Voronoi conjecture for five-dimensional parallelohedra

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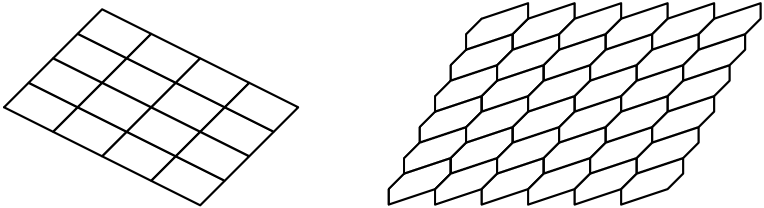
Parallelohedra

Definition

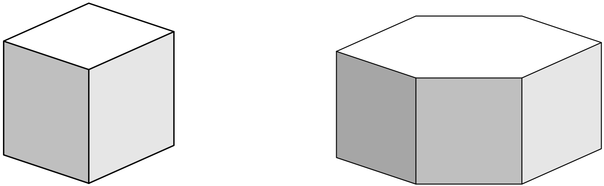
Convex d-dimensional polytope P is called a **parallelohedron** if $\mathbb{R}^d$ can be (face-to-face) tiled into parallel copies of P.



Two types of two-dimensional parallelohedra

Three-dimensional parallelohedra

In 1885 Russian crystallographer Fedorov listed all types of three-dimensional parallelohedra.



Parallelepiped and hexagonal prism with centrally symmetric base.

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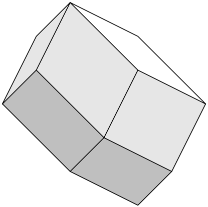
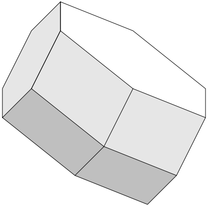
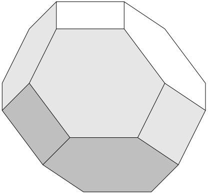
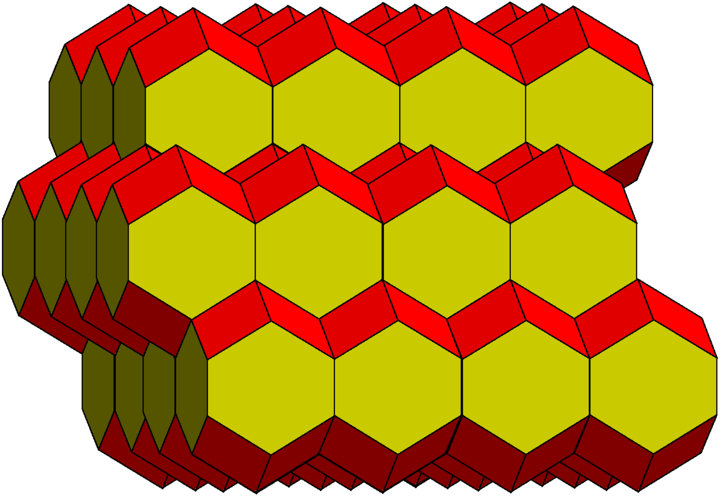






Rhombic dodecahedron, elongated dodecahedron, and truncated octahedron

Tiling by elongated dodecahedra (from Wikipedia)



Minkowski-Venkov conditions

Theorem (Minkowski, 1897; Venkov, 1954; and McMullen, 1980)

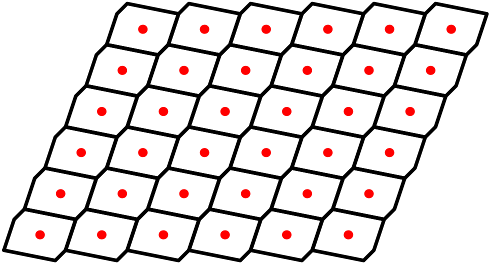
P is a d-dimensional parallelohedron iff it satisfies the following conditions:

1. *P is centrally symmetric;*
2. *Any facet of P is centrally symmetric;*
3. *Projection of P along any its $(d-2)$-dimensional face is parallelogram or centrally symmetric hexagon.*

Particularly, if P tiles $\mathbb{R}^d$ in a non-face-to-face way, then it satisfies Minlowski-Venkov conditions, and hence tiles $\mathbb{R}^d$ in a face-to-face way as well.

Parallelohedra to Lattices

- Let P be a parallelohedron, i.e. centrally symmetric convex polytope with symmetric facets and 4- or 6-belts;
- Let $\mathcal{T}_P$ be the unique face-to-face tiling of $\mathbb{R}^d$ into parallel copies of P. Then the centers of the tiles form a lattice Λ_P.



