# On $A_\infty$ algebras and modules

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October 1, 2020

# Motivation and history

A **topological group**, $G$, is a topological space that is also a group such that the group product:

$$m : G \times G \to G\,,\ (g, h) \mapsto g \cdot h$$

and the inversion map

$$G \to G\,,\ g \mapsto g^{-1}$$

are continuous maps. Examples of topological groups include: $\mathbb{R}, S^1, S^3$, Lie groups, algebraic groups, topological vector spaces etc.

There are a number of important results in various areas of mathematics about topological groups which one would like to generalize to more general topological spaces.

One of such generalizations in algebraic topology are $H$-**spaces**.

It is a topological space $X$ together with a continuous map $m : X \times X \to X$ with the identity element $e$.

These spaces where introduced and studied by I.M. James, J.P.Serre, J.Milnor, J.F. Adams and J. Stasheff from 1953-1962.

Characterization of $H$-spaces was one of the central problems. In particular, Adams showed that the only spheres that are $H$-spaces are $S^0, S^1, S^3, S^7$.

As one can see $H$-spaces lack associativity and inverses. But from algebraic topology point of view associativity property is an essential feature of $H$-spaces making them closer to topological groups.

In 1963, J. Stasheff introduced $A_\infty$-**spaces** which are $H$-spaces with multiplication being associative up to *homotopy*.

# Homotopy in topology and homological algebra

Let $X, Y$ be topological spaces, and let $f, g : X \to Y$ be continuous maps. Then $f$ and $g$ are **homotopic** if there exists a continuous map

$$H : [0,1] \times X \to Y$$

so that $H(0,x) = f(x)$ and $H(1,x) = g(x)$ for all $x \in X$.

One can think of homotopy as a family of maps

$$\{f_t : X \to Y\}_{t \in [0,1]}$$

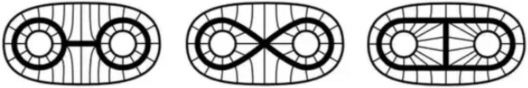
$f_t(x) = H(t,x)$ connecting $f_0 = f$ and $f_1 = g$. We denote $f \sim g$.

**Lemma**

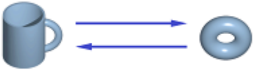
*$\sim$ is an equivalence relation.*

**Defintion:** $X$ is **homotopy equivalent** to $Y$ if there exist $f : X \to Y$ and $g : Y \to X$ s.t. $f \circ g \sim 1_Y$ and $g \circ f \sim 1_X$. We denote by $X \sim Y$ and say $X$ and $Y$ have the same homotopy type.

**Lemma:** $X \sim Y$ is an equivalence relation.



**Remark:** Roughly, $X \sim Y$ if one can be transformed into the other by bending, shrinking and expanding



Let

$$C_\bullet : \cdots \to C_{n+1} \xrightarrow{d^C_{n+1}} C_n \xrightarrow{d^C_n} C_{n-1} \to \dots$$

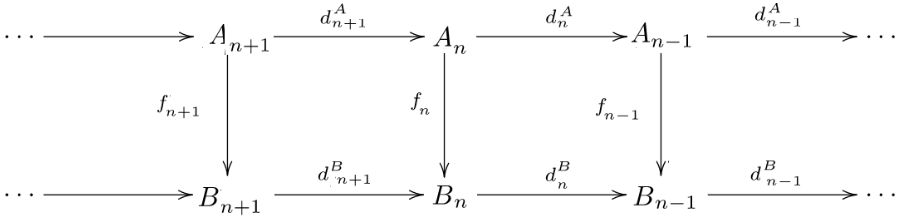
be a chain complex, that is, $d^C_n \circ d^C_{n-1} = 0$. Then the $n^{th}$ **homology** defined as

$$H_n(C_\bullet) = \ker d^C_n / Im d^C_{n+1}.$$

Let $A_\bullet, B_\bullet$ be chain complexes. Then a **morphism**

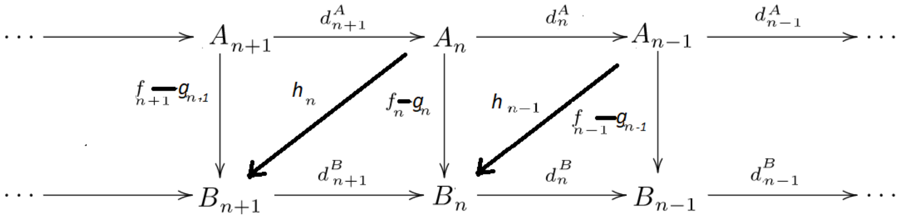
$$f : A_\bullet \to B_\bullet$$

is a family of maps $f_n : A_n \to B_n$ s.t. $f_{n-1} \circ d^A_n = d^B_n \circ f_n$ for all $n$.



