Linearly Converging Error Compensated SGD

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The talk is based on our paper «Linearly Converging Error Compensated SGD» accepted to **NeurIPS 2020**

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MIPT

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Outline

1. The problem
2. Parallel SGD
3. Communication bottleneck
4. Error-compensated SGD

15 minutes

5. New methods: EC-GDstar, EC-SGD-DIANA and EC-L-SVRG-DIANA

10 minutes

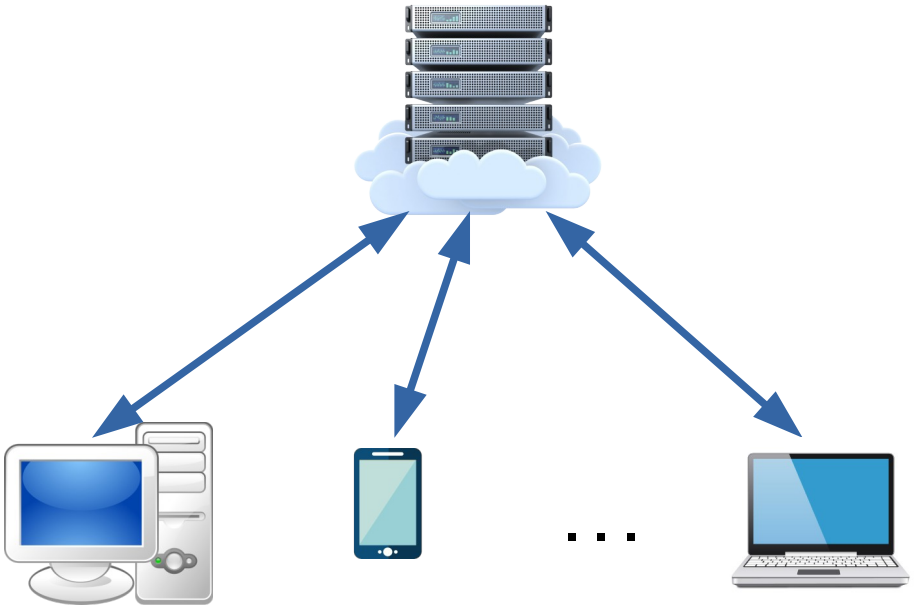
6. Unified convergence analysis

10 minutes

7. Experiments
8. Other applications of the framework: methods without error-compensation and methods with delayed updates

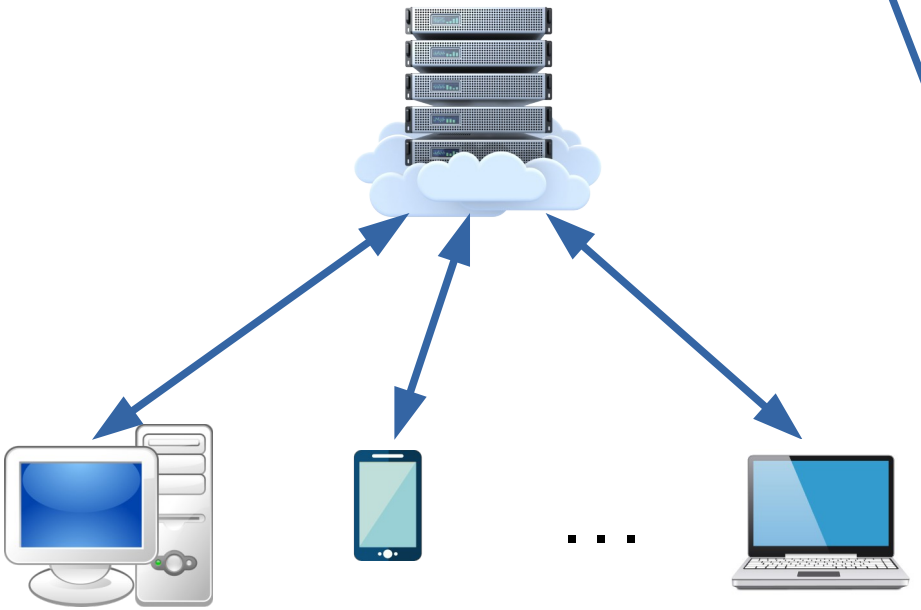
5 minutes

1. The Problem



$$\min_{x \in \mathbb{R}^d} \left\{ f(x) := \frac{1}{n} \sum_{i=1}^{n} f_i(x) \right\}$$

parameters of the model



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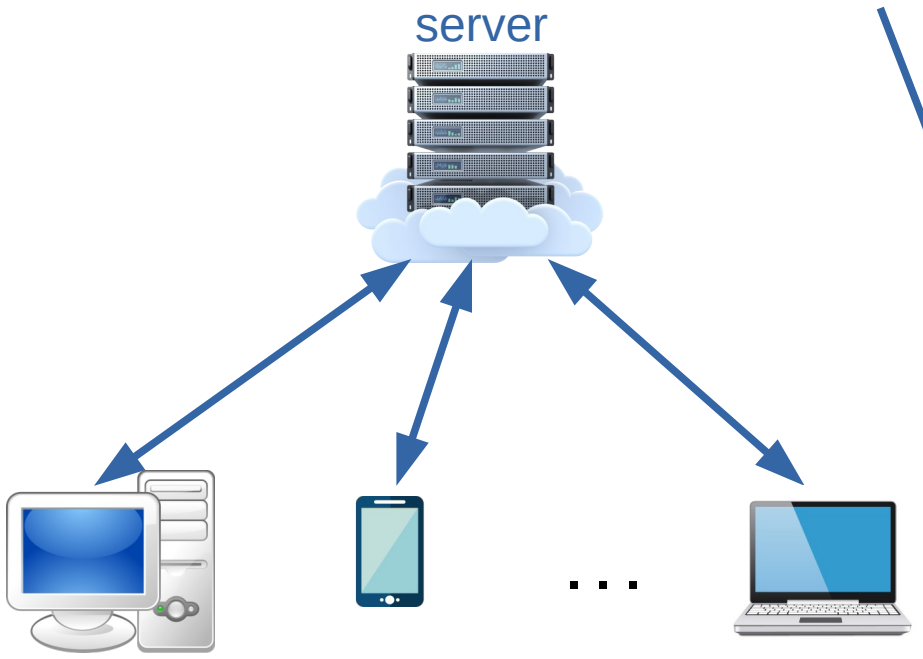

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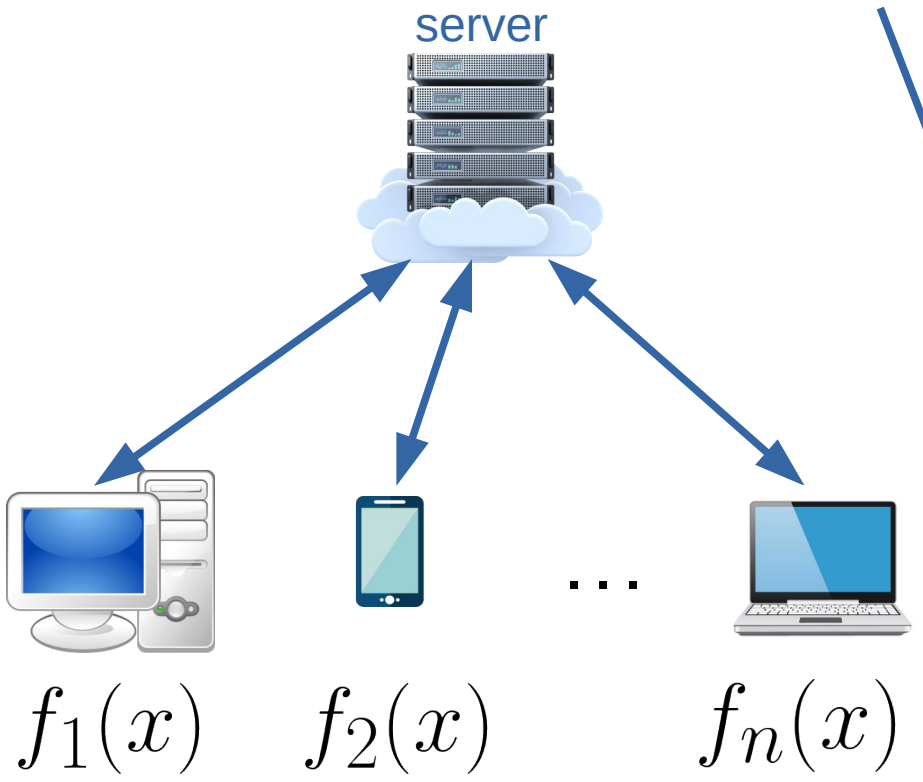
# of features

# of workers/devices

n workers/devices



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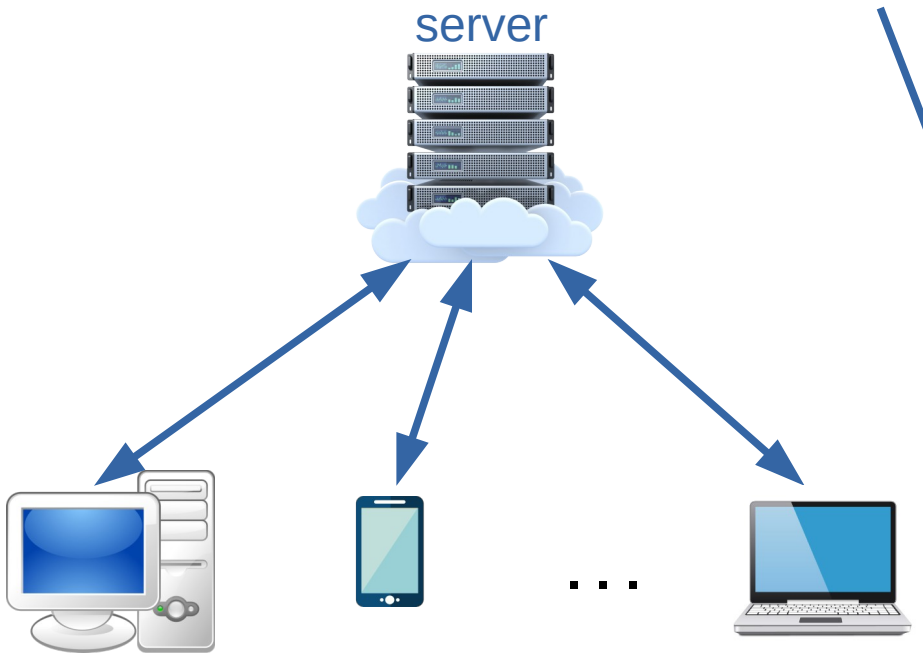
loss on the data accessible by worker *i*



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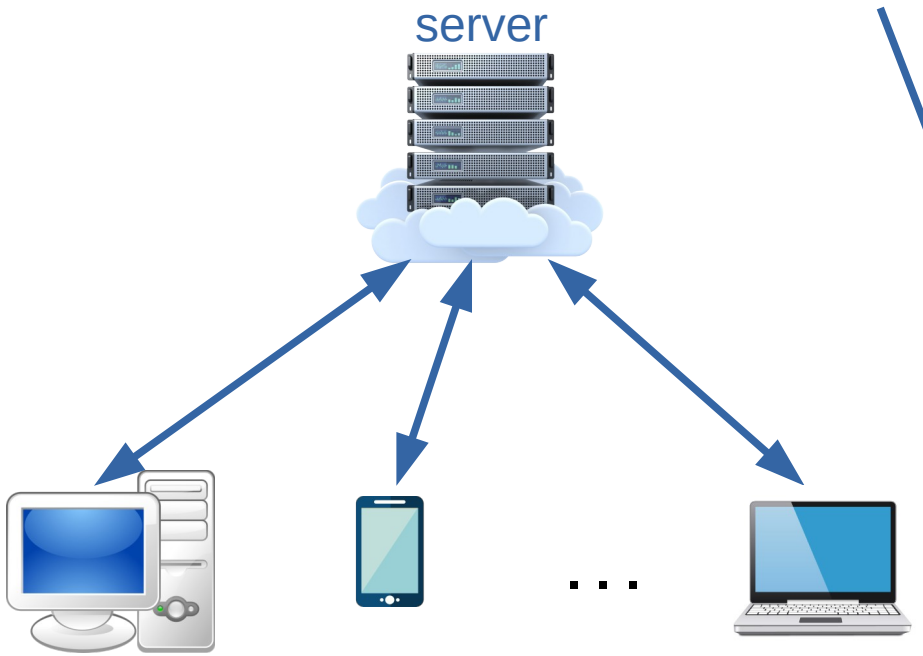
$$f_i(x) = \mathbf{E}_{\xi_i \sim \mathcal{D}_i} \left[f_{\xi_i}(x) \right]$$



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# of features

# of workers/devices

loss on the data accessible by worker *i*

- $f_i(x) = \mathbf{E}_{\xi_i \sim \mathcal{D}_i} \left[f_{\xi_i}(x) \right]$
- $f_i(x) = \frac{1}{m} \sum_{j=1}^{m} f_{ij}(x)$

Assumptions

- $f_1, f_2, \ldots, f_n$ – L-smooth and convex

Assumptions

$$\|\nabla f_i(x) - \nabla f_i(y)\| \leq L\|x - y\|$$

$$f_i(x) - f_i(y) \geq \langle \nabla f_i(y), x - y \rangle$$

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- $f_1, f_2, \ldots, f_n$ – L-smooth and convex
- f – strongly quasi-convex

$$f(x^*) \geq f(x) + \langle \nabla f(x), x^* - x \rangle + \frac{\mu}{2}\|x^* - x\|^2$$

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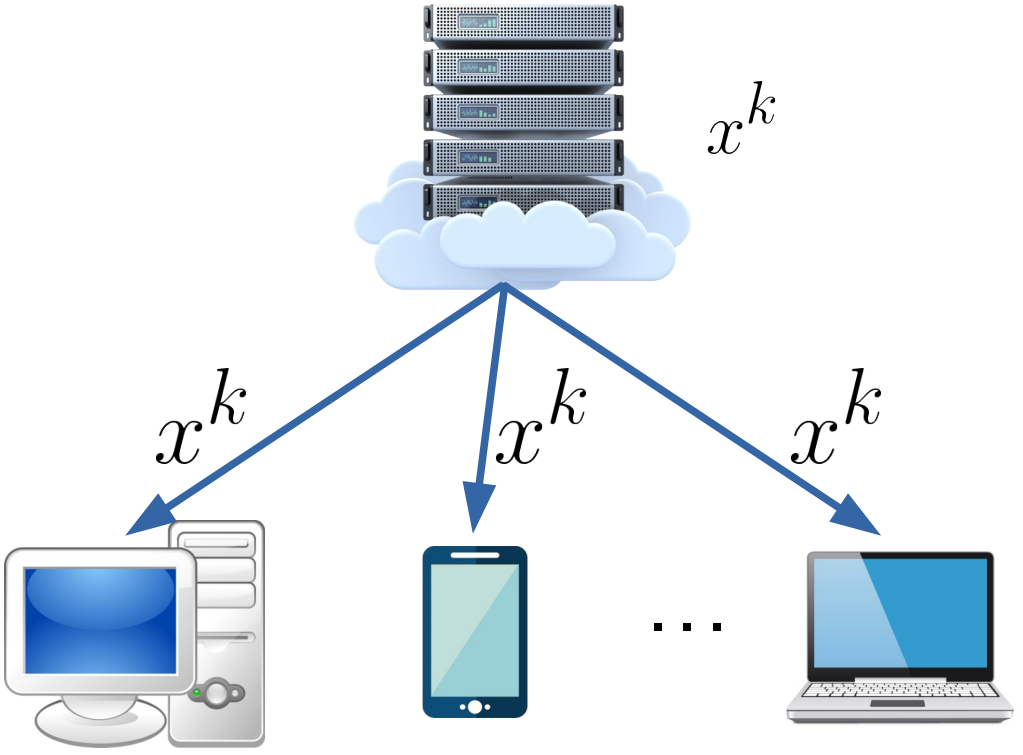
the solution of the problem

2. Parallel SGD

1 Server broadcasts the parameters



1. Server broadcasts the parameters
2. Devices compute **stochastic gradients** in parallel

x^k

$x^k \to g_1^k \quad x^k \to g_2^k \quad \cdots \quad x^k \to g_n^k$

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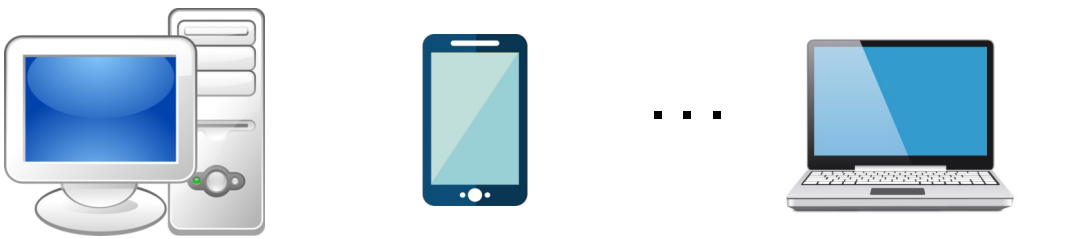

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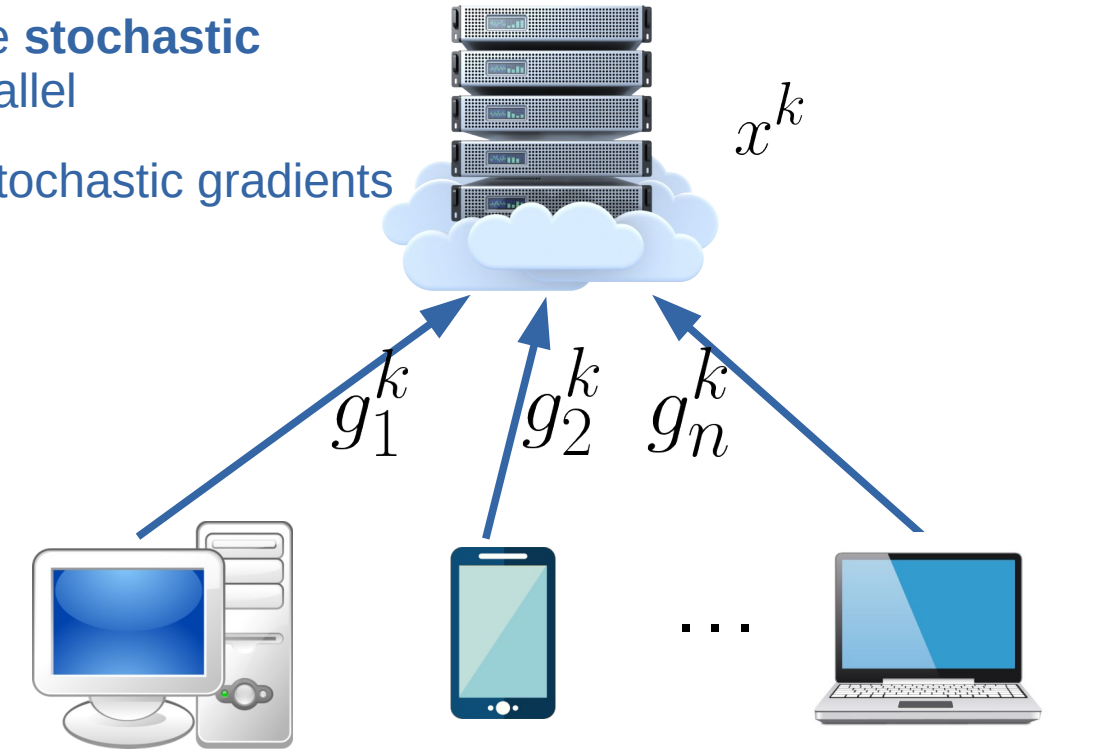