Lattices to Paralleohedra

- Let Λ be an arbitrary d-dimensional lattice and let O be a point of Λ.

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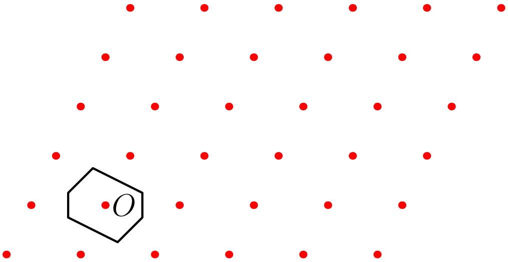
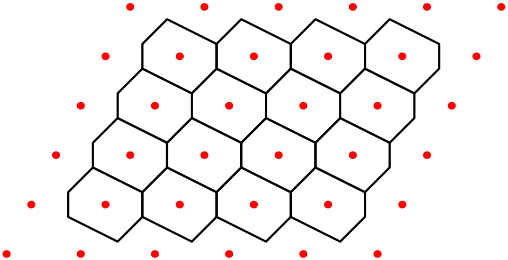
- We construct the polytope consisting of points that are closer to O than to any other point of Λ (the Dirichlet-Voronoi polytope of Λ).



Lattices to Paralleohedra

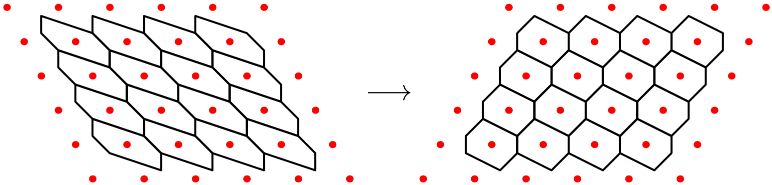
- Let Λ be an arbitrary d-dimensional lattice and let O be a point of Λ.
- We construct the polytope consisting of points that are closer to O than to any other point of Λ (the Dirichlet-Voronoi polytope of Λ).
- Then DV_Λ is a parallelohedron and the points of Λ are centers of the corresponding tiles.



The Voronoi conjecture

Conjecture (Voronoi, 1909)

Every parallelohedron is affinely equivalent to the Dirichlet-Voronoi polytope of some lattice Λ.



VORONOI CONJECTURE IN $\mathbb{R}^2$

- Each parallelogram can be transformed into some rectangle and all rectangles are Voronoi polygons.
- Each centrally-symmetric hexagon can be transformed into some hexagon inscribed in a circle. This transformation is **unique** modulo isometry and/or homothety. Similarly, all centrally-symmetric hexagons inscribed in circles are Voronoi polygons.

The Voronoi conjecture: small dimensions

- $\mathbb{R}^2$: folklore.
- $\mathbb{R}^3$: kind of folklore. All three-dimensional parallelohedra are known due to Fedorov, and then one can **check** that they satisfy the Voronoi conjecture.

Theorem (Delone, 1929)

The Voronoi conjecture is true in $\mathbb{R}^4$.

Classification: there are 52 four-dimensional parallelohedra; Delone, 1929 and Stogrin, 1974.

Theorem (G., Magazinov, 2019+)

The Voronoi conjecture is true in $\mathbb{R}^5$.

Hilbert's 18th problem: lattices in $\mathbb{R}^d$

- Finiteness of the family of crystallographic groups
- Existence of a polytope that tiles $\mathbb{R}^d$ but can't be obtained as a fundamental region of a crystallographic group
- Densest sphere packing in $\mathbb{R}^3$ (Kepler conjecture)

Hilbert's 18th problem: lattices in $\mathbb{R}^d$

- Finiteness of the family of crystallographic groups
  - Bieberbach, 1911-12;

- Existence of a polytope that tiles $\mathbb{R}^d$ but can't be obtained as a fundamental region of a crystallographic group
  - Reinhardt, 1928 in $\mathbb{R}^3$ and Heesch, 1935 in $\mathbb{R}^2$;

- Densest sphere packing in $\mathbb{R}^3$ (Kepler conjecture)
  - Hales, 2005 and 2017.

Which (convex) polytopes may tile the space?

- $\mathbb{R}^2$: If $n \geq 7$, then a convex n-gon cannot tile the plane;

  Rao (2017+): full classification of pentagons (15 types).

- $\mathbb{R}^3$: the maximal number of facets for **stereohedron** is unknown.

  Engel (1981): There exists a stereohedron with 38 facets;

  Santos et. al. (2001-2011): **Dirichlet stereohedron** cannot have more than 92 facets.