We say two morphisms $f_\bullet, g_\bullet : A_\bullet \to B_\bullet$ are **homotopic** if there exist maps $h_n : A_n \to B_{n+1}$ s.t.

$$f_n - g_n = d^B_{n+1} \circ h_n + h_{n-1} \circ d^A_n$$




In this case we call $f_\bullet$ and $g_\bullet$ are **chain homotopic**, $f_\bullet \sim g_\bullet$.

A morphism $f_\bullet : A_\bullet \to B_\bullet$ induces the maps on homologies

$$H_n(f_\bullet) : H_n(A_\bullet) \to H_n(B_\bullet).$$

Indeed, $f_n(\ker d_n^A) \subset \ker d_n^B$

If $a \in \ker d_n^A \Rightarrow d_n^A(a) = 0 \Rightarrow f_n d_n^A(a) = 0$

$$\Rightarrow d_n^B f_n(a) = 0 \Rightarrow f_n(a) \in \ker d_n^B.$$

Similarly, $f_n(Imd_{n+1}^A) \subset Imd_{n+1}^B$.

**Lemma**

*$f_\bullet \sim g_\bullet$ is an equivalence relation and they induce the same map $H_n(A_\bullet) \to H_n(B_\bullet)$.*

We say $A_\bullet$ and $B_\bullet$ are **chain homotopic** if there exist $f_\bullet : A_\bullet \to B_\bullet$ and $g_\bullet : B_\bullet \to A_\bullet$ s.t. $f_\bullet \circ g_\bullet \sim 1_{B_\bullet}$ and $g_\bullet \circ f_\bullet \sim 1_{A_\bullet}$.

We say $f_\bullet : A_\bullet \to B_\bullet$ is a **quasi-isomorphism** if $H_n(f_\bullet)$ is an isomorphism for all $n$.

**Lemma**

*If $A_\bullet$ and $B_\bullet$ are chain homotopic then they are quasi-isomorphic.*

# $A_\infty$-spaces (after J. Stasheff 1963)

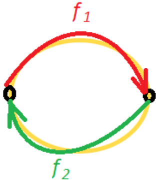
Let $X$ be a connected topological space, and let $x_0$ be a base point in $X$. Consider **based** loopspace

$$\Omega X = \{f : S^1 \to X \ continuous \ \ and \ \ f(0) = x_0\}.$$

$\Omega X$ is a topological space and there is the composition map

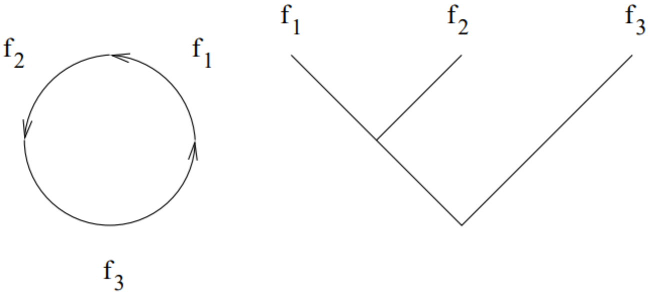
$$m_2 : \Omega X \times \Omega X \to \Omega X, \quad (f_1, f_2) \mapsto f_1 * f_2.$$

$$(f_1 * f_2)(t) = \begin{cases} f_1(2t), & 0 \leq t \leq \frac{1}{2} \\ f_2(2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

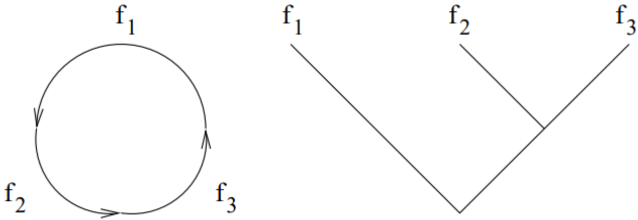


This composition is not associative $f_1 * (f_2 * f_3) \neq (f_1 * f_2) * f_3$.