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4. Server updates the parameters

$$x^k \rightarrow x^{k+1} = x^k - \gamma \cdot \frac{1}{n} \sum_{i=1}^{n} g_i^k$$

$g_1^k \quad g_2^k \quad g_n^k$

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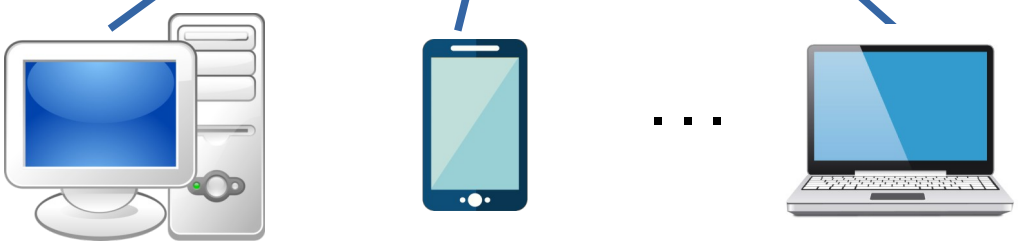

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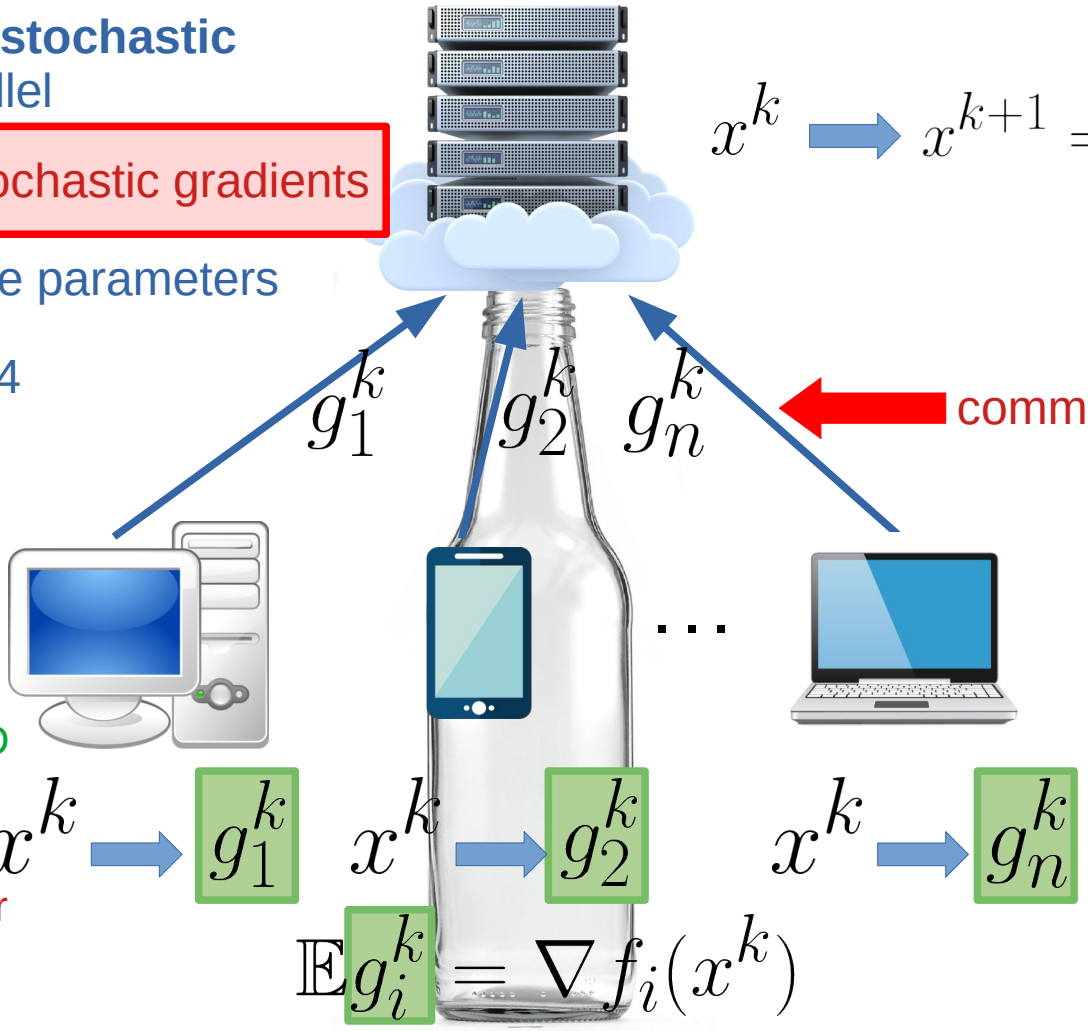

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Issues:

- Overload of the server

3. Communication Bottleneck

How to Handle Communication Bottleneck?

- Change the topology of the network → Decentralized optimization

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Example:

Send $g_i^k = \begin{pmatrix} 1 \\ -15 \\ 0.2 \\ -7 \\ 10 \end{pmatrix}$ → Send $\mathcal{C}(g_i^k) = \begin{pmatrix} 0 \\ -15 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

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- **Send less information to reduce the communication cost**

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Compression operator

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What are the options for choosing this?

Compression Operators

Unbiased compressors
(quantizations)

$$x \to \mathcal{Q}(x) \qquad \mathbb{E}[\mathcal{Q}(x)] = x$$

Biased compressors

$$x \to \mathcal{C}(x)$$

Compression Operators

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$$x \to \mathcal{Q}(x) \qquad \mathbb{E}[\mathcal{Q}(x)] = x$$

$$\mathbb{E}\|\mathcal{Q}(x) - x\|^2 \leq \omega\|x\|^2$$

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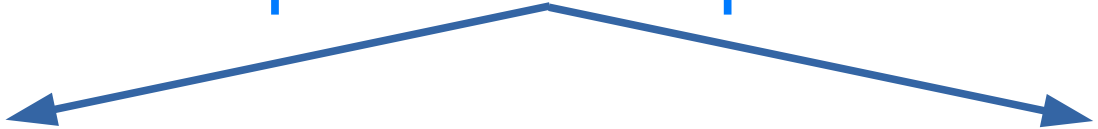
Compression Operators



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Example: RandK (for K = 2)

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Pick K = 2 components uniformly at random

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$$\begin{pmatrix} 1 \\ -15 \\ 0.2 \\ -7 \\ 10 \end{pmatrix} \quad \Rightarrow \quad \frac{5}{2} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ -7 \\ 0 \end{pmatrix}$$

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Example: TopK (for K = 2)

$$\begin{pmatrix} 1 \\ -15 \\ 0.2 \\ -7 \\ 10 \end{pmatrix} \to$$

Pick K = 2 components with largest absolute value

Compression Operators

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Methods with Unbiased Compressors

QSGD

Alistarh, Dan, Demjan Grubic, Jerry Li, Ryota Tomioka, and Milan Vojnovic. **"QSGD: Communication-efficient SGD via gradient quantization and encoding."** *In Advances in Neural Information Processing Systems*, pp. 1709-1720. 2017.

TernGrad

Wen, Wei, Cong Xu, Feng Yan, Chunpeng Wu, Yandan Wang, Yiran Chen, and Hai Li. **"Terngrad: Ternary gradients to reduce communication in distributed deep learning."** *In Advances in neural information processing systems*, pp. 1509-1519. 2017.

DQGD

Khirirat, Sarit, Hamid Reza Feyzmahdavian, and Mikael Johansson. **"Distributed learning with compressed gradients."** arXiv preprint arXiv:1806.06573 (2018).

DIANA

Mishchenko, Konstantin, Eduard Gorbunov, Martin Takáč, and Peter Richtárik. **"Distributed learning with compressed gradient differences."** arXiv preprint arXiv:1901.09269 (2019).

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Sublinear convergence rates even in the case when workers quantize full gradients

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Converges **linearly** when workers quantize full gradients

Parallel SGD with Biased Compressor Can Diverge at Exponential Rate

Beznosikov, Aleksandr, Samuel Horváth, Peter Richtárik, and Mher Safaryan. **"On Biased Compression for Distributed Learning."** arXiv preprint arXiv:2002.12410 (2020).

$$n = d = 3$$

$$f_1(x) = \langle a, x\rangle^2 + \frac{1}{4}\|x\|^2 \qquad f_2(x) = \langle b, x\rangle^2 + \frac{1}{4}\|x\|^2 \qquad f_3(x) = \langle c, x\rangle^2 + \frac{1}{4}\|x\|^2$$

$$a = (-3, 2, 2)^\top \qquad b = (2, -3, 2)^\top \qquad c = (2, 2, -3)^\top$$

$$x^0 = (t, t, t)^\top$$

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In this case Parallel SGD with Top1 compression operator satisfies

$$x^k = \left(1 + \frac{11\gamma}{6}\right)^k x^0$$

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One can fix this using one special trick called ***error-compensation***

4. Error-Compensated SGD

Seide, Frank, Hao Fu, Jasha Droppo, Gang Li, and Dong Yu. **"1-bit stochastic gradient descent and its application to data-parallel distributed training of speech dnns."** *In Fifteenth Annual Conference of the International Speech Communication Association*. 2014.

Stich, Sebastian U., Jean-Baptiste Cordonnier, and Martin Jaggi. **"Sparsified SGD with memory."** *In Advances in Neural Information Processing Systems*, pp. 4447-4458. 2018.

Karimireddy, Sai Praneeth, Quentin Rebjock, Sebastian Stich, and Martin Jaggi. **"Error Feedback Fixes SignSGD and other Gradient Compression Schemes."** *In International Conference on Machine Learning*, pp. 3252-3261. 2019.

Stich, Sebastian U., and Sai Praneeth Karimireddy. **"The error-feedback framework: Better rates for SGD with delayed gradients and compressed communication."** arXiv preprint arXiv:1909.05350 (2019).

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Step 1

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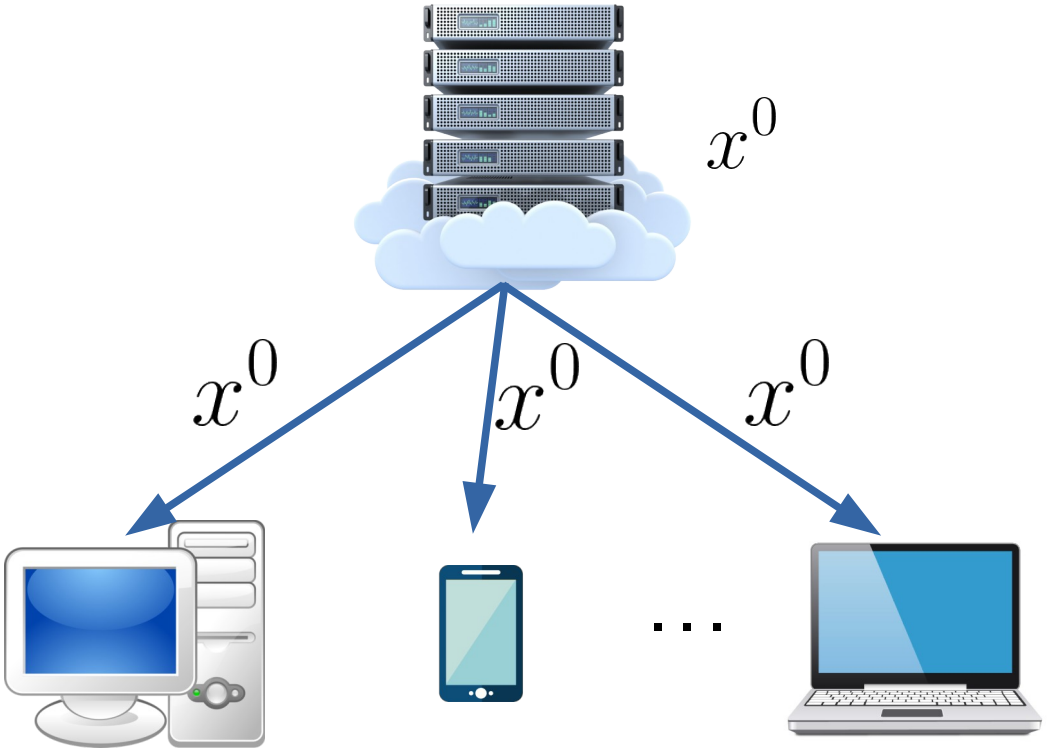

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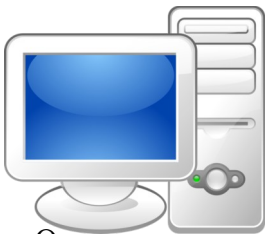
x^0



$x^0 \rightarrow g_1^0$

$x^0 \rightarrow g_2^0$

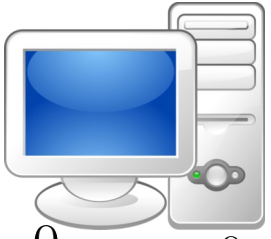
$\cdots$

$x^0 \rightarrow g_n^0$

Step 1

1. Server broadcasts the parameters
2. Devices compute **stochastic gradients** in parallel
3. Compression

x^0

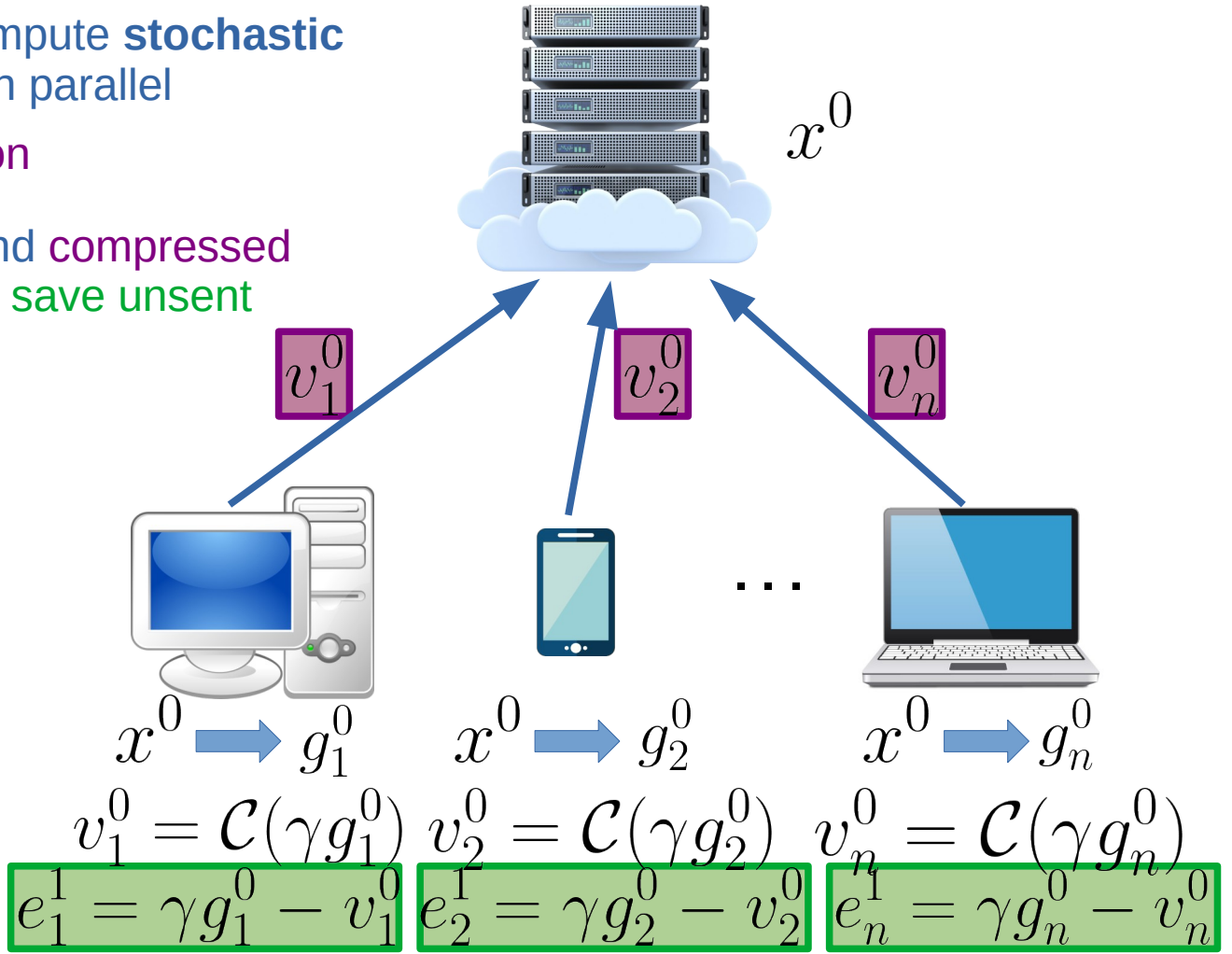


$$x^0 \rightarrow g_1^0 \qquad x^0 \rightarrow g_2^0 \qquad x^0 \rightarrow g_n^0$$

$$v_1^0 = \mathcal{C}(\gamma g_1^0) \qquad v_2^0 = \mathcal{C}(\gamma g_2^0) \qquad v_n^0 = \mathcal{C}(\gamma g_n^0)$$

Step 1

1. Server broadcasts the parameters
2. Devices compute **stochastic gradients** in parallel
3. Compression
4. Devices send compressed vectors and save unsent information



Step 1

1. Server broadcasts the parameters
2. Devices compute **stochastic gradients** in parallel
3. Compression
4. Devices send compressed vectors and save unsent information
5. Server gathers the information and updates the parameters

$$x^0 \rightarrow x^1 = x^0 - \frac{1}{n}\sum_{i=1}^{n} v_i^0$$

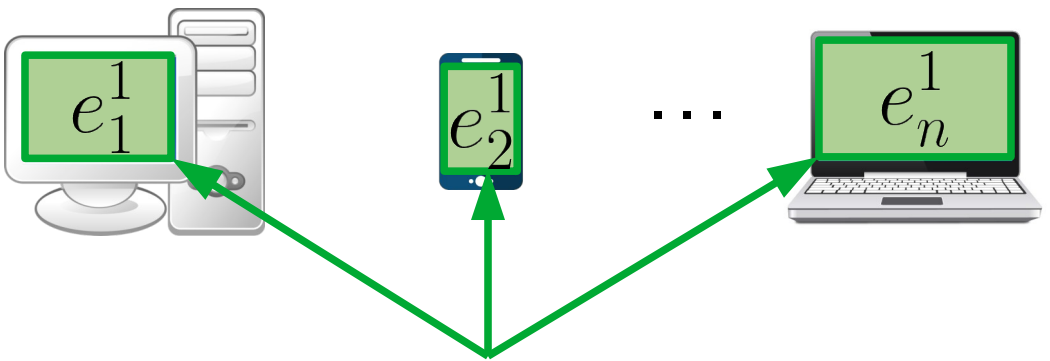
$$x^0 \rightarrow g_1^0 \qquad x^0 \rightarrow g_2^0 \qquad x^0 \rightarrow g_n^0$$

$$v_1^0 = \mathcal{C}(\gamma g_1^0) \quad v_2^0 = \mathcal{C}(\gamma g_2^0) \quad v_n^0 = \mathcal{C}(\gamma g_n^0)$$

$$e_1^1 = \gamma g_1^0 - v_1^0 \quad e_2^1 = \gamma g_2^0 - v_2^0 \quad e_n^1 = \gamma g_n^0 - v_n^0$$