Parallelohedra and lattice covering problem

Problem: for a given d, find a lattice that gives **optimal** covering of $\mathbb{R}^d$ with balls of equal radius.

- $\mathbb{R}^2$: Kershner, 1939 and A_2^*;
- $\mathbb{R}^3$: Bambah, 1954 and A_3^*;
- $\mathbb{R}^4$: Delone and Ryshkov, 1963 and A_4^*;
- $\mathbb{R}^5$: Ryshkov and Baranovskii, 1976 and A_5^*;
- $\mathbb{R}^d$, $d = 6, 7, 8$ (examples only): Schürmann and Vallentin, 2006 and lattices different from A_d^*.

The results in dimensions 4 through 8 rely on **reduction theory** for lattices, or (partial) classification of Voronoi parallelohedra.

SVP and CVP: using parallelohedra for lattice algorithms

- SVP (Shortest Vector Problem): find a shortest non-zero vector of a given lattice Λ;
- CVP (Closest Vector Problem): for a given target vector $\mathbf{t}$ and a lattice Λ, find the vector $\mathbf{x} \in \Lambda$ that minimizes $||\mathbf{t} - \mathbf{x}||$.

  - LLL-algorithm for lattice reduction and polynomial fatorization over $\mathbb{Q}$;
  - Solvability in radicals;
  - Cryptography;
  - Integer optimization.

Spectral sets

Let $\Omega \subset \mathbb{R}^d$ be a bounded measurable set with positive measure.

Definition

The set Ω is called **spectral** if there is an **orthogonal** basis of exponential functions in $L^2(\Omega)$.

Conjecture (Fuglede, 1974)

A set $\Omega \subseteq \mathbb{R}^d$ is spectral if and only if Ω tiles $\mathbb{R}^d$ with translations.

Spectral Sets II

There are non-convex counterexamples in

- $\mathbb{R}^5$: Tao, 2004; and in
- $\mathbb{R}^4$ and $\mathbb{R}^3$: Matolcsi, 2005 and Kolountzakis and Matolcsi, 2010.

Theorem (Lev and Matolcsi, 2019+)

The Fuglede conjecture holds for convex sets in $\mathbb{R}^d$.

That is, all convex spectral sets are parallelohedra and each parallelohedron is a spectral set.

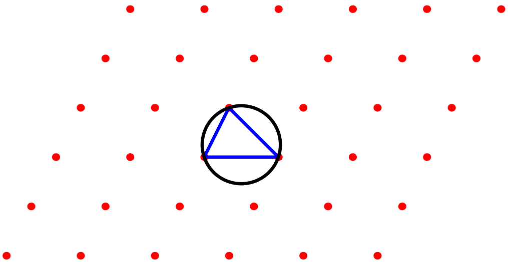
Reduction theory

- **Reduction theory for lattices**: find an **"optimal"** basis.
- "Dual" view: for a given positive definite matrix Q, find an invertible integer transformation A, such that A^tQA is "optimal".
- Voronoi's reduction theory: find an optimal basis for the representation of the Voronoi parallelohedron and for the tiling dual to the Voronoi tiling.

DELONE TILING

Delone tiling is the tiling with "empty spheres".

A polytope P is in the Delone tiling $\mathrm{Del}(\Lambda)$ iff it is inscribed in an empty sphere.



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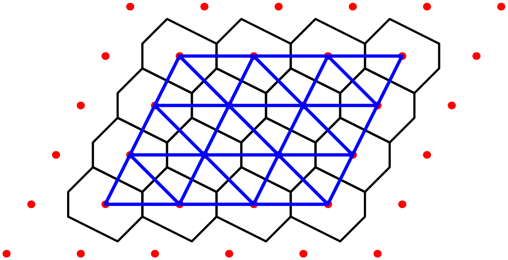
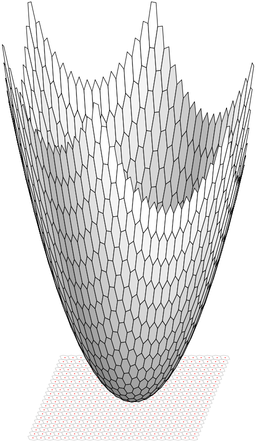


The Delone tiling is dual to the Voronoi tiling.

Constructing the Voronoi and Delone tilings

- **Lifting** construction for a point set X.
- Lift the points of X to paraboloid $y = \mathbf{x}^t\mathbf{x}$ in $\mathbb{R}^{d+1}$.
- Construct the tangent hyperplanes and take the intersection of the upper half-spaces; project this infinite polyhedron back to $\mathbb{R}^d$ to get the **Voronoi** tiling.
- Take the convex hull of points on $y = \mathbf{x}^t\mathbf{x}$ and project this (infinite) polyhedron back to $\mathbb{R}^d$ to get the **Delone** tiling.