$(f_1 * f_2) * f_3$




$f_1 * (f_2 * f_3)$



We have two maps

$$f : (\Omega X)^3 \to \Omega X \qquad (f_1, f_2, f_3) \mapsto (f_1 * f_2) * f_3$$
$$g : (\Omega X)^3 \to \Omega X \qquad (f_1, f_2, f_3) \mapsto f_1 * (f_2 * f_3)$$

It is easy to show there is a homotopy

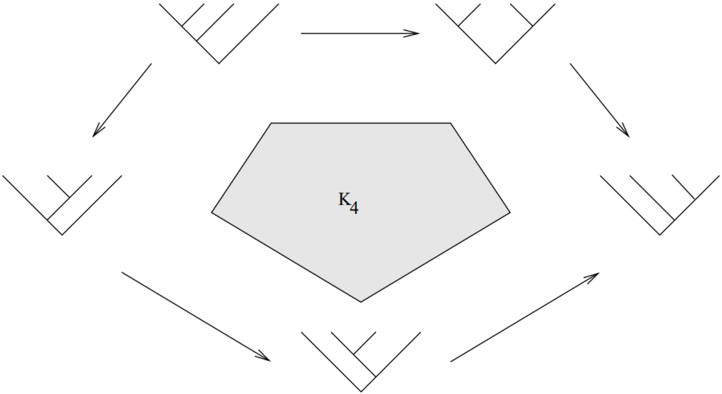
$$m_3 : [0,1] \times (\Omega X)^3 \to \Omega X$$

s.t. $m_3(0, -) = f$ and $m_3(1, -) = g$.

Next, if we want to compose 4 loops $f_1, f_2, f_3, f_4$ then we have 5 possibilities.

$w_1 = ((f_1 * f_2) * f_3) * f_4, \quad w_2 = (f_1 * f_2) * (f_3 * f_4), \quad w_3 = f_1 * (f_2 * (f_3 * f_4)),$
$w_4 = (f_1 * (f_2 * f_3)) * f_4, \quad w_5 = f_1 * ((f_2 * f_3) * f_4).$
Then there exists a continuous map

$$m_4 : K_4 \times (\Omega X)^4 \to \Omega X$$



We can continue and get a sequence of maps

$$m_n : K_n \times Y^n \to Y, \quad n \geq 2$$

where $K_2 = \{x_0\}, \quad K_3 = [0, 1], \ldots \qquad \dim K_n = n - 2.$

Such spaces are called $A_\infty$-**spaces**.
$Y = \Omega X$ is a prime example of such spaces.

# $A_\infty$-algebras

Let $k$ be a field, and let $A = \bigoplus_{m\in\mathbb{Z}} A_m$ is a $\mathbb{Z}$-graded vector space.
Then $A^{\otimes n}$ is also a graded vector space

$$A^{\otimes n} = \bigoplus_{i\in\mathbb{Z}} A_i^{\otimes n}, \quad A_i^{\otimes n} := \bigoplus_{i_1+\cdots+i_n=i} (A_{i_1} \otimes \cdots \otimes A_{i_n})$$

Let $V = \bigoplus_{i\in\mathbb{Z}} V_i$ and $W = \bigoplus_{i\in\mathbb{Z}} W_i$ be two graded vector spaces. Then a map $\varphi : V \to W$ is a **graded map of degree** $m$ if $\varphi(V_i) \subset W_{i+m}$ for all $i \in \mathbb{Z}$.

If $(A_\bullet, d^A)$ and $(B_\bullet, d^B)$ be chain complexes, then $A_\bullet \otimes B_\bullet$ is a chain complex with the differential $d^A \otimes 1_B + 1_A \otimes d^B$.

## Definition

*An $A_\infty$-**algebras** is a $\mathbb{Z}$-graded vector space $A = \bigoplus_{m\in\mathbb{Z}} A_m$ together with graded maps*

$$m_n : A^{\otimes n} \to A, \quad n \geq 1.$$

*of degree $2-n$ subject to the following relations:*

- $m_1 \circ m_1 = 0$

**Explanation:** $m_1 : A \to A, \quad deg(m_1) = 1 \Rightarrow m_1 : A_i \to A_{i+1}$

$$\cdots \to A_i \xrightarrow{m_1} A_{i+1} \xrightarrow{m_1} A_{i+2} \to \dots$$

so $(A, m_1)$ is a chain complex.

- $m_1 \circ m_2 = m_2(m_1 \otimes 1 + 1 \otimes m_1)$

**Explanation:** Let $a, b \in A$ and denote $a \cdot b := m_2(a, b)$. Then

$$m_1(a \cdot b) = m_1(a) \cdot b + a \cdot m_1(b).$$

So $m_1$ is a derivation with respect to operation $m_2$

- $m_2(1 \otimes m_2 - m_2 \otimes 1) = m_1 \circ m_3 + m_3 \circ (m_1 \otimes 1 \otimes 1 + 1 \otimes m_1 \otimes 1 + 1 \otimes 1 \otimes m_1)$

**Explanation:** If $f := m_2 \circ (1 \otimes m_2)$ and $g := m_2(m_2 \otimes 1)$ then $f, g : A^{\otimes 3} \to A$. Here $A^{\otimes 3}$ is a chain complex with differential given by

$$d^{A^3} := m_1 \otimes 1 \otimes 1 + 1 \otimes m_1 \otimes 1 + 1 \otimes 1 \otimes m_1 ,$$

and $A$ is a chain complex with differential $d^A := m_1$. Then $f$ and $g$ are chain maps and the above equation can be written as

$$f - g = d^A \circ m_3 + m_3 \circ d^{A^3} .$$

Thus $m_3$ can be interpreted as a homotopy between $f$ and $g$. Note that

$$(f - g)(a, b, c) = a \cdot (b \cdot c) - (a \cdot b) \cdot c$$

This means $m_2$ multiplication is associative up to homotopy $m_3$.