Step 1

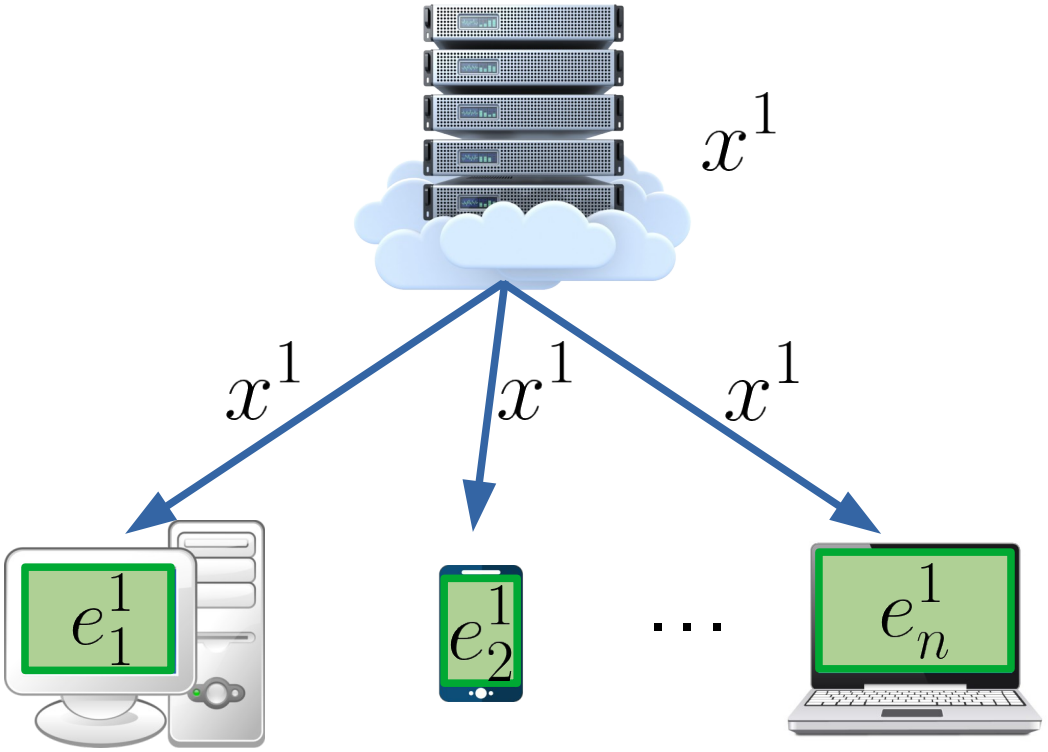
x^1



devices keep these vectors for the next iterations
to *partially* send them later

Step 2

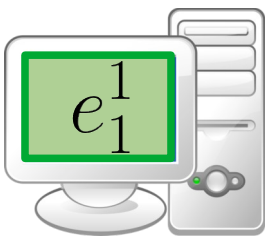
1. Server broadcasts new parameters



Step 2

1. Server broadcasts new parameters
2. Devices compute **stochastic gradients** in parallel

x^1



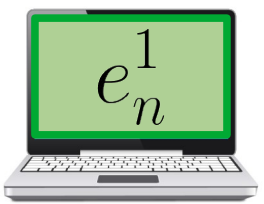
e_1^1

$x^1 \rightarrow g_1^1$

e_2^1

$x^1 \rightarrow g_2^1$

$\cdots$



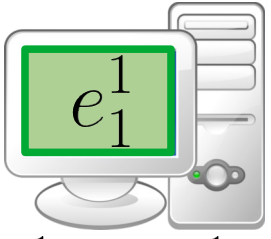
e_n^1

$x^1 \rightarrow g_n^1$

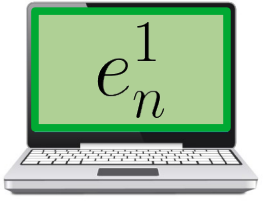
Step 2

1. Server broadcasts new parameters
2. Devices compute **stochastic gradients** in parallel
3. Compression

x^1



$\cdots$



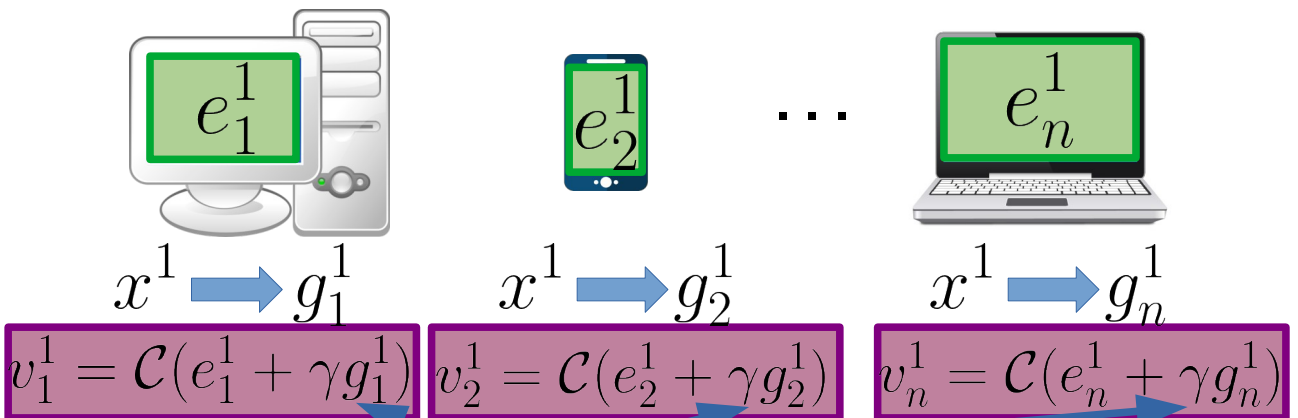
$x^1 \rightarrow g_1^1$ $\quad$ $x^1 \rightarrow g_2^1$ $\quad$ $x^1 \rightarrow g_n^1$

$v_1^1 = \mathcal{C}(e_1^1 + \gamma g_1^1)$ $\quad$ $v_2^1 = \mathcal{C}(e_2^1 + \gamma g_2^1)$ $\quad$ $v_n^1 = \mathcal{C}(e_n^1 + \gamma g_n^1)$

Step 2

1. Server broadcasts new parameters
2. Devices compute **stochastic gradients** in parallel
3. Compression

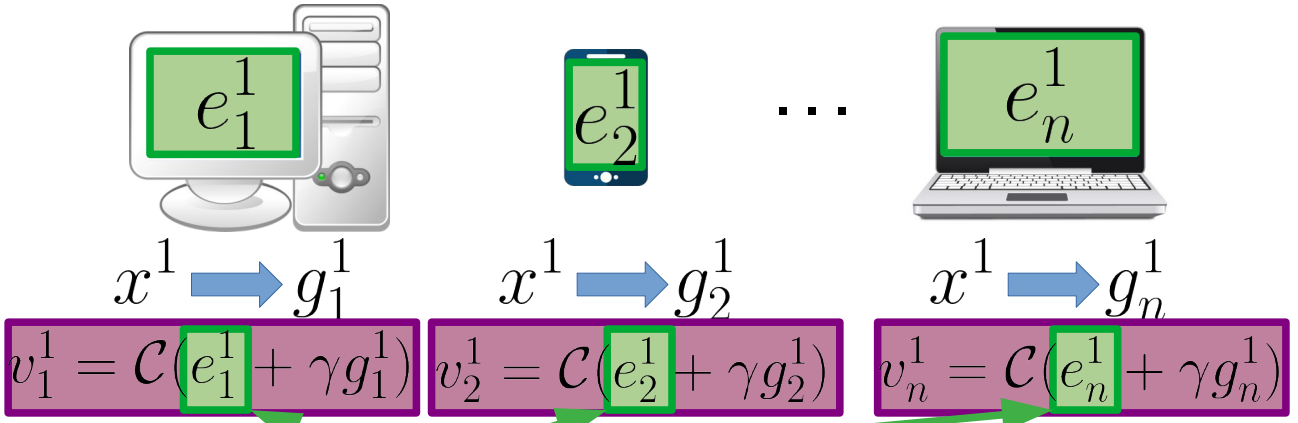
x^1



Step 2

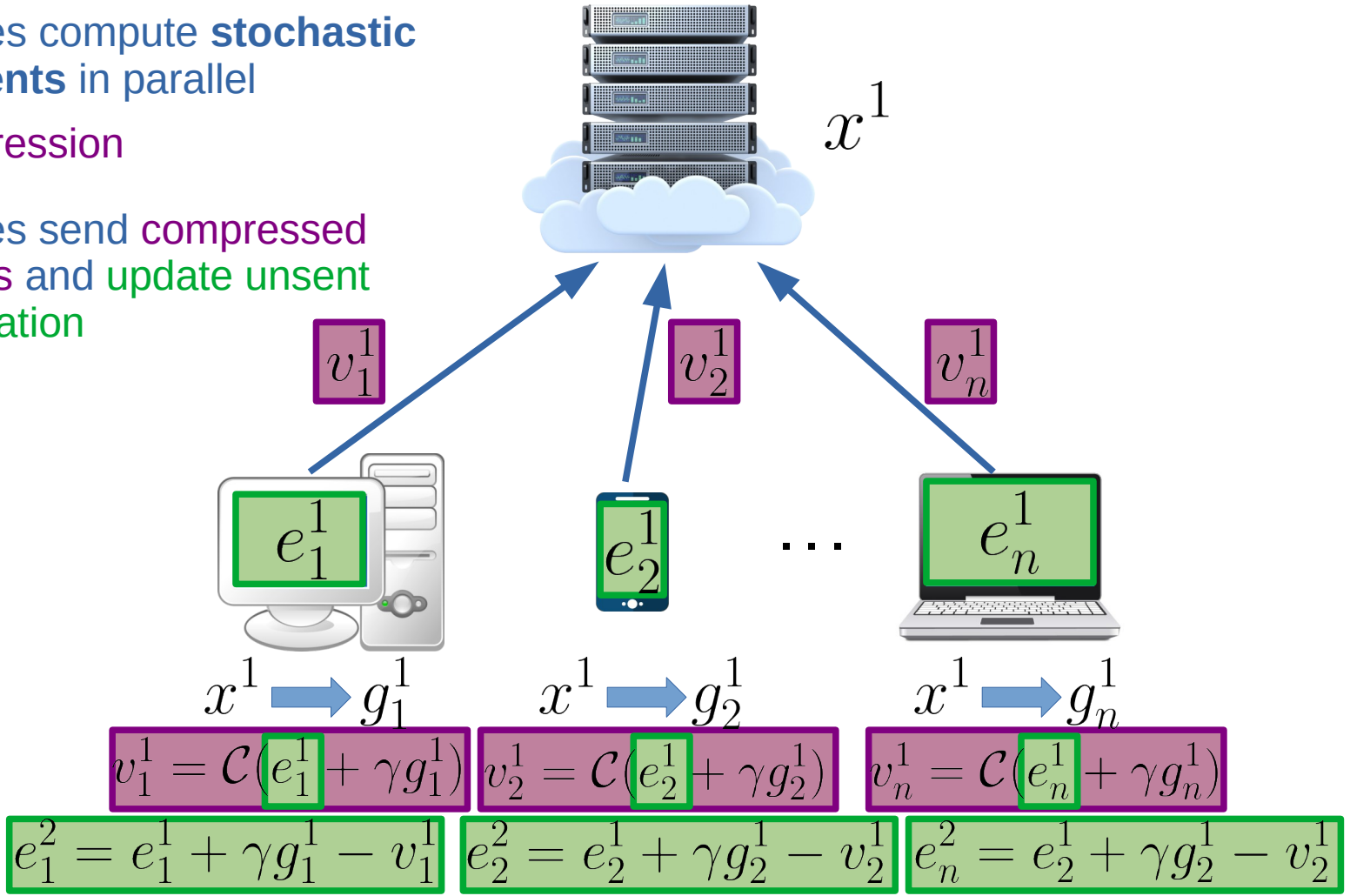
1. Server broadcasts new parameters
2. Devices compute **stochastic gradients** in parallel
3. Compression

x^1



Step 2

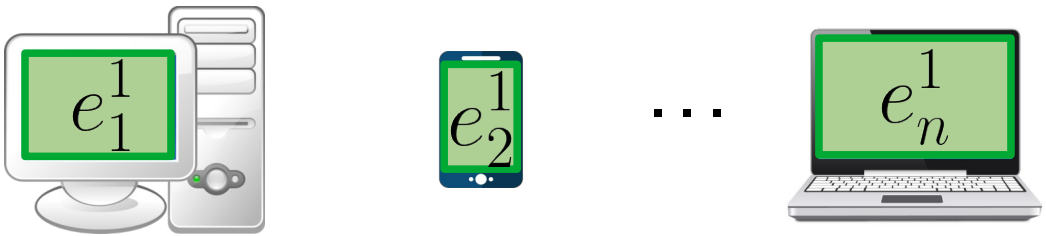
1. Server broadcasts new parameters
2. Devices compute **stochastic gradients** in parallel
3. Compression
4. Devices send compressed vectors and update unsent information



Step 2

1. Server broadcasts new parameters
2. Devices compute **stochastic gradients** in parallel
3. Compression
4. Devices send compressed vectors and update unsent information
5. Server gathers the information and updates the parameters

$$x^1 \rightarrow x^2 = x^1 - \frac{1}{n}\sum_{i=1}^{n} v_i^1$$



$$x^1 \rightarrow g_1^1 \qquad x^1 \rightarrow g_2^1 \qquad x^1 \rightarrow g_n^1$$

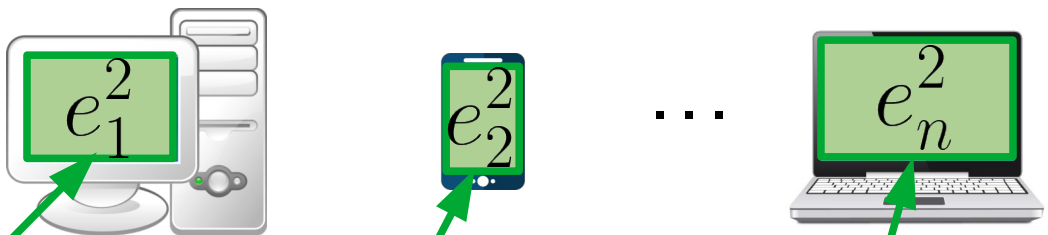
$$v_1^1 = \mathcal{C}(e_1^1 + \gamma g_1^1) \quad v_2^1 = \mathcal{C}(e_2^1 + \gamma g_2^1) \quad v_n^1 = \mathcal{C}(e_n^1 + \gamma g_n^1)$$

$$e_1^2 = e_1^1 + \gamma g_1^1 - v_1^1 \quad e_2^2 = e_2^1 + \gamma g_2^1 - v_2^1 \quad e_n^2 = e_2^1 + \gamma g_2^1 - v_2^1$$

Step 2

1. Server broadcasts new parameters
2. Devices compute **stochastic gradients** in parallel
3. Compression
4. Devices send compressed vectors and update unsent information
5. Server gathers the information and updates the parameters
6. Devices update their memory

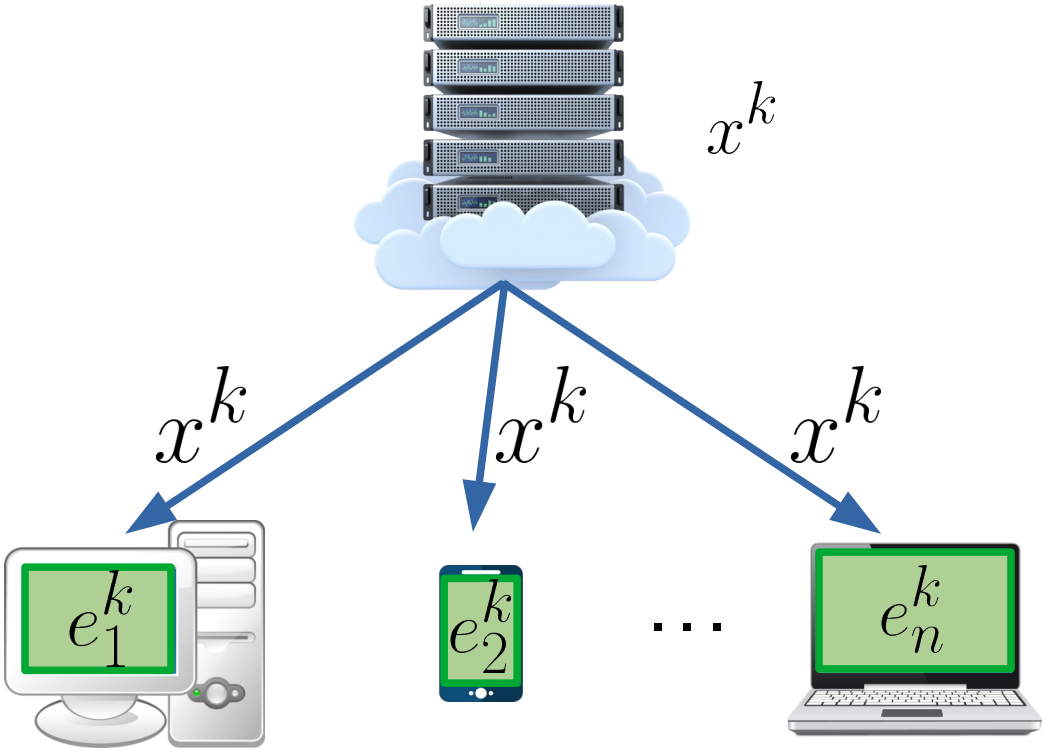
$$x^1 \rightarrow x^2 = x^1 - \frac{1}{n}\sum_{i=1}^{n} v_i^1$$



$$e_1^2 = e_1^1 + \gamma g_1^1 - v_1^1 \qquad e_2^2 = e_2^1 + \gamma g_2^1 - v_2^1 \qquad e_n^2 = e_2^1 + \gamma g_2^1 - v_2^1$$