From lattices to PQF

An affine transformation can take a lattice to $\mathbb{Z}^d$, but it changes metrics from $\mathbf{x}^t\mathbf{x}$ to $\mathbf{x}^t Q\mathbf{x}$ for some positive definite quadratic form Q.

Task

Find all combinatorially **different** *Delone tilings of $\mathbb{Z}^d$.*

Definition

The Delone tiling $\mathrm{Del}(\mathbb{Z}^d, Q)$ of the lattice $\mathbb{Z}^d$ with respect to PQF Q is the tiling of $\mathbb{Z}^d$ with empty ellipsoids determined by Q (spheres in the metric $\mathbf{x}^t Q\mathbf{x}$).

Secondary cones

Let $\mathcal{S}^d \subset \mathbb{R}^{\frac{d(d+1)}{2}}$ be the cone of all PQF.

Definition

The **secondary cone** of a Delone tiling $\mathcal{D}$ is the set of all PQFs Q with Delone tiling equal to $\mathcal{D}$.

$$\mathrm{SC}(\mathcal{D}) = \left\{ Q \in \mathcal{S}^d | \mathcal{D} = \mathrm{Del}(\mathbb{Z}^d, Q) \right\}$$

Theorem (Voronoi, 1909)

$\mathrm{SC}(\mathcal{D})$ *is a convex polyhedron in* $\mathcal{S}^d$.

Secondary cones II

Theorem (Voronoi, 1909)

The set of closures all secondary cones gives a face-to-face tiling of the closure of $\mathcal{S}^d$ (that is the cone of positive semidefinite quadratic forms).

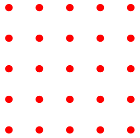
- Full-dimensional secondary cones correspond to Delone triangulations
- One-dimensional secondary cones are called **extreme rays**

Lemma

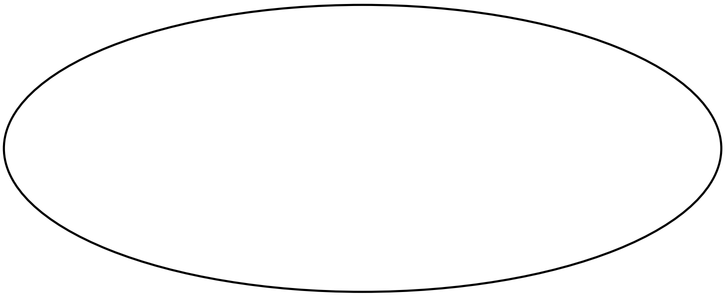
Two Delone tilings $\mathcal{D}$ and $\mathcal{D}'$ are affinely equivalent iff there is a matrix $\mathcal{A} \in GL_d(\mathbb{Z})$ such that

$$\mathcal{A}(\mathrm{SC}(\mathcal{D})) = \mathrm{SC}(\mathcal{D}').$$

Secondary cones in dimension 2

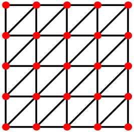


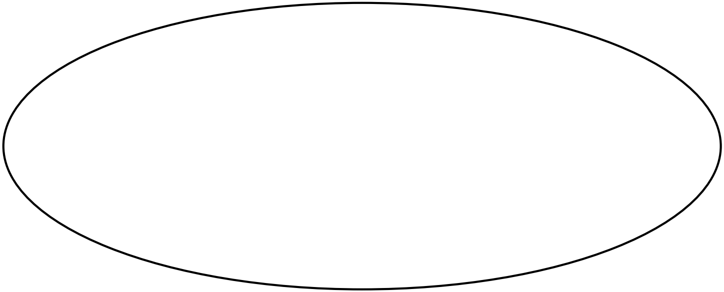
Any PQF $Q = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ can be represented by a point in a cone over open disc.



Secondary cones in dimension 2

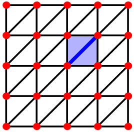
We will find the secondary cone of Delone triangulation on the right.

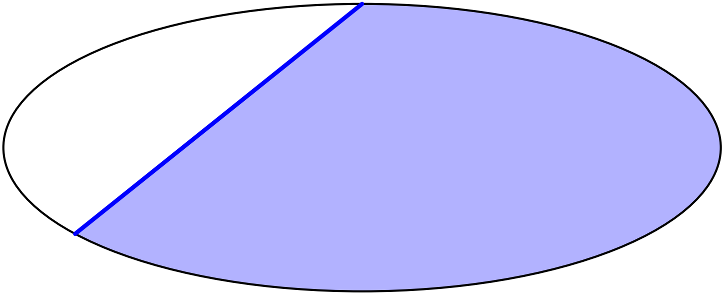




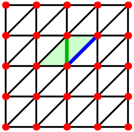
Secondary cones in dimension 2

Each pair of adjacent triangles defines one linear inequality for the secondary cone. For the **blue** pair the inequality is $b < 0$.

