1}$ such that $\tau_{\mathcal{M}}(z_i) < +\infty$;
- Φ maps $S(Z(\mathcal{M}))$ onto $S(Z(\mathcal{N}))$, $i \in I$ there exists a family $\{z_{i,j}\}_{j\in J}$ of mutually orthogonal central projections in $z_i\mathcal{M}$ with $\bigvee\limits_{j\in J} z_{i,j} = z_i$ such that $\tau_{\mathcal{N}}(\Phi(z_{i,j})) < +\infty$;
- Φ maps each $S(z_{i,j}\mathcal{M})$ onto $S(\Phi(z_{i,j})\mathcal{N}) \equiv \Phi(z_{i,j})S(\mathcal{N})$ for all $i \in I, j \in J$.
- So, it suffices to consider the type II_1 vNa $\mathcal{M}$ and $\mathcal{N}$ with faithful normal finite traces $\tau_{\mathcal{M}}$ and $\tau_{\mathcal{N}}$.

Outline

Real-linearity and continuity of ring isomorphisms in the measure topology

Theorem 2.1

Let $\mathcal{M}$ and $\mathcal{N}$ be type II_1 von Neumann algebras. Then any ring isomorphism from $S(\mathcal{M})$ onto $S(\mathcal{N})$ is a real algebra isomorphism.

Theorem 2.2

Let $\mathcal{M}$ and $\mathcal{N}$ be type II_1 von Neumann algebras. Then any ring isomorphism from $S(\mathcal{M})$ onto $S(\mathcal{N})$ is continuous in the local measure topology.[a]

[a] Sh.A. Ayupov, K.K.Kudaybergenov, Ring isomorphisms of Murray-von Neumann algebras, Preprint (2020).

Outline

Partial order

- Let $\mathcal{M}$ and $\mathcal{N}$ be arbitrary type II_1 von Neumann algebras with faithful normal finite traces $\tau_{\mathcal{M}}$ and $\tau_{\mathcal{N}}$;
- $\Phi : \mathcal{M} \to \mathcal{N}$ be a ring isomorphism;
- Φ is a continuous real algebra isomorphism according to Theorems 2.1 and 2.2;
- for $x, y \in S(\mathcal{M})$ set

$$x \prec y \iff y = x + z,\ s(x)s(z) = 0;$$

- $\prec$ is a partial order on $S(\mathcal{M})$.

Lemma 2.3

There exists a sequence of projections $\{q_n\}$ in $\mathcal{M}$ with $\tau_{\mathcal{M}}(\mathbf{1} - q_n) \to 0$ such that Φ maps $q_n \mathcal{M} q_n$ into $\mathcal{N}$.

Range projection

Let $e \in S(\mathcal{M})$ be an idempotent, i.e., $e^2 = e$. Then

$$l(e)e = e, \ el(e) = l(e). \quad (1)$$

The first equality is the definition of the left projection. Using equality $ei(e) = l(e)$ we obtain that

$$el(e) = e(ei(e)) = e^2 i(e) = ei(e) = l(e).$$

Lemma 2.4

Let $\{e_n\} \subset S(\mathcal{M})$ be a sequence of idempotents such that $e_n \to e \in P(\mathcal{M})$ in the measure topology. Then $l(e_n) \to e$ in the same topology.

Cutting of derivation

Let $D : S(\mathcal{M}) \to S(\mathcal{M})$ be a derivation and let $e \in P(\mathcal{M})$. Then the mapping $D^{(e)} : S(e\mathcal{M}e) \to S(e\mathcal{M}e)$ defined as

$$D^{(e)}(x) = eD(x)e,\ x \in S(e\mathcal{M}e)$$

is a derivation. Here, essentially used the equality $eD(e)e = 0$.

But in general, for isomorphisms such type cutting do not work even in the case $\mathcal{M} = \mathcal{N}$, because, in general, $\Phi(e)$ is an idempotent, but is not a projection.

Cutting of isomorphism

- $p \in \mathcal{M}$ be an arbitrary projection;
- $\Phi(p)^2 = \Phi(p^2) = \Phi(p)$. But in general, $\Phi(p)^* \neq \Phi(p)$.
- $\widetilde{p} = l(\Phi(p))$ is the range projection of $\Phi(p)$;
- the mapping $\Phi_{p,\widetilde{p}} : S(p\mathcal{M}p) \to S(\widetilde{p}\mathcal{N}\widetilde{p})$ defined as

$$\Phi_{p,\widetilde{p}}(x) = \Phi(x)\widetilde{p}, \; x \in S(p\mathcal{M}p) \tag{2}$$

is a real algebra isomorphism from $S(p\mathcal{M}p)$ onto $S(\widetilde{p}\mathcal{N}\widetilde{p})$.

Lemma 2.5

There exists a sequence of projections $\{p_n\}$ in $\mathcal{M}$ with $\tau_{\mathcal{M}}(\mathbf{1} - p_n) \to 0$ such that $\Phi_{p_n,\widetilde{p_n}}$ maps $p_n\mathcal{M}p_n$ onto $\widetilde{p_n}\mathcal{N}\widetilde{p_n}$ for all $n \in \mathbb{N}$, where $\Phi_{p_n,\widetilde{p_n}}$ is defined by (2).

Main Theorem

The following main result confirms the Conjecture 5.1[1] and answers the above Question 1.2 for the type II_1 case.

Theorem 2.6

Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras of type II_1. Suppose that $\Phi : S(\mathcal{M}) \to S(\mathcal{N})$ is a ring isomorphism. Then there exist an invertible element $a \in S(\mathcal{N})$ and a real $*$-isomorphism $\Psi : \mathcal{M} \to \mathcal{N}$ (which extends to a real $*$-isomorphism from $S(\mathcal{M})$ onto $S(\mathcal{N})$) such that $\Phi(x) = a\Psi(x)a^{-1}$ for all $x \in S(\mathcal{M})$.[a]

[a] Sh.A. Ayupov, K.K.Kudaybergenov, Ring isomorphisms of Murray-von Neumann algebras, Preprint (2020).

[1] M. Mori, Lattice isomorphisms between projection lattices of von Neumann algebras, arXiv:2006.08959 (2020).

Main Theorem

Corollary 2.7

Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras of type II_1. The projection lattices $P(\mathcal{M})$ and $P(\mathcal{N})$ are lattice isomorphic, if and only if the von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ are real $*$-isomorphic (or equivalently, $\mathcal{M}$ and $\mathcal{N}$ are Jordan $*$-isomorphic).

Example 2.8

Let $\mathcal{R}$ be a hyperfinite factor of type II_1. There exists an $*$-regular subalgebra $\mathcal{A} \subset S(\mathcal{R})$ which admits an algebra automorphism, discontinuous in the measure topology.