Step k+1

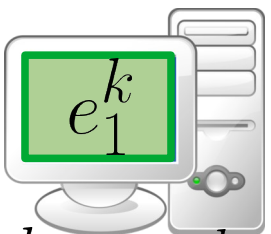
1. Server broadcasts new parameters



Step k+1

1. Server broadcasts new parameters
2. Workers compute **stochastic gradients** in parallel

x^k



$x^k \rightarrow g_1^k$

$x^k \rightarrow g_2^k$

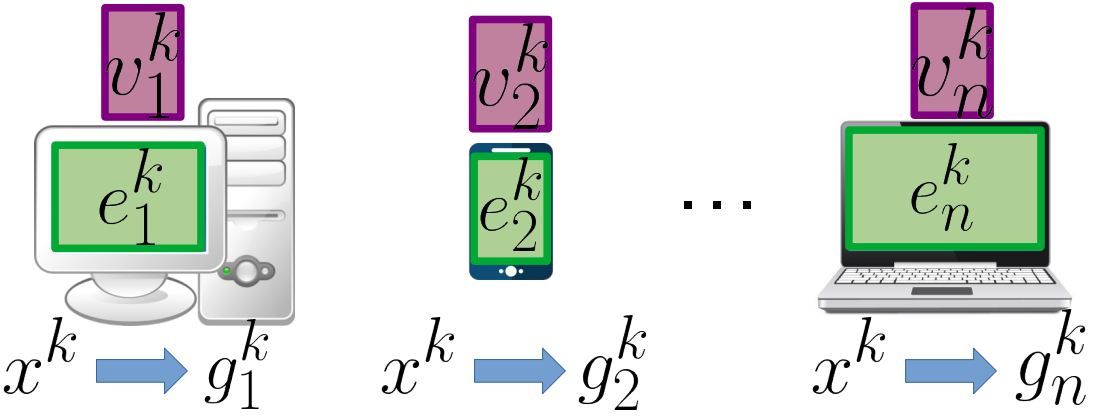
$\cdots$

$x^k \rightarrow g_n^k$

Step k+1

1. Server broadcasts new parameters
2. Workers compute **stochastic gradients** in parallel
3. Compression

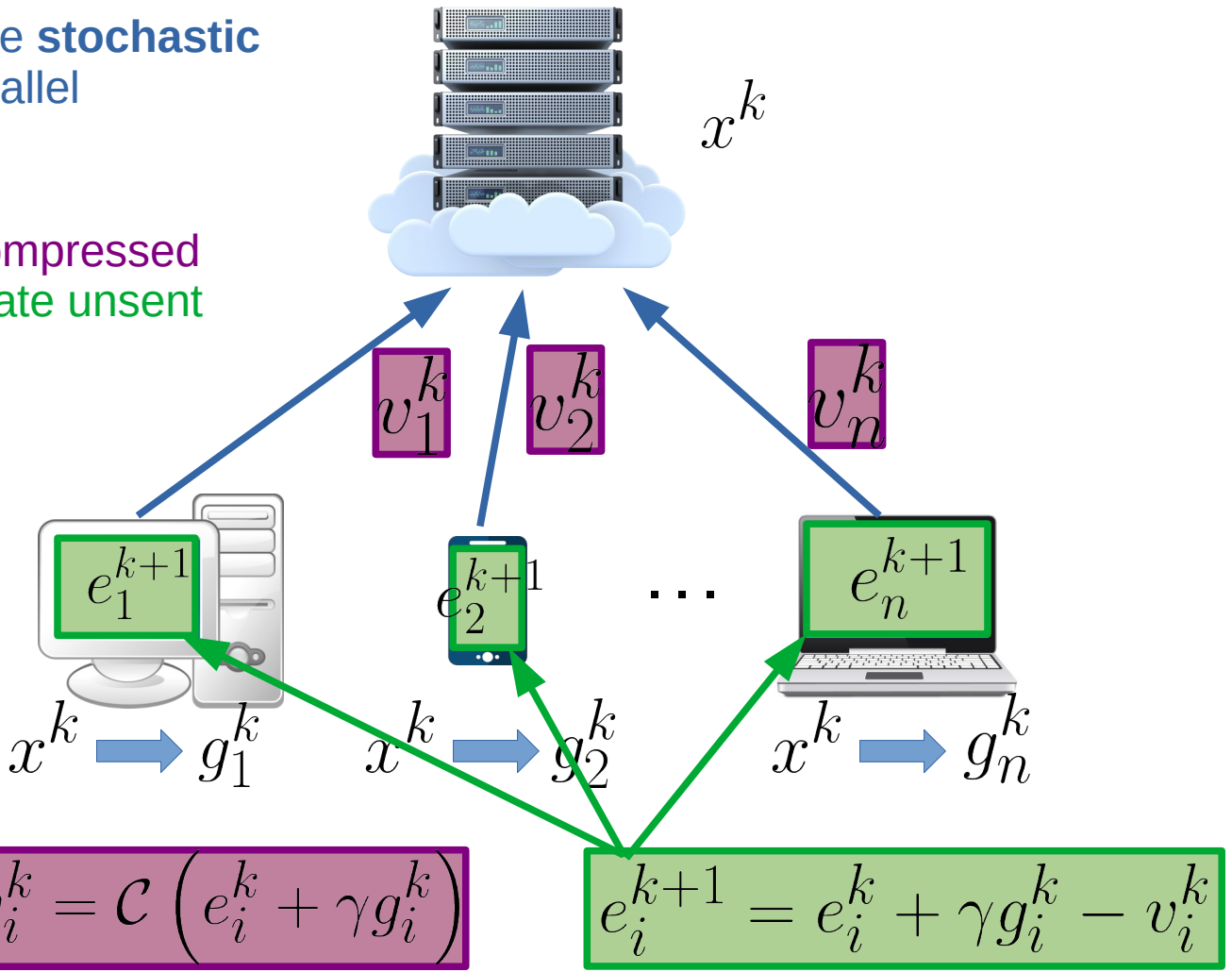
x^k



$$v_i^k = \mathcal{C}\left(e_i^k + \gamma g_i^k\right)$$

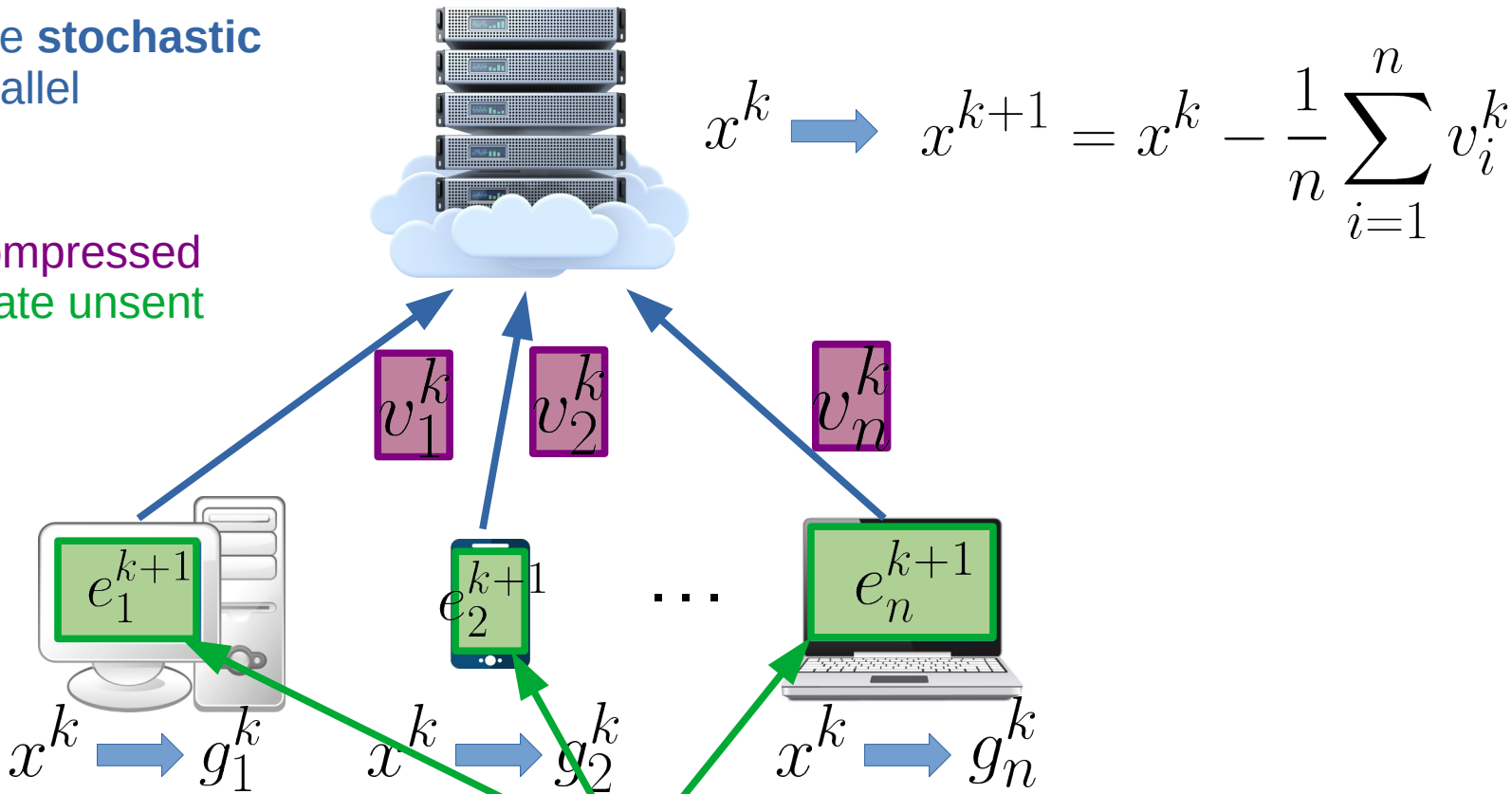
Step k+1

1. Server broadcasts new parameters
2. Workers compute **stochastic gradients** in parallel
3. Compression
4. Devices send compressed vectors and update unsent information



Step k+1

1. Server broadcasts new parameters
2. Workers compute **stochastic gradients** in parallel
3. Compression
4. Devices send compressed vectors and update unsent information
5. Server gathers the information and updates the parameters
6. Repeat steps 1 – 5



Error-Compensated SGD

Converges even with biased compression operators

EC-SGD finds such $\hat{x}$ that $\mathbb{E}[f(\hat{x})] - f(x^*) \leq \varepsilon$ after

$$\widetilde{\mathcal{O}}\left(\frac{L}{\delta\mu} + \frac{\sigma^2}{n\mu\varepsilon} + \frac{\sqrt{L(\sigma^2 + \zeta_*^2/\delta)}}{\mu\sqrt{\delta\varepsilon}}\right)$$

iterations

Error-Compensated SGD

Converges even with biased compression operators

EC-SGD finds such $\hat{x}$ that $\mathbb{E}[f(\hat{x})] - f(x^*) \leq \varepsilon$ after

Hides logarithmical factors

$$\widetilde{\mathcal{O}}\left(\frac{L}{\delta\mu} + \frac{\sigma^2}{n\mu\varepsilon} + \frac{\sqrt{L(\sigma^2 + \zeta_*^2/\delta)}}{\mu\sqrt{\delta\varepsilon}}\right)$$

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$$\widetilde{\mathcal{O}}\left(\frac{L}{\delta\mu} + \frac{\sigma^2}{n\mu\varepsilon} + \frac{\sqrt{L(\sigma^2 + \zeta_*^2/\delta)}}{\mu\sqrt{\delta\varepsilon}}\right)$$ iterations

$$\mathbb{E}\|\mathcal{C}(x) - x\|^2 \leq (1-\delta)\|x\|^2$$

Error-Compensated SGD

Converges even with biased compression operators

EC-SGD finds such $\hat{x}$ that $\mathbb{E}[f(\hat{x})] - f(x^*) \leq \varepsilon$ after

Hides logarithmical factors

$$\widetilde{\mathcal{O}}\left(\frac{L}{\delta\mu} + \frac{\sigma^2}{n\mu\varepsilon} + \frac{\sqrt{L(\sigma^2 + \zeta_*^2/\delta)}}{\mu\sqrt{\delta\varepsilon}}\right)$$ iterations

$$\mathbb{E}\|\mathcal{C}(x) - x\|^2 \leq (1-\delta)\|x\|^2$$

$$\mathbb{E}\left[\left\|g_i^k - \nabla f_i\left(x^k\right)\right\|^2 \mid x^k\right] \leq \sigma^2$$

Error-Compensated SGD

Converges even with biased compression operators

EC-SGD finds such $\hat{x}$ that $\mathbb{E}[f(\hat{x})] - f(x^*) \leq \varepsilon$ after

Hides logarithmical factors

$$\widetilde{\mathcal{O}}\left(\frac{L}{\delta\mu} + \frac{\sigma^2}{n\mu\varepsilon} + \frac{\sqrt{L(\sigma^2 + \zeta_*^2/\delta)}}{\mu\sqrt{\delta\varepsilon}}\right) \text{ iterations}$$

$$\mathbb{E}\|\mathcal{C}(x) - x\|^2 \leq (1-\delta)\|x\|^2$$

$$\mathbb{E}\left[\left\|g_i^k - \nabla f_i\left(x^k\right)\right\|^2 \mid x^k\right] \leq \sigma^2$$

$$\zeta_*^2 = \frac{1}{n}\sum_{i=1}^{n}\|\nabla f_i(x^*)\|^2$$

Error-Compensated SGD

Converges even with biased compression operators

Fails to converge with **linear rate** even when workers compute full gradients

EC-SGD finds such $\hat{x}$ that $\mathbb{E}[f(\hat{x})] - f(x^*) \leq \varepsilon$ after

Hides logarithmical factors

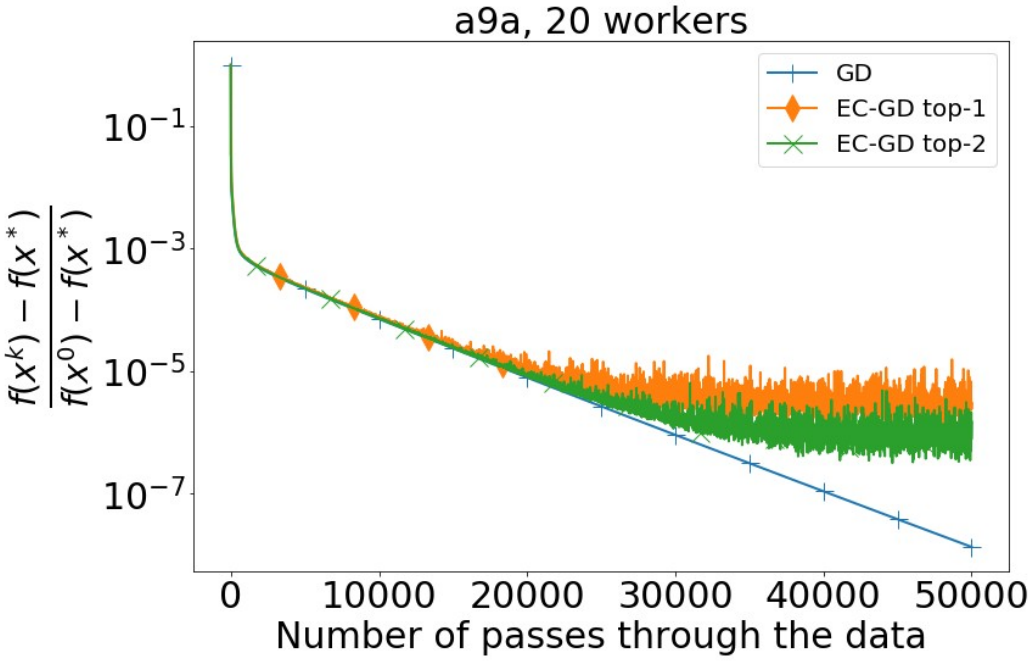
$$\widetilde{\mathcal{O}}\left(\frac{L}{\delta\mu} + \frac{\sigma^2}{n\mu\varepsilon} + \frac{\sqrt{L(\sigma^2 + \zeta_*^2/\delta)}}{\mu\sqrt{\delta\varepsilon}}\right) \text{ iterations}$$

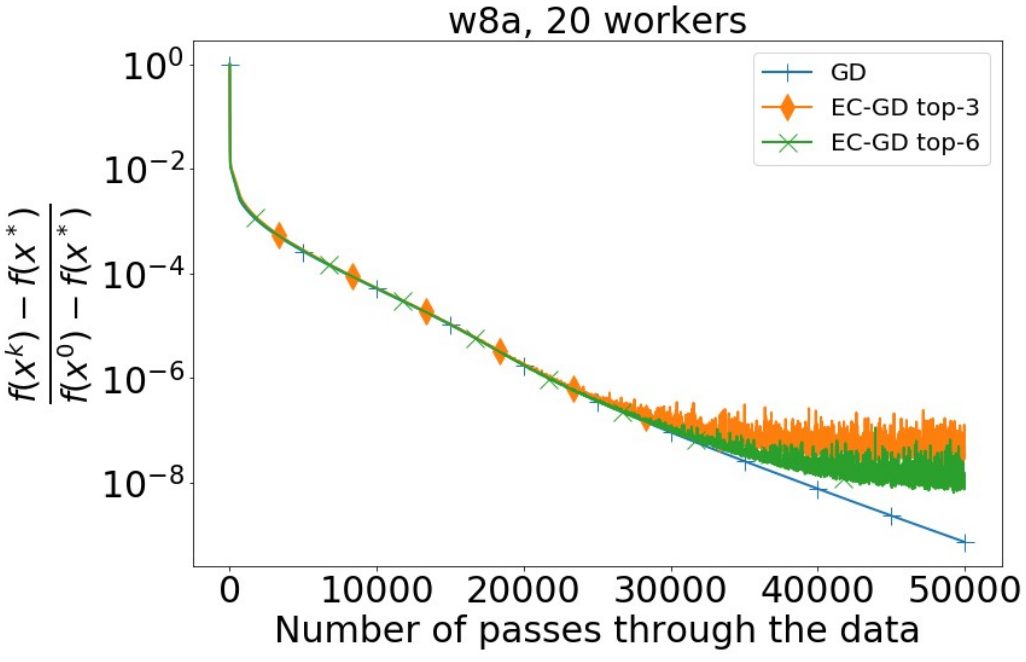
$$\mathbb{E}\|\mathcal{C}(x) - x\|^2 \leq (1-\delta)\|x\|^2 \qquad \mathbb{E}\left[\left\|g_i^k - \nabla f_i\left(x^k\right)\right\|^2 \mid x^k\right] \leq \sigma^2$$

$$\zeta_*^2 = \frac{1}{n}\sum_{i=1}^{n} \|\nabla f_i(x^*)\|^2$$

EC-GD and Logistic Regression

$$\min_{x\in\mathbb{R}^d}\left\{f(x)=\frac{1}{N}\sum_{i=1}^{N}\log\left(1+\exp\left(-y_i\cdot(Ax)_i\right)\right)+\frac{\mu}{2}\|x\|^2\right\}$$