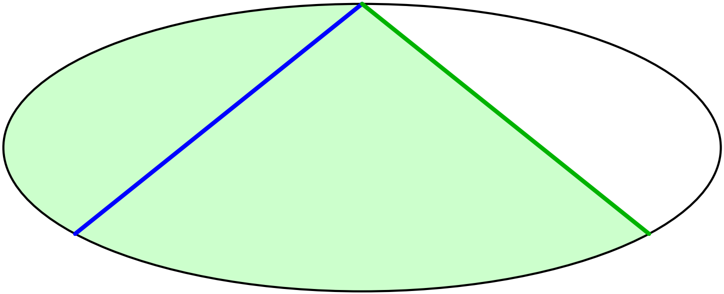




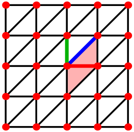
Secondary cones in dimension 2



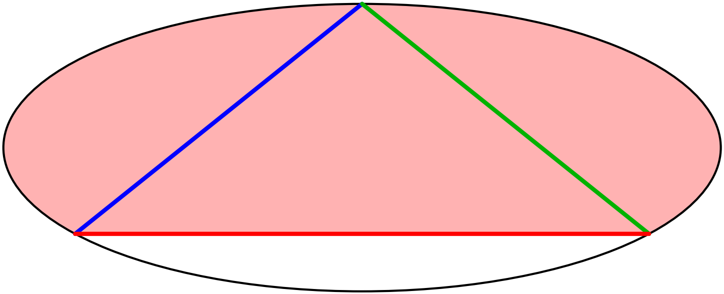
The **green** pair of triangles gives us the inequality $b + c > 0$.



Secondary cones in dimension 2

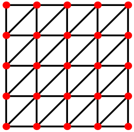


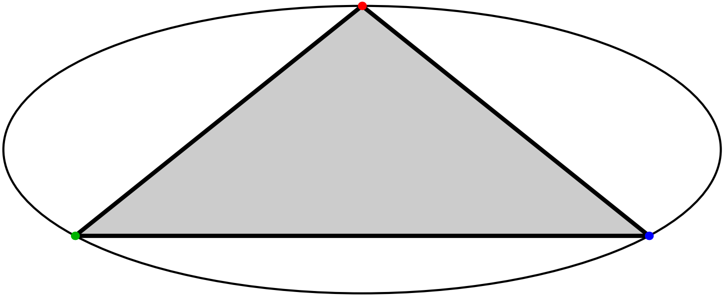
The **red** pair gives us the inequality $a+b>0$.



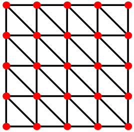
Secondary cones in dimension 2

The secondary cone is a cone over triangle with vertices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

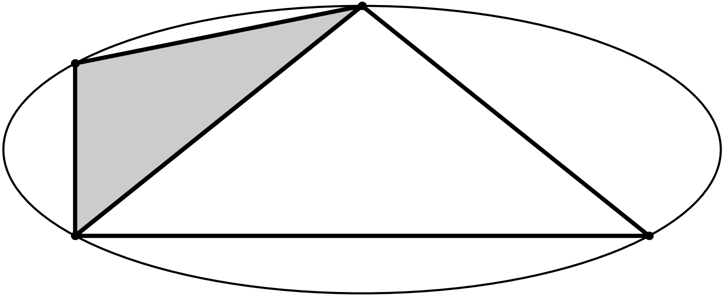




Secondary cones in dimension 2

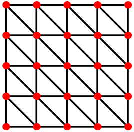


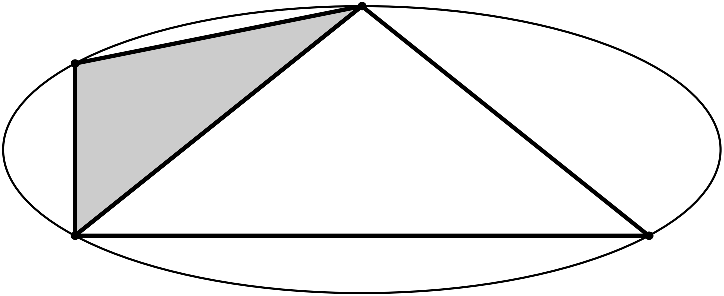
Similarly we can construct secondary cones for other