More generally, for $n \geq 1$, we have

• $\sum_n (-1)^{r+st} m_u \circ (1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0$, where $n = r + s + t$ and $u = r + 1 + t$. Note the following consequences:

1. If $A_m = 0$ for $m \neq 0$, then $A = A_0$ is a usual associative algebra, since $m_1 = 0$ and $m_i = 0$ for $i \geq 3$.

2. If we take the homology of $A$ with respect to $m_1$, then $H_\bullet$ is an associative algebra because $\overline{m}_1 = 0$ in $H_\bullet(A)$.

3. If $m_i = 0$ for $i \geq 3$, then $A$ is also an associative algebra.

# Singular homology of $A_\infty$ spaces

Let $v_0, \dots, v_n$ be points in $\mathbb{R}^m$, so that the vectors $v_1 - v_0, \dots v_n - v_0$ are linearly independent. Then the $n$-simplex with vertices $v_0, v_1, \dots, v_n$ is the smallest convex set in $\mathbb{R}^m$, which we denote by $[v_0, \dots, v_n]$.

The standard $n$-simplex

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0\}$$

is identified with $[v_0, \dots, v_n]$ via

$$(t_0, \dots, t_n) \mapsto \sum_{i=0}^{n} t_i v_i.$$

Let $X$ be a topological space. Then a singular $n$-simplex in $X$ is a continuous map $\sigma\colon \Delta^n \to X$. We set

$$C_n(X) := \Big\{ \sum_i n_i\sigma_i - \text{ finite sum, } n_i \in \mathbb{Z},\ \sigma_i \text{ is } n\text{-simplex} \Big\}.$$

One can define a boundary map $\partial_n\colon C_n(X) \to C_{n-1}(X)$ as

$$\partial_n(\sigma) = \sum (-1)^i \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]},$$

where $[v_0, \ldots, \hat{v}_i, \ldots, v_n]$ is $(n-1)$-simplex $v_i$ vertex missing.

Lemma: $\partial_n \circ \partial_{n+1} = 0$.

We define $H_n(X) := \ker \partial_n / Im\partial_{n+1}$ the $n$-th singular homology group of $X$.

**Proposition**

If $Y$ is an $A_\infty$-space, then $A = \bigoplus_n C_n(Y)$ is an $A_\infty$-algebra.

# Morphisms of $A_\infty$-algebras

A *morphism of* $A_\infty$*-algebras* $f : A \to B$ is a family

$$f_n : A^{\otimes n} \to B$$

of graded maps of degree $1 - n$ satisfying

- $f_1 \circ m_1^A = m_1^B \circ f_1$, that is, $f_1$ is a morphism of complexes.
- $f_1 \circ m_2^A = m_2^B \circ (f_1 \otimes f_1) + m_1^B \circ f_2 + f_2(m_1^A \otimes 1 + 1 \otimes m_1^A)$

meaning $f_1$ commutes with the multiplication $m_2$ up to homotopy $f_2$.

More generally, for $n \geq 1$, we have

- $\sum(-1)^{r+st} f_u \circ (1^{\otimes r} \otimes m_s^A \otimes 1^{\otimes t}) = \sum(-1)^s m_r^B \circ (f_{i_1} \otimes \ldots \otimes f_{i_r})$.

A morphism $f$ is a *quasi-isomorphism* if $f_1$ is a quasi-isomorphism of chain complexes.

**Problem**. Given a a chain complex $M_\bullet$, what additional structure is needed to reconstruct $M_\bullet$ from its homology.

**Theorem**

*(T.Kadeishvili,1982) Let $A$ be an $A_\infty$ algebra. Then the homology $H_*(A)$ has an $A_\infty$ algebra structure s.t.*

*1) $m_1 = 0$, $m_2$ is induced by $m_2^A$,*
*2) there is a quasi-isomorphism of $A_\infty$ algebras $H_*(A) \to A$ lifting the identity of $H_*(A)$ .*

*This structure is unique up to $A_\infty$ isomorphism.*