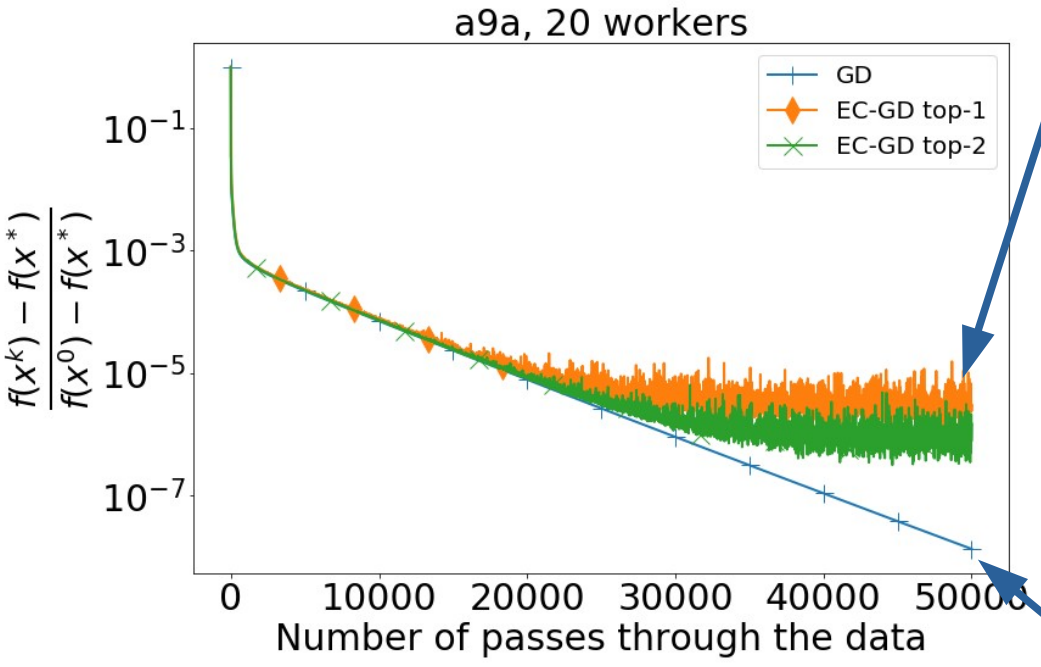


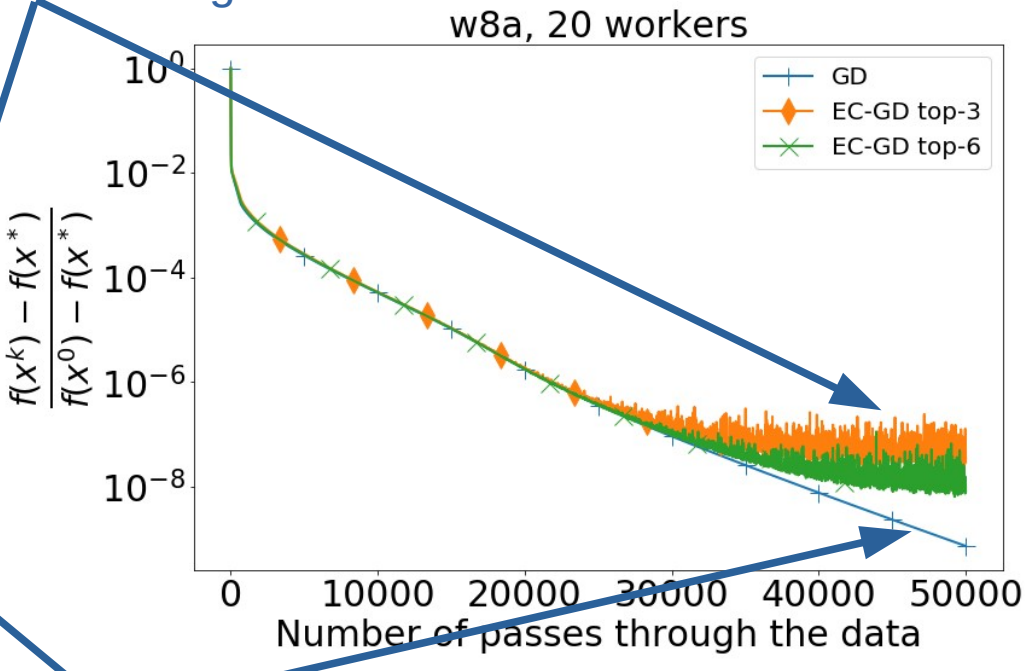


EC-GD and Logistic Regression

$$\min_{x\in\mathbb{R}^d}\left\{ f(x) = \frac{1}{N}\sum_{i=1}^{N}\log\left(1+\exp\left(-y_i\cdot(Ax)_i\right)\right) + \frac{\mu}{2}\|x\|^2 \right\}$$

sub-linear convergence





linear convergence

Error-Compensated SGD

✓ Converges even with biased compression operators

✗ Fails to converge with **linear rate** even when workers compute full gradients

Questions:

1. *Is it possible to design **linearly converging** SGD with error compensation when workers compute full gradients, i.e., linearly converging **EC-GD**?*

2. *Is it possible to design **linearly converging SGD** with error compensation when **the local loss functions have a finite-sum form**?*

Error-Compensated SGD

✓ Converges even with biased compression operators

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Questions:

1. *Is it possible to design **linearly converging** SGD with error compensation when workers compute full gradients, i.e., linearly converging **EC-GD**?*

2. *Is it possible to design **linearly converging SGD** with error compensation when **the local loss functions have a finite-sum form**?*

The answer is *Yes* for both questions

5.1. New method: EC-GDstar

Error-Compensated GD

EC-GD finds such $\hat{x}$ that $\mathbb{E}[f(\hat{x})] - f(x^*) \leq \varepsilon$ after

Hides logarithmical factors →

$$\widetilde{\mathcal{O}}\left(\frac{L}{\delta\mu} + \frac{\sqrt{L\zeta_*^2}}{\mu\delta\sqrt{\varepsilon}}\right)$$ iterations

$$\zeta_*^2 = \frac{1}{n}\sum_{i=1}^{n} \|\nabla f_i(x^*)\|^2$$

Error-Compensated GD

EC-GD finds such $\hat{x}$ that $\mathbb{E}[f(\hat{x})] - f(x^*) \leq \varepsilon$ after

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$$\widetilde{\mathcal{O}}\left(\frac{L}{\delta\mu} + \frac{\sqrt{L\zeta_*^2}}{\mu\delta\sqrt{\varepsilon}}\right)$$

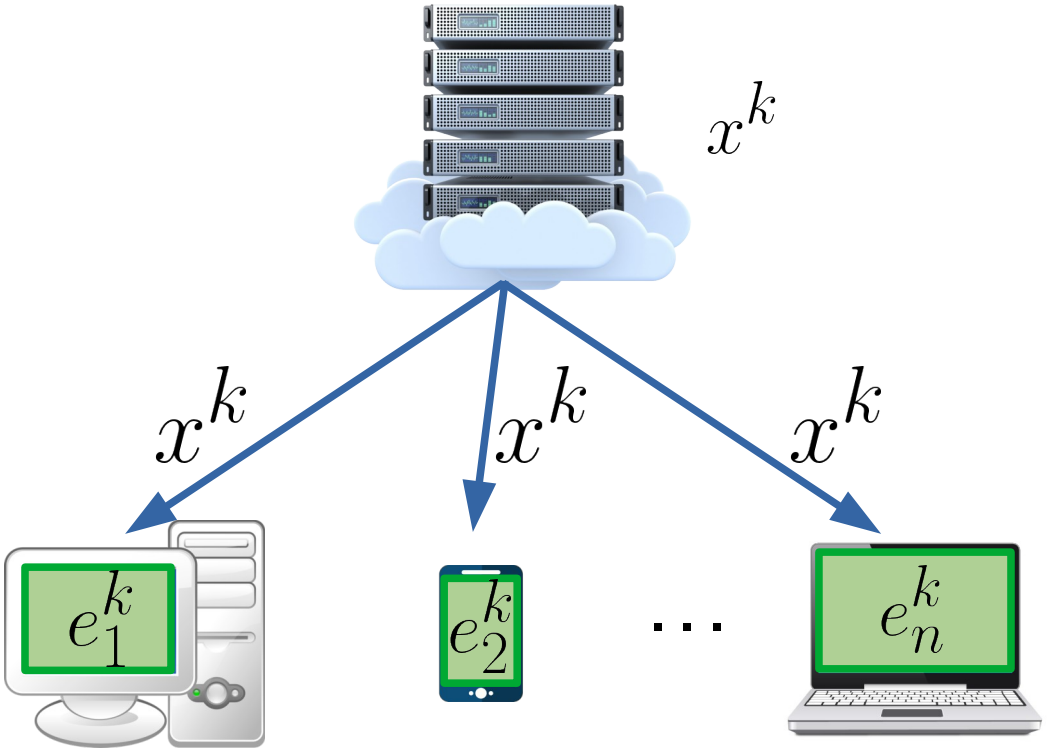
iterations

$$\zeta_*^2 = \frac{1}{n}\sum_{i=1}^{n} \|\nabla f_i(x^*)\|^2$$

What if devices know these vectors from the beginning?

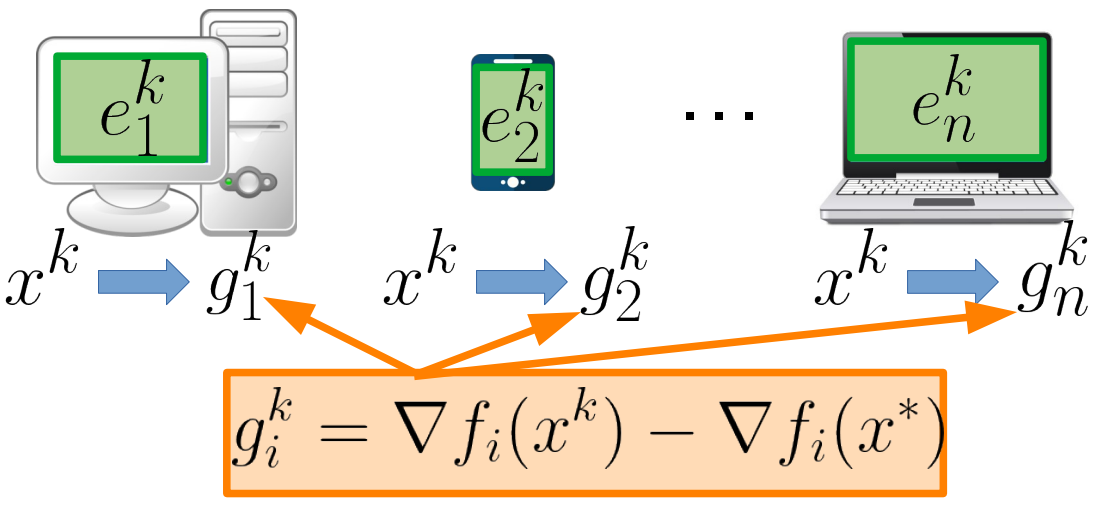
EC-GDstar

1. Server broadcasts new parameters



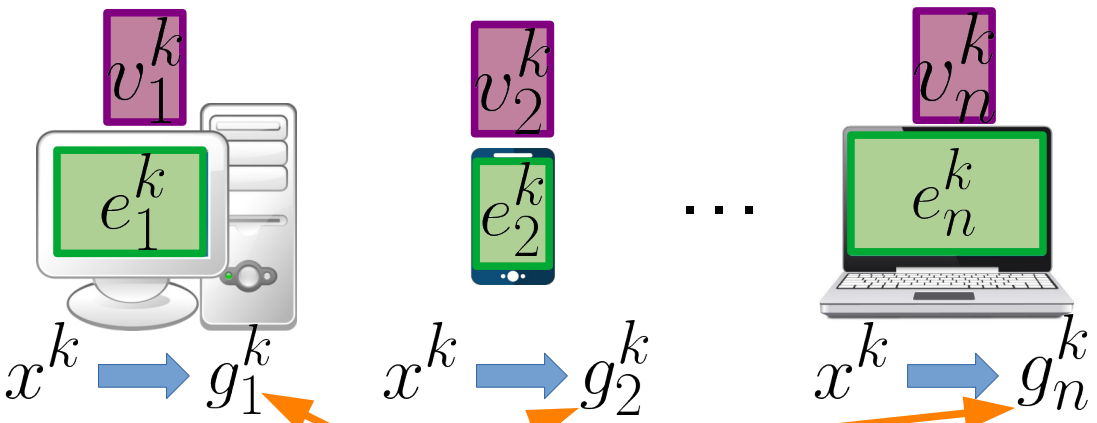
EC-GDstar

1. Server broadcasts new parameters
2. Workers compute **shifted gradients** in parallel



EC-GDstar

1. Server broadcasts new parameters
2. Workers compute **shifted gradients** in parallel
3. Compression

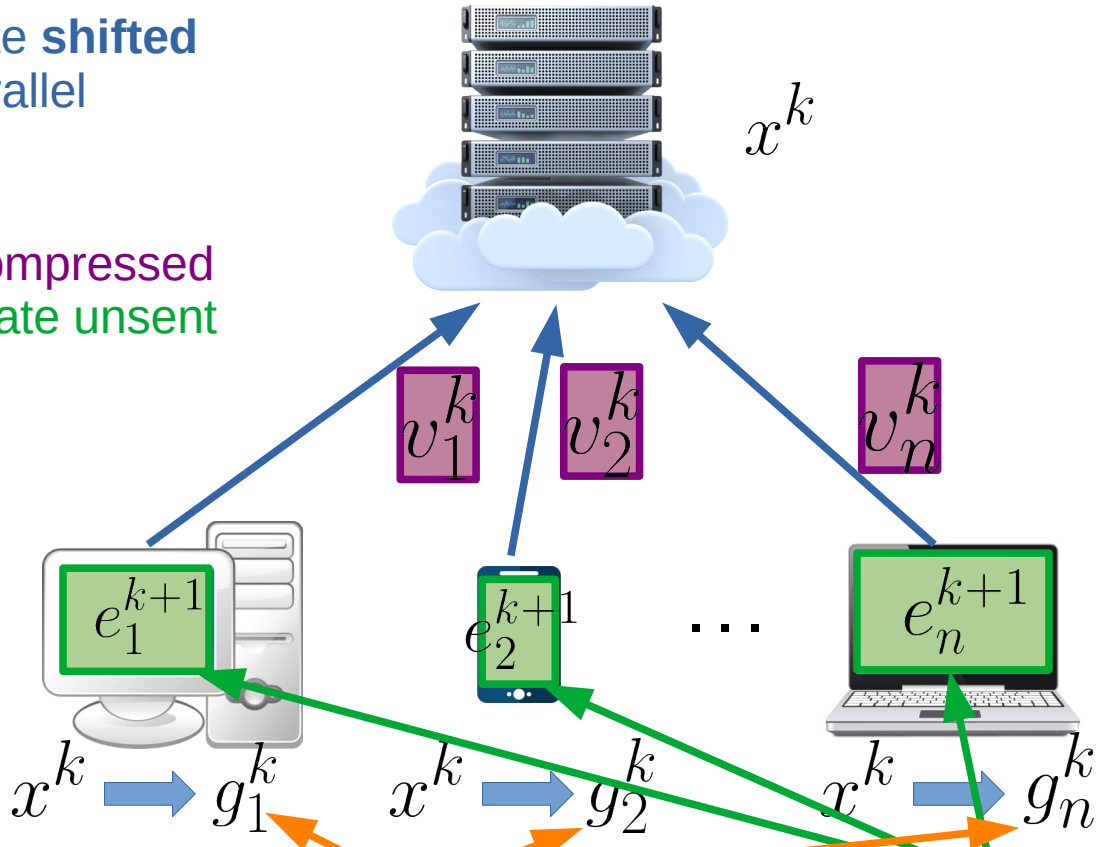


$$v_i^k = \mathcal{C}\left(e_i^k + \gamma g_i^k\right)$$

$$g_i^k = \nabla f_i(x^k) - \nabla f_i(x^*)$$

EC-GDstar

1. Server broadcasts new parameters
2. Workers compute **shifted gradients** in parallel
3. Compression
4. Devices send compressed vectors and update unsent information



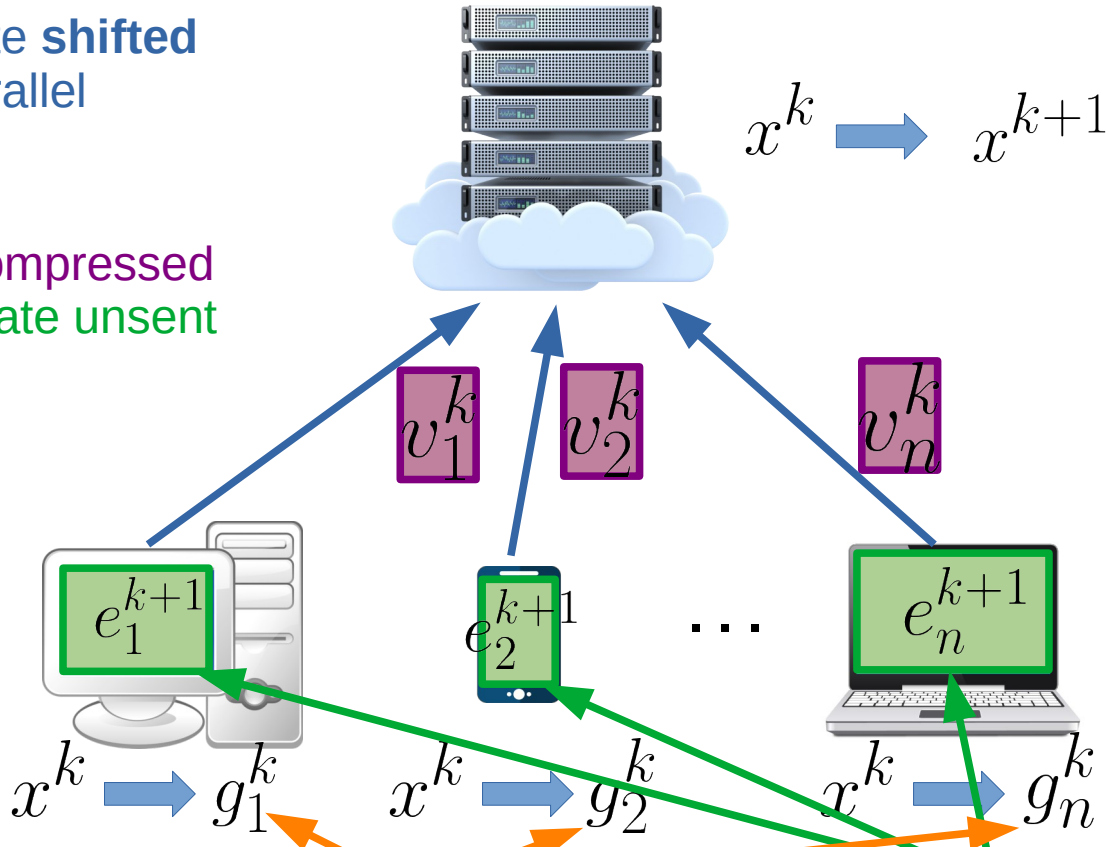
$$v_i^k = \mathcal{C}\left(e_i^k + \gamma g_i^k\right)$$

$$g_i^k = \nabla f_i(x^k) - \nabla f_i(x^*)$$

$$e_i^{k+1} = e_i^k + \gamma g_i^k - v_i^k$$

Step k+1

1. Server broadcasts new parameters
2. Workers compute **shifted gradients** in parallel
3. Compression
4. Devices send compressed vectors and update unsent information
5. Server gathers the information and updates the parameters
6. Repeat steps 1 – 5



$$v_i^k = \mathcal{C}\left(e_i^k + \gamma g_i^k\right)$$

$$g_i^k = \nabla f_i(x^k) - \nabla f_i(x^*)$$

$$e_i^{k+1} = e_i^k + \gamma g_i^k - v_i^k$$

EC-GDstar: Rate of Convergence

EC-GDstar finds such $\hat{x}$ that $\mathbb{E}[f(\hat{x})] - f(x^*) \leq \varepsilon$ after

$$\mathcal{O}\left(\frac{L}{\delta\mu}\ln\frac{1}{\varepsilon}\right)$$

iterations

EC-GDstar: Rate of Convergence

EC-GDstar finds such $\hat{x}$ that $\mathbb{E}[f(\hat{x})] - f(x^*) \leq \varepsilon$ after

$$\mathcal{O}\left(\frac{L}{\delta\mu}\ln\frac{1}{\varepsilon}\right)$$ iterations

- ✓ Linear rate
- ✗ The method is impractical: it uses the gradients at the solution

EC-GDstar: Rate of Convergence

EC-GDstar finds such $\hat{x}$ that $\mathbb{E}[f(\hat{x})] - f(x^*) \leq \varepsilon$ after

$$\mathcal{O}\left(\frac{L}{\delta\mu}\ln\frac{1}{\varepsilon}\right)$$ iterations

Linear rate

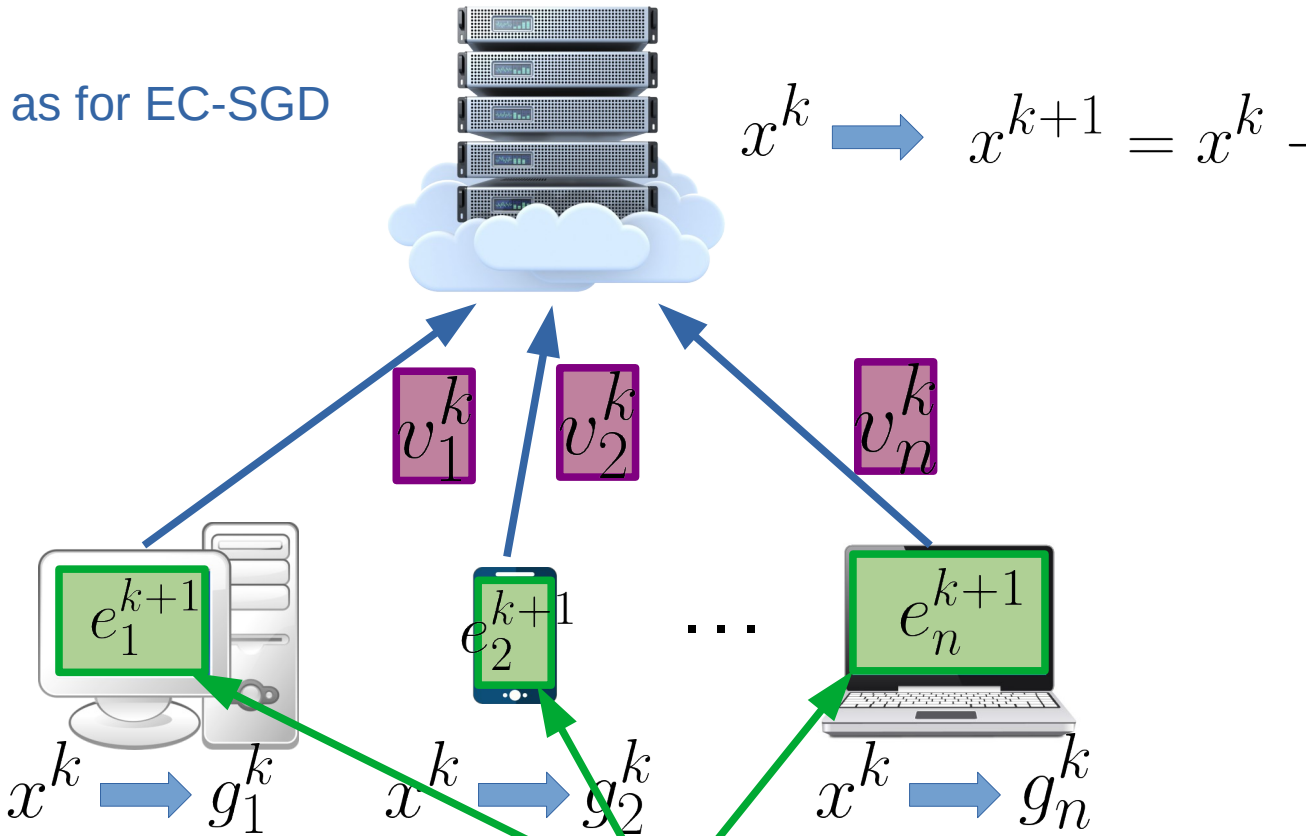
The method is impractical: it uses the gradients at the solution

Can we develop a practical analog?

5.2. New method: EC-SGD-DIANA

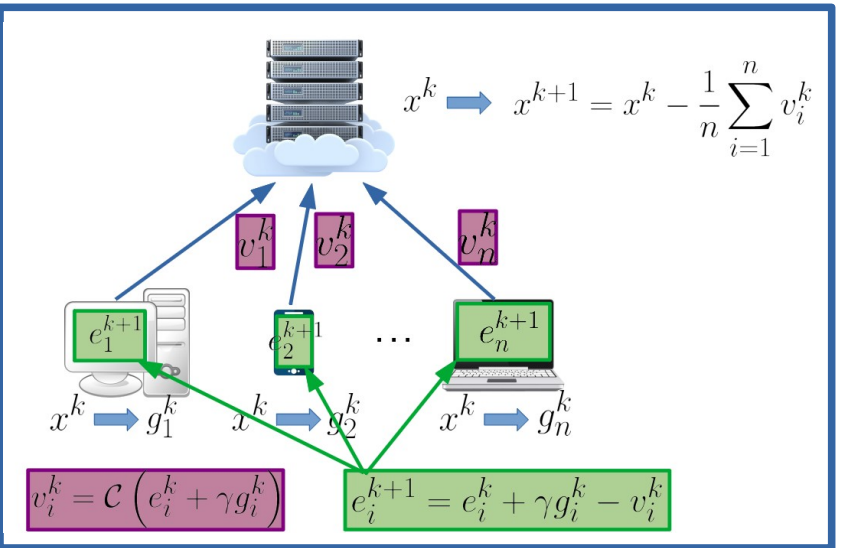
EC-SGD-DIANA

The same scheme as for EC-SGD



EC-SGD-DIANA

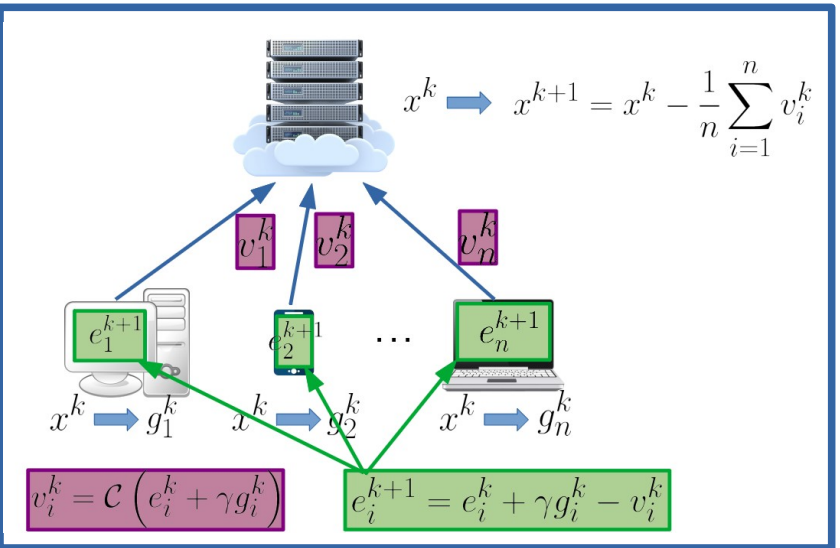
$$g_i^k = \hat{g}_i^k - h_i^k + h^k$$



EC-SGD-DIANA

$$g_i^k = \hat{g}_i^k - h_i^k + h^k$$

$$\mathbb{E}\left[\hat{g}_i^k \mid x^k\right] = \nabla f_i(x^k)$$



EC-SGD-DIANA

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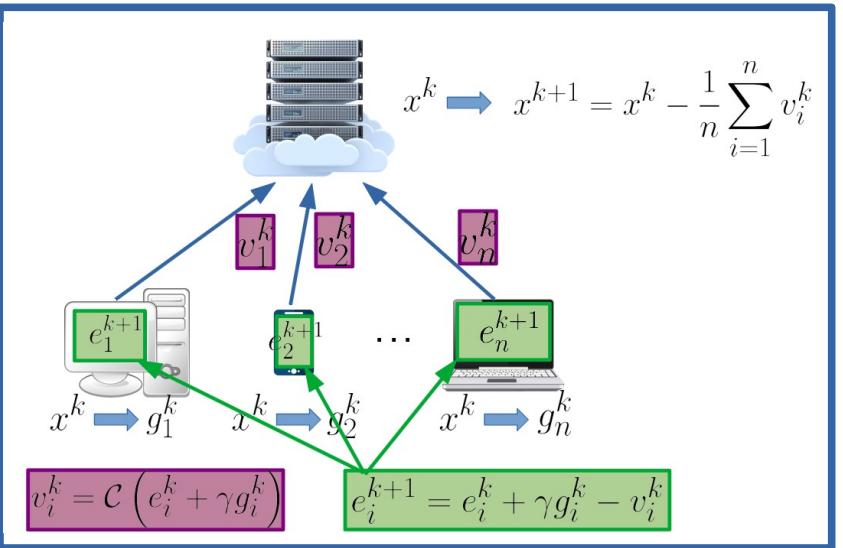


Works for both cases:

- $f_i(x) = \mathbf{E}_{\xi_i \sim \mathcal{D}_i}\left[f_{\xi_i}(x)\right]$ $\quad \hat{g}_i^k = \nabla f_{\xi_i}(x^k)$
- $f_i(x) = \frac{1}{m}\sum_{j=1}^m f_{ij}(x)$ $\quad \hat{g}_i^k = \nabla f_{il}(x^k)$

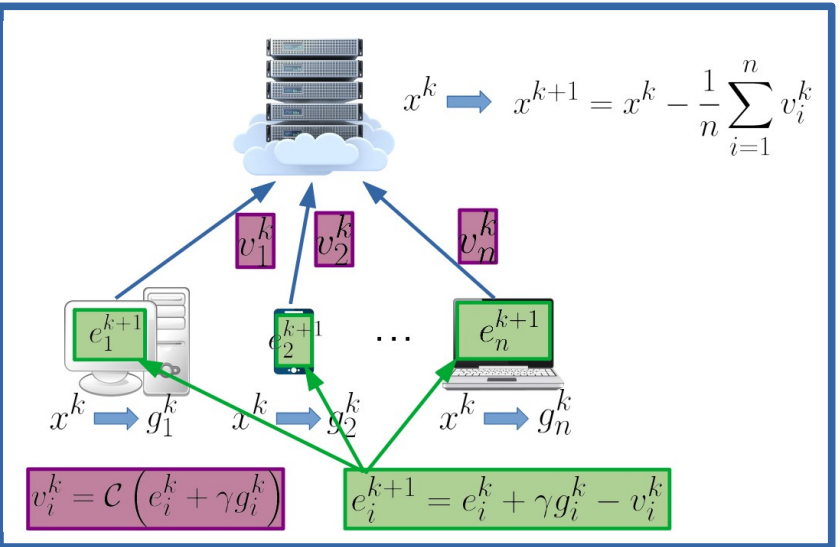
$l \sim [m]$ uniformly at random

EC-SGD-DIANA

$$g_i^k = \hat{g}_i^k - h_i^k + h^k$$

stepsize

$$\mathbb{E}\left[\hat{g}_i^k \mid x^k\right] = \nabla f_i(x^k) \qquad h_i^{k+1} = h_i^k + \alpha \mathcal{Q}\left(\hat{g}_i^k - h_i^k\right)$$



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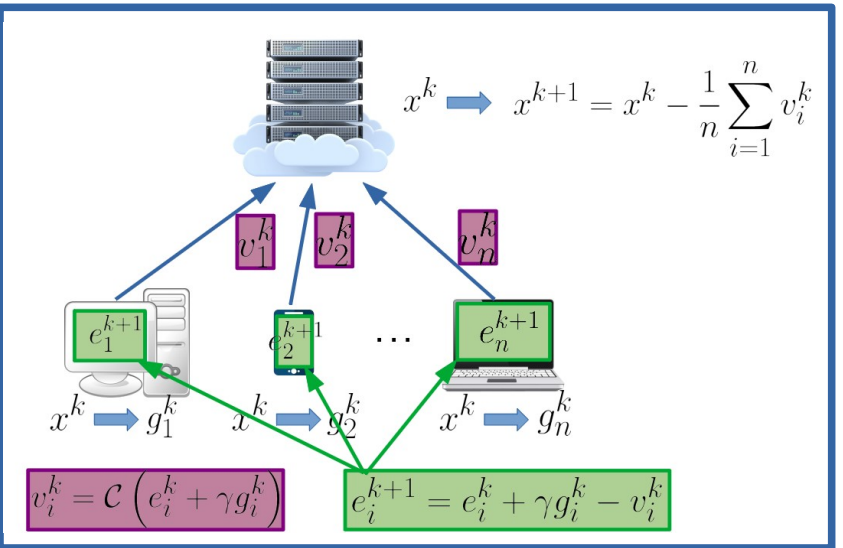
Workers send these vectors to the server



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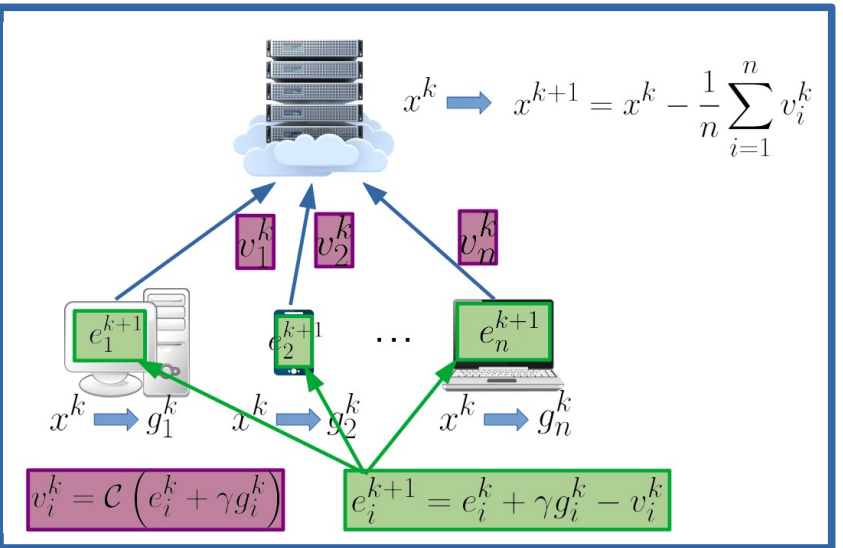

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EC-SGD-DIANA

The key insight why do we need $\{h_i^k\}_{i=1}^n$:

$$g_i^k = \hat{g}_i^k - h_i^k + h^k \qquad h^k = \frac{1}{n}\sum_{i=1}^n h_i^k$$

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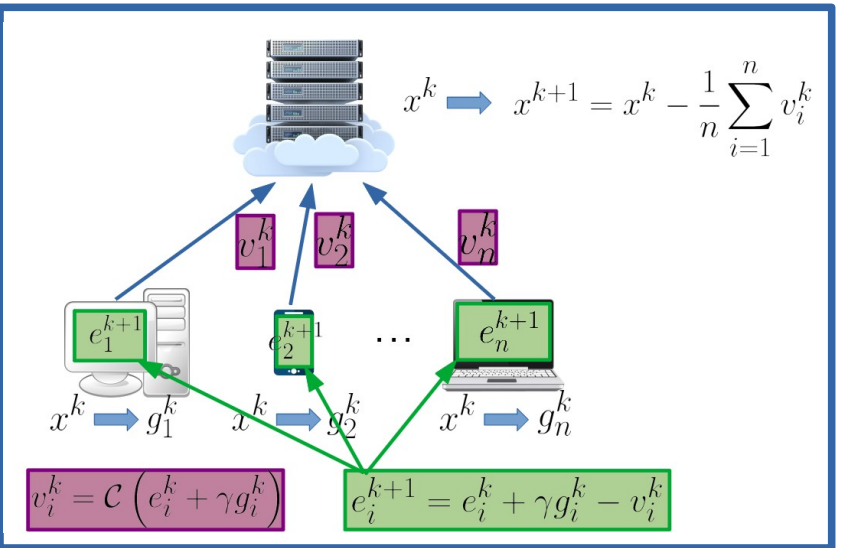

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$l \sim [m]$ uniformly at random

EC-SGD-DIANA

The key insight why do we need $\{h_i^k\}_{i=1}^n$:

it reduces the variance coming from compressions via learning the gradients at the solution!

$$g_i^k = \hat{g}_i^k - h_i^k + h^k \qquad h^k = \frac{1}{n}\sum_{i=1}^n h_i^k$$

Server broadcasts this vector to the workers

$$\mathbb{E}\left[\hat{g}_i^k \mid x^k\right] = \nabla f_i(x^k)$$

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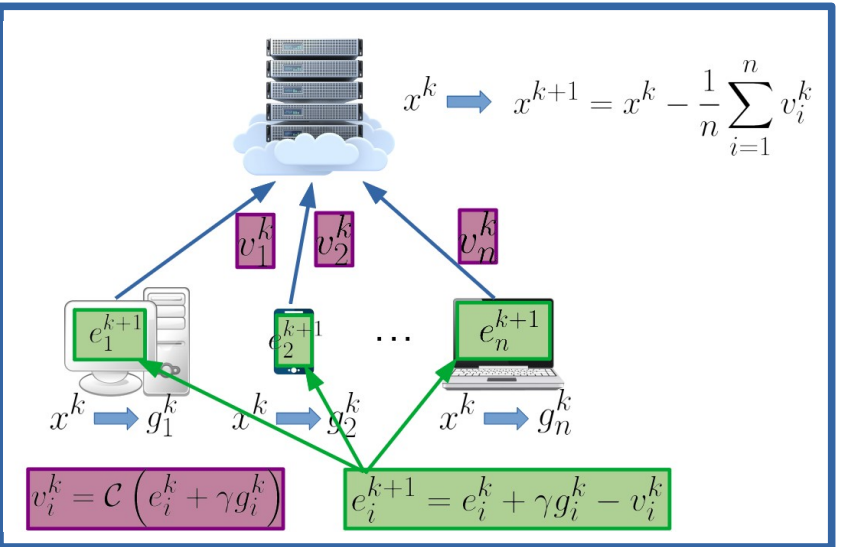

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EC-SGD-DIANA: Rate of Convergence

EC-SGD-DIANA finds such $\hat{x}$ that $\mathbb{E}[f(\hat{x})] - f(x^*) \leq \varepsilon$ after

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Option I: $$\widetilde{\mathcal{O}}\left(\omega + \frac{L}{\delta\mu} + \frac{\sigma^2}{n\mu\varepsilon} + \frac{\sqrt{L\sigma^2}}{\delta\mu\sqrt{\varepsilon}}\right)$$ iterations

Option II: $$\widetilde{\mathcal{O}}\left(\frac{1+\omega}{\delta} + \frac{L}{\delta\mu} + \frac{\sigma^2}{n\mu\varepsilon} + \frac{\sqrt{L\sigma^2}}{\mu\sqrt{\delta\varepsilon}}\right)$$

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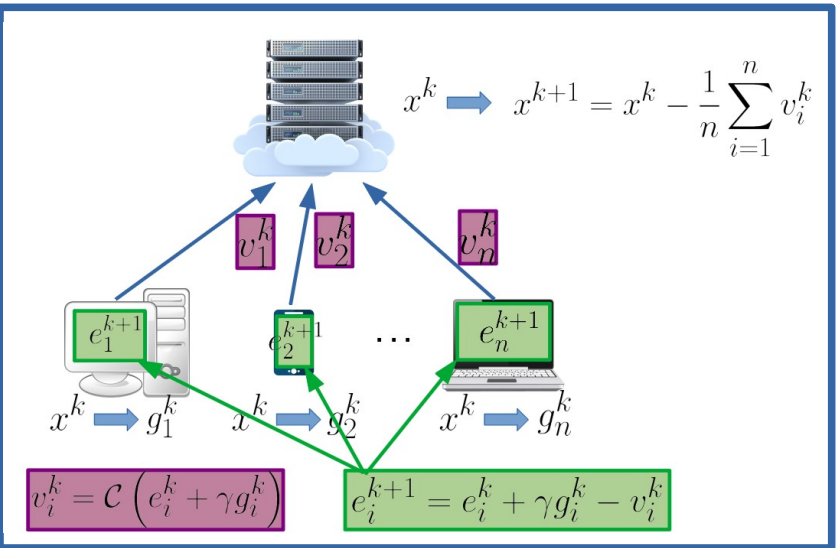
Moreover, if workers compute **full gradients**, then the rate of convergence is linear

$$\mathcal{O}\left(\left(\omega + \frac{L}{\delta\mu}\right)\log\frac{1}{\varepsilon}\right)$$

5.3. New method: EC-LSVRG-DIANA

EC-LSVRG-DIANA

$$g_i^k = \hat{g}_i^k - h_i^k + h^k$$



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EC-LSVRG-DIANA

$$g_i^k = \hat{g}_i^k - h_i^k + h^k$$

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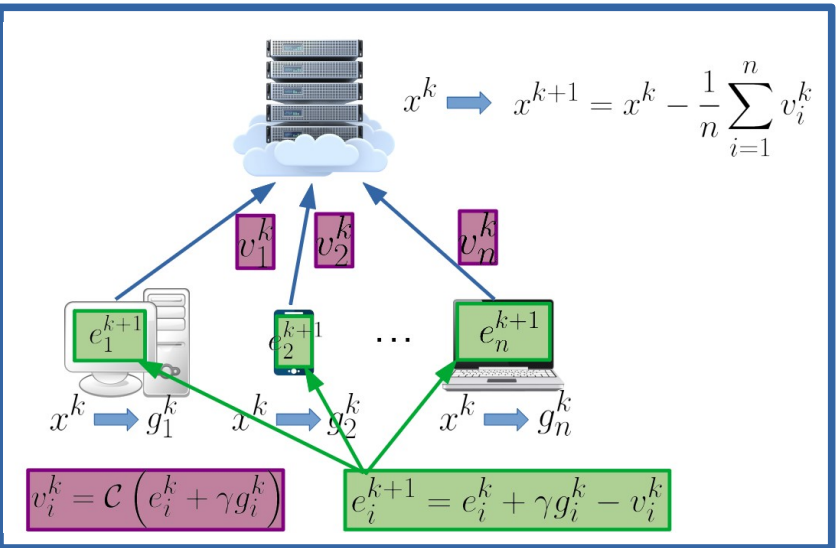

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EC-LSVRG-DIANA

$$g_i^k = \hat{g}_i^k - h_i^k + h^k \qquad l \sim [m] \quad \text{uniformly at random}$$

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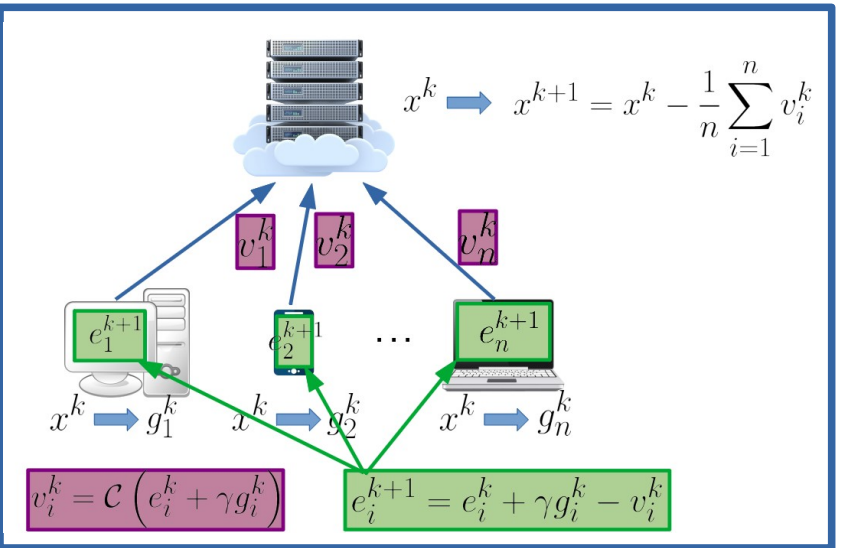

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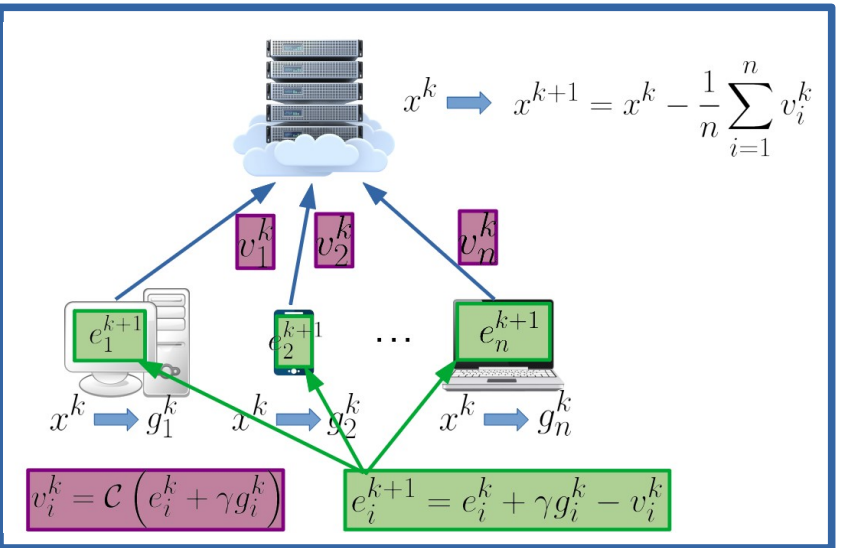
$$w_i^{k+1} = \begin{cases} x^k, & \text{with probability } p \\ w_i^k, & \text{with probability } 1-p \end{cases}$$



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$l \sim [m]$ uniformly at random

$$\hat{g}_i^k = \nabla f_{il}\left(x^k\right) - \nabla f_{il}\left(w_i^k\right) + \nabla f_i\left(w_i^k\right)$$

Reduction of the variance introduced due to the stochasticity of the gradients

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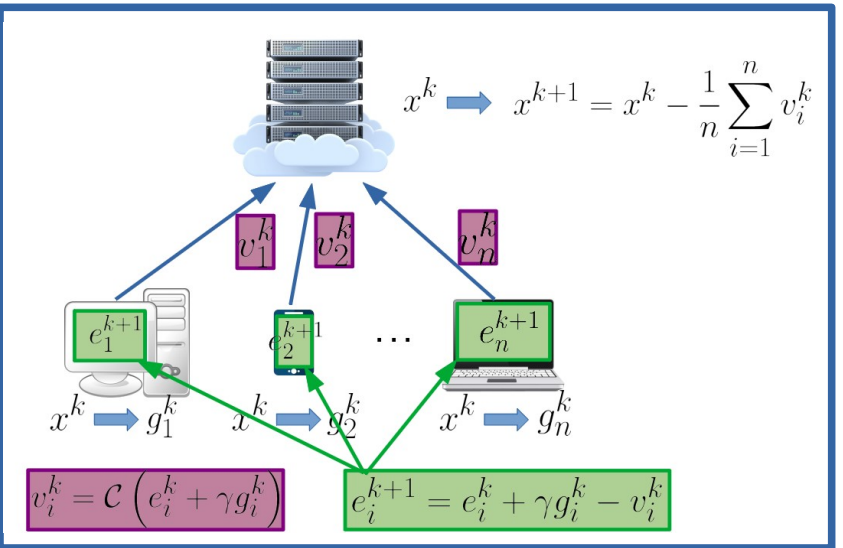

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$$\mathcal{O}\left(\left(\omega + m + \frac{L}{\delta\mu}\right)\log\frac{1}{\epsilon}\right)$$ iterations

6.1. Unified Theory of SGD

Gorbunov, Eduard, Filip Hanzely, and Peter Richtárik. **"A Unified Theory of SGD: Variance reduction, Sampling, Quantization and Coordinate Descent."** In International Conference on Artificial Intelligence and Statistics, pp. 680-690. 2020.

Unified Theory of SGD

The Problem:

$$\min_{x \in \mathbb{R}^d} f(x)$$

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The Method: $$x^{k+1} = x^k - \gamma g^k$$

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The Method: $x^{k+1} = x^k - \gamma g^k$

The Assumption: $\mathbb{E}\left[g^k \mid x^k \right] = \nabla f\left(x^k \right)$

$$\mathbb{E}\left[\left\| g^k \right\|^2 \mid x^k \right] \leq 2A \left(f\left(x^k \right) - f\left(x^* \right) \right) + B\sigma_k^2 + D_1$$

Reflects smoothness properties of the problem and noises introduced by stochastic gradients

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$$\mathbb{E}\left[\sigma_{k+1}^2 \mid \sigma_k^2 \right] \leq (1-\rho)\sigma_k^2 + 2C\left(f\left(x^k \right) - f\left(x^* \right) \right) + D_2$$

Reflects smoothness properties of the problem and noises introduced by stochastic gradients

Describes the process of variance reduction

Method	A	B	ρ	C	D_1	D_2
SGD	$2L$	0	1	0	$2\sigma^2$	0
SGD-SR	$2\mathcal{L}$	0	1	0	$2\sigma^2$	0
SGD-MB	$\frac{A'+L(\tau-1)}{\tau}$	0	1	0	$\frac{D'}{\tau}$	0
SGD-star	$2\mathcal{L}$	0	1	0	0	0
SAGA	$2L$	2	$1/n$	L/n	0	0
N-SAGA	$2L$	2	$1/n$	L/n	$2\sigma^2$	$\frac{\sigma^2}{n}$
SEGA	$2dL$	$2d$	$1/d$	L/d	0	0
N-SEGA	$2dL$	$2d$	$1/d$	L/d	$2d\sigma^2$	$\frac{\sigma^2}{d}$
SVRG[a]	$2L$	2	0	0	0	0
L-SVRG	$2L$	2	p	Lp	0	0
DIANA	$\left(1+\frac{2\omega}{n}\right)L$	$\frac{2\omega}{n}$	α	$L\alpha$	$\frac{(1+\omega)\sigma^2}{n}$	$\alpha\sigma^2$
DIANA[b]	$(1+2\omega)L$	2ω	α	$L\alpha$	0	0
Q-SGD-SR	$2(1+\omega)\mathcal{L}$	0	1	0	$2(1+\omega)\sigma^2$	0
VR-DIANA	$\left(1+\frac{4\omega+2}{n}\right)L$	$\frac{2(\omega+1)}{n}$	α	$\left(\frac{1}{m}+4\alpha\right)L$	0	0
JacSketch	$2\mathcal{L}_1$	$\frac{2\lambda_{\max}}{n}$	$\lambda_{\min}$	$\frac{\mathcal{L}_2}{n}$	0	0

Main Theorem

If the stepsize satisfies

$$0 < \gamma \leq \min\left\{\frac{1}{\mu}, \frac{1}{A + CM}\right\}, \quad \text{where} \quad M > \frac{B}{\rho}$$

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then the iterates of SGD satisfy

$$\mathbb{E}\left[V^k\right] \leq \max\left\{(1-\gamma\mu)^k, \left(1 + \frac{B}{M} - \rho\right)^k\right\} V^0 + \frac{\left(D_1 + MD_2\right)\gamma^2}{\min\left\{\gamma\mu, \rho - \frac{B}{M}\right\}}$$

where $V^k \overset{\text{def}}{=} \left\|x^k - x^*\right\|^2 + M\gamma^2\sigma_k^2$

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where $V^k \overset{\text{def}}{=} \left\|x^k - x^*\right\|^2 + M\gamma^2\sigma_k^2$

$$\left\|x^k - x^*\right\|^2 \leq V^k$$

Table 1: List of specific existing (in some cases generalized) and new methods which fit our general analysis framework. VR = variance reduced method, AS = arbitrary sampling, Quant = supports gradient quantization, RCD = randomized coordinate descent type method. [a] Special case of SVRG with 1 outer loop only; [b] Special case of DIANA with 1 node and quantization of exact gradient.

Problem	Method	Alg #	Citation	VR?	AS?	Quant?	RCD?	Section	Result
(1)+(2)	SGD	Alg 1	Nguyen et al. (2018)	✗	✗	✗	✗	A.1	Cor A.1
(1)+(3)	SGD-SR	Alg 2	Gower et al. (2019)	✗	✓	✗	✗	A.2	Cor A.2
(1)+(3)	SGD-MB	Alg 3	**NEW**	✗	✗	✗	✗	A.3	Cor A.3
(1)+(3)	SGD-star	Alg 4	**NEW**	✓	✓	✗	✗	A.4	Cor A.4
(1)+(3)	SAGA	Alg 5	Defazio et al. (2014)	✓	✗	✗	✗	A.5	Cor A.5
(1)+(3)	N-SAGA	Alg 6	**NEW**	✗	✗	✗	✗	A.6	Cor A.6
(1)	SEGA	Alg 7	Hanzely et al. (2018)	✓	✗	✗	✓	A.7	Cor A.7
(1)	N-SEGA	Alg 8	**NEW**	✗	✗	✗	✓	A.8	Cor A.8
(1)+(3)	SVRG[a]	Alg 9	Johnson and Zhang (2013)	✓	✗	✗	✗	A.9	Cor A.9
(1)+(3)	L-SVRG	Alg 10	Hofmann et al. (2015)	✓	✗	✗	✗	A.10	Cor A.10
(1)+(3)	DIANA	Alg 11	Mishchenko et al. (2019a)	✗	✗	✓	✗	A.11	Cor A.11
(1)+(3)	DIANA[b]	Alg 12	Mishchenko et al. (2019a)	✓	✗	✓	✗	A.11	Cor A.12
(1)+(3)	Q-SGD-SR	Alg 13	**NEW**	✗	✓	✓	✗	A.12	Cor A.13
(1)+(3)+(4)	VR-DIANA	Alg 14	Horváth et al. (2019)	✓	✗	✓	✗	A.13	Cor A.15
(1)+(3)	JacSketch	Alg 15	Gower et al. (2018)	✓	✓✗	✗	✗	A.14	Cor A.16

6.2. Unified Convergence Analysis of Methods with Error Compensation

Key Assumption

$$g^k = \frac{1}{n}\sum_{i=1}^{n} g_i^k, \quad \mathbb{E}\left[g^k \mid x^k\right] = \nabla f\left(x^k\right) \quad \bar{g}_i^k = \mathbb{E}\left[g_i^k \mid x^k\right]$$

$$\frac{1}{n}\sum_{i=1}^{n} \left\|\bar{g}_i^k\right\|^2 \leq 2A\left(f\left(x^k\right) - f\left(x^*\right)\right) + B_1\sigma_{1,k}^2 + B_2\sigma_{2,k}^2 + D_1$$

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Reflects smoothness properties of the problem and noises introduced by compressions and stochastic gradients

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Describes the process of variance reduction of the variance coming from compressions

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Main Theorem

$$\mathbb{E}\left[f\left(\bar{x}^K\right) - f\left(x^*\right)\right] \leq (1-\eta)^K \frac{\Psi\left(x^0, \gamma\right)}{\gamma} + \gamma\Phi\left(D_1, \widetilde{D}_1, D_1', D_2\right)$$

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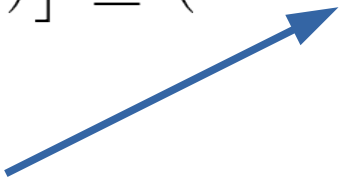

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Main Theorem

Some quantity depending only on the starting point and stepsize

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Linear function

Methods with Error Compensation Covered by Our Framework

Problem	Method	Alg #	Citation	Sec #	Rate (constants ignored)
(1)+(3)	EC-SGDsr	Alg 3	new	H.1	$\widetilde{\mathcal{O}}\left(\frac{\mathcal{L}}{\mu}+\frac{\sqrt{L\mathcal{L}}}{\delta\mu}+\frac{\mathrm{Var}}{n\mu\varepsilon}+\frac{\sqrt{L(\mathrm{Var}+\zeta_*^2/\delta)}}{\mu\sqrt{\delta\varepsilon}}\right)$
(1)+(2)	EC-SGD	Alg 4	[45]	H.2	$\widetilde{\mathcal{O}}\left(\frac{\kappa}{\delta}+\frac{\mathrm{Var}}{n\mu\varepsilon}+\frac{\sqrt{L(\mathrm{Var}+\zeta_*^2/\delta)}}{\delta\mu\sqrt{\varepsilon}}\right)$
(1)+(3)	EC-GDstar	Alg 5	new	H.3	$\mathcal{O}\left(\frac{\kappa}{\delta}\log\frac{1}{\varepsilon}\right)$
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(1)+(2)	EC-GD-DIANA†	Alg 6	new	H.4	$\mathcal{O}\left(\left(\omega+\frac{\kappa}{\delta}\right)\log\frac{1}{\varepsilon}\right)$
(1)+(3)	EC-LSVRG	Alg 8	new	H.6	$\widetilde{\mathcal{O}}\left(m+\frac{\kappa}{\delta}+\frac{\sqrt{L\zeta_*^2}}{\delta\mu\sqrt{\varepsilon}}\right)$
(1)+(3)	EC-LSVRGstar	Alg 9	new	H.7	$\mathcal{O}\left(\left(m+\frac{\kappa}{\delta}\right)\log\frac{1}{\varepsilon}\right)$
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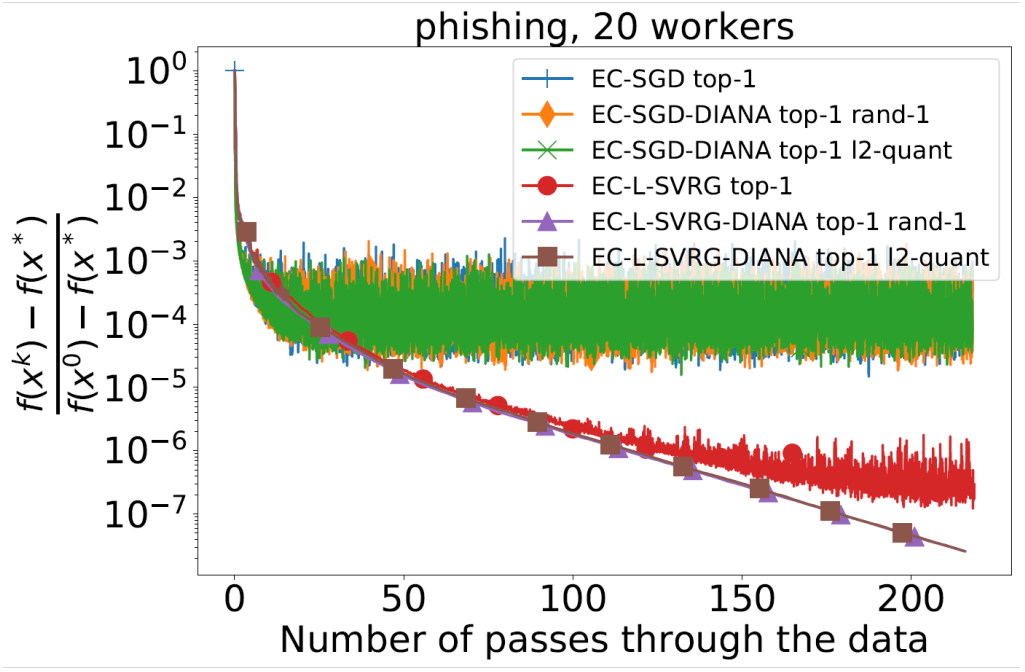
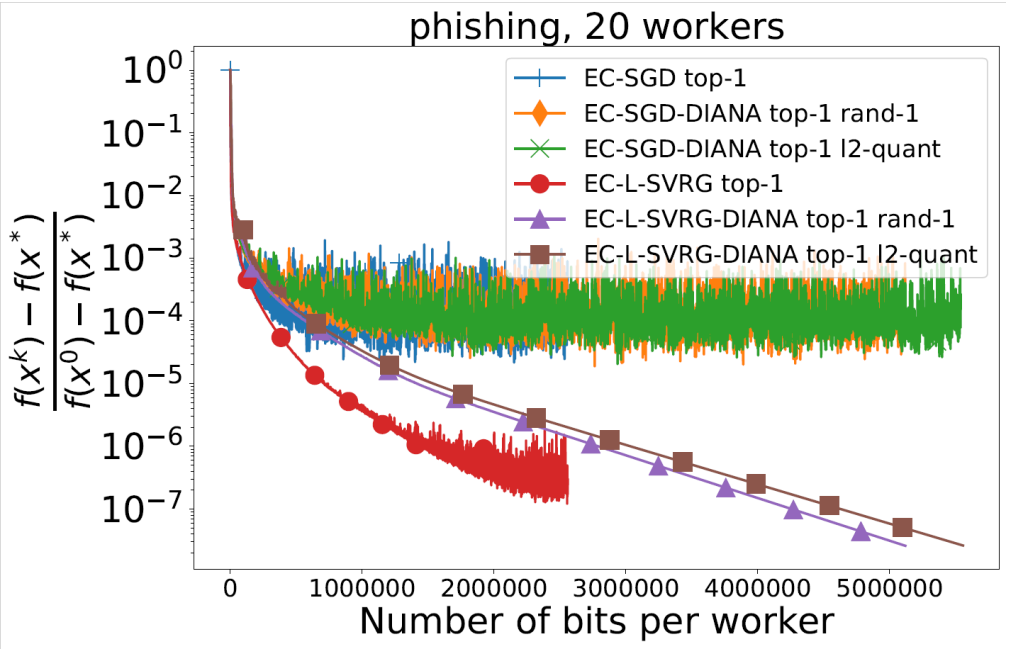
Our framework covers even methods without error compensation and methods with delayed updates

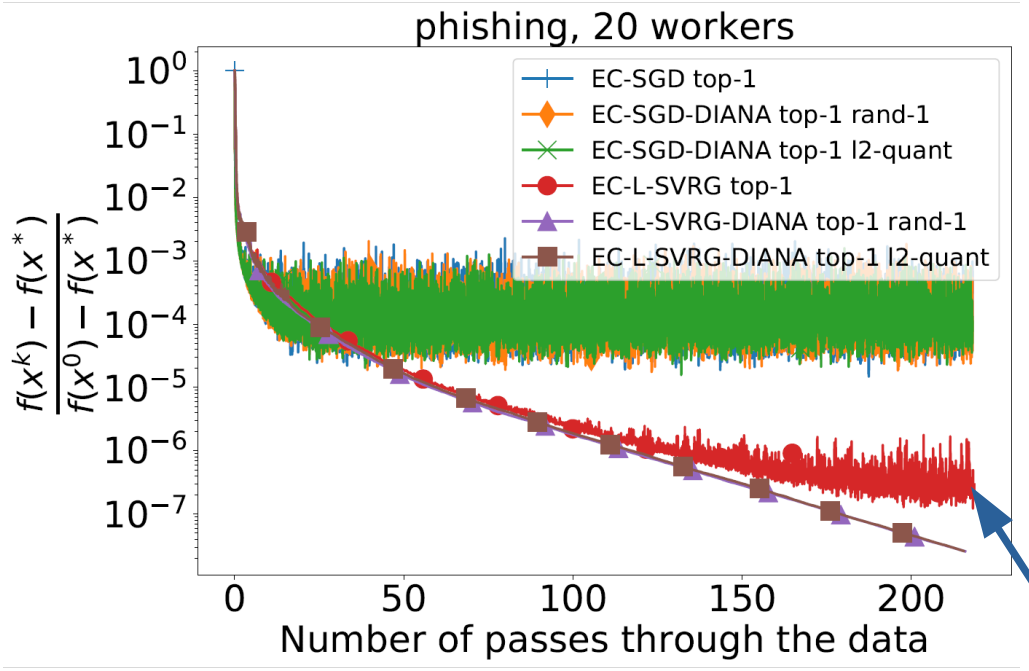
7. Experiments

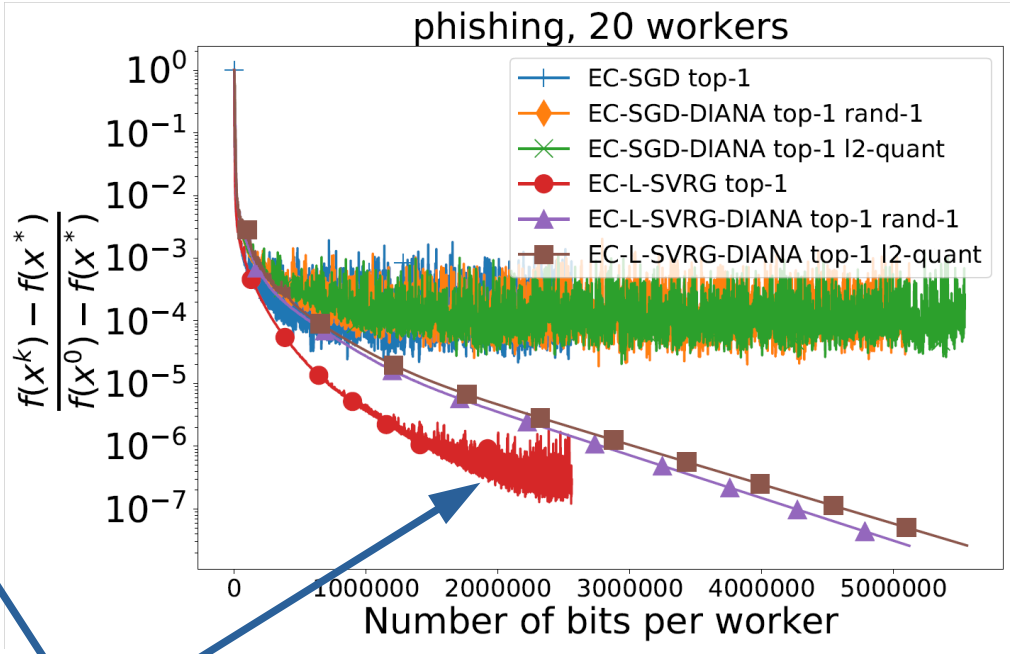
Logistic Regression with l2-regularization





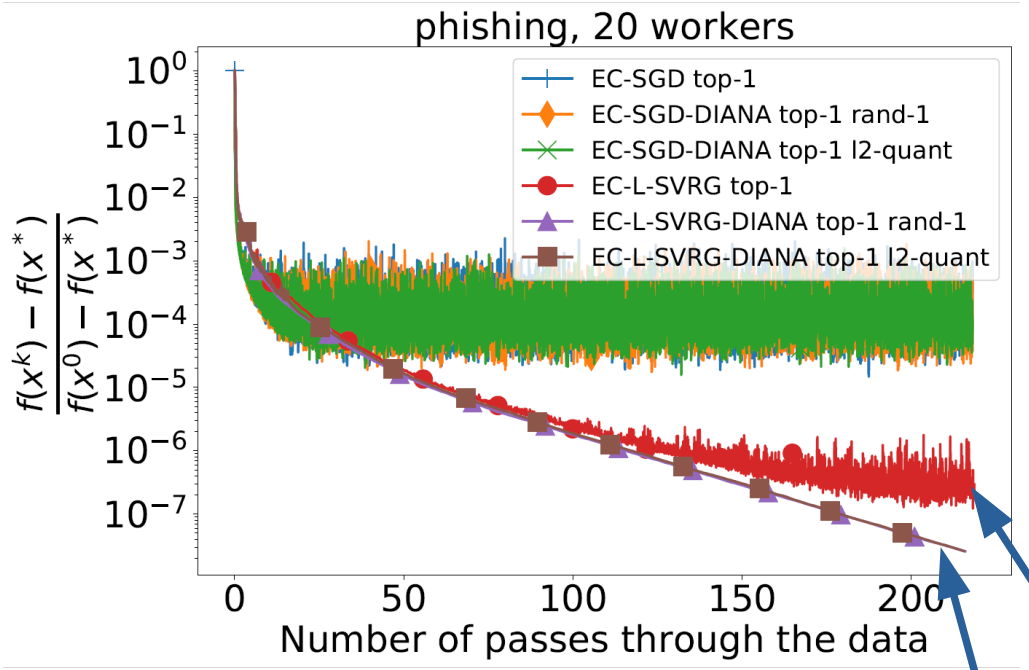
Logistic Regression with l2-regularization

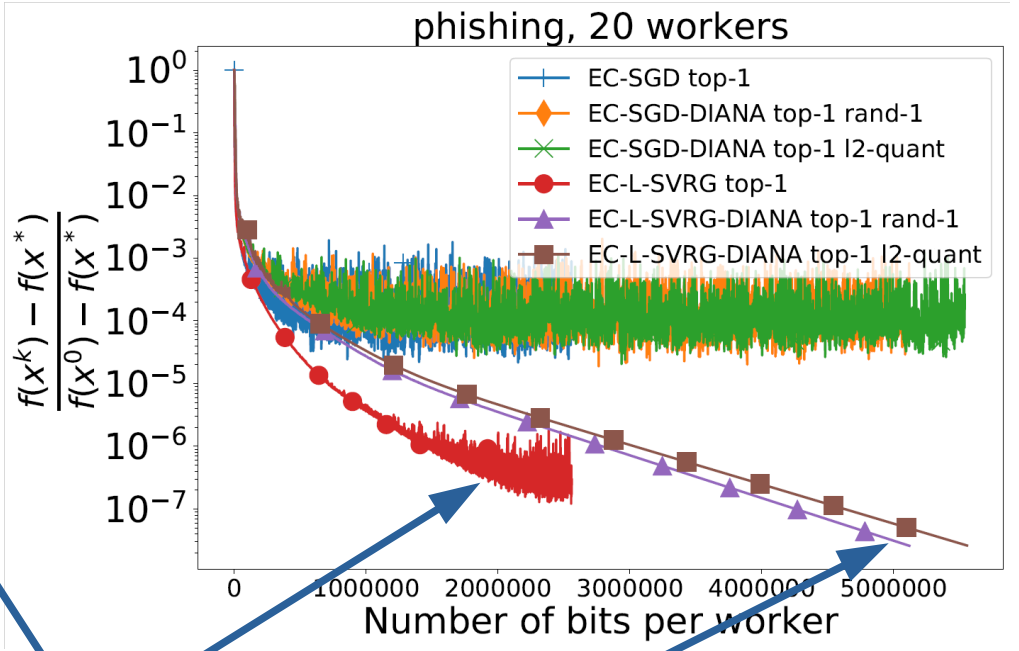




partial variance reduction

Logistic Regression with l2-regularization





partial variance reduction

full variance reduction

8. More Methods

More Methods Fitting our Framework

The generality of our approach helps to obtain convergence guarantees for a big number of different stochastic methods (even without error compensation). Here are some examples.

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- Methods with delayed updates: D-SGD, **D-SGD-SR** (arbitrary sampling), **D-QSGD, D-SGD-DIANA, D-LSVRG, D-QLSVRG, D-LSVRG-DIANA**

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- In one theorem, we recover the sharpest rates for all known special cases

- Our analysis works for non-strongly convex objectives as well

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Thank you for your attention!

bold font = new method

9. Extra Examples on Unified SGD

Gorbunov, Eduard, Filip Hanzely, and Peter Richtárik. **"A Unified Theory of SGD: Variance reduction, Sampling, Quantization and Coordinate Descent."** In International Conference on Artificial Intelligence and Statistics, pp. 680-690. 2020.

9.1. SGD-SR

Gower, R. M., Loizou, N., Qian, X., Sailanbayev, A., Shulgin, E., & Richtárik, P. (2019, May).
SGD: General Analysis and Improved Rates. *In International Conference on Machine Learning* (pp. 5200-5209).

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$$\min_{x \in \mathbb{R}^d} f(x) \qquad f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x)$$

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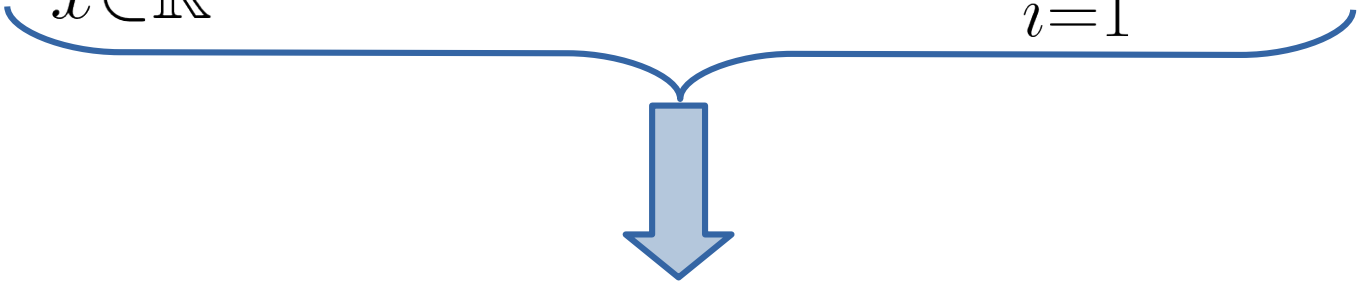

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$$f(x) = \mathbb{E}_{\xi \sim \mathcal{D}} \left[f_\xi(x) \right] \qquad f_\xi(x) \overset{\text{def}}{=} \frac{1}{n} \sum_{i=1}^{n} \xi_i f_i(x)$$

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- $\xi \sim \mathcal{D} : \mathbb{E}_{\xi \sim \mathcal{D}} \left[\xi_i \right] = 1$
- $f_i(x)$ is smooth, but can be non-convex

SGD-SR

1. **for** $k=0,1,2,\ldots$ **do**
2. Sample $\xi \sim \mathcal{D}$

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4. $x^{k+1} = x^k - \gamma g^k$
5. **end for**

Expected Smoothness

$$\mathbb{E}_{\xi\sim\mathcal{D}}\left[\left\|\nabla f_\xi(x)-\nabla f_\xi\left(x^*\right)\right\|^2\right]\leq 2\mathcal{L}\left(f(x)-f(x^*)\right)$$

Expected Smoothness

Can be significantly smaller than the maximal smoothness constant of $f_i(x)$

Some positive constant depending on $\mathcal{D}$

$$\mathbb{E}_{\xi\sim\mathcal{D}}\left[\left\|\nabla f_\xi(x)-\nabla f_\xi\left(x^*\right)\right\|^2\right]\leq 2\mathcal{L}\left(f(x)-f(x^*)\right)$$

Main Result for SGD-SR

Young inequality and expected smoothness imply:

$$\mathbb{E}_{\xi\sim\mathcal{D}}\left[\|\nabla f_\xi(x)\|^2\right] \leq 4\mathcal{L}\left(f(x)-f(x^*)\right)+2\sigma_*^2$$

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Main Theorem: Reminder

If the stepsize satisfies

$$0 < \gamma \leq \min\left\{\frac{1}{\mu}, \frac{1}{A + CM}\right\}, \quad \text{where} \quad M > \frac{B}{\rho}$$

then the iterates of SGD satisfy

$$\mathbb{E}\left[V^k\right] \leq \max\left\{(1-\gamma\mu)^k, \left(1 + \frac{B}{M} - \rho\right)^k\right\} V^0 + \frac{(D_1 + MD_2)\,\gamma^2}{\min\left\{\gamma\mu, \rho - \frac{B}{M}\right\}}$$

where $V^k \overset{\text{def}}{=} \left\|x^k - x^*\right\|^2 + M\gamma^2\sigma_k^2$

$$\left\|x^k - x^*\right\|^2 \leq V^k$$

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$$A=2\mathcal{L}, B=0, D_1=2\sigma_*^2, \sigma_k=0, \rho=1, C=0, D_2=0$$

$$\mathbb{E}\left\|x^k-x^*\right\|^2 \leq (1-\gamma\mu)^k\left\|x^0-x^*\right\|^2+\frac{2\gamma\sigma_*^2}{\mu}$$

$$\gamma \leq \frac{1}{2\mathcal{L}}$$

Main theorem:

$$\mathbb{E}\left[V^k\right] \leq \max\left\{(1-\gamma\mu)^k, \left(1+\frac{B}{M}-\rho\right)^k\right\}V^0+\frac{\left(D_1+MD_2\right)\gamma^2}{\min\left\{\gamma\mu,\rho-\frac{B}{M}\right\}}$$

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9.2. SAGA

Defazio, Aaron, Francis Bach, and Simon Lacoste-Julien. **SAGA: A fast incremental gradient method with support for non-strongly convex composite objectives.** *In Advances in neural information processing systems*, pp. 1646-1654. 2014.

The Problem

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$$\|\nabla f_i(x) - \nabla f_i(y)\| \leq L\|x - y\|$$

$$f_i(y) \geq f_i(x) + \langle \nabla f_i(x), y - x \rangle$$

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5. Set $\phi_j^{k+1} = x^k$ and $\phi_i^{k+1} = \phi_i^k$ for $i \neq j$
6. $x^{k+1} = x^k - \gamma g^k$
7. **end for**

Main Result for SAGA

$$\mathbb{E}\left[\left\|g^k\right\|^2 \mid x^k\right] \leq 4L\left(f(x^k) - f(x^*)\right) + 2\sigma_k^2$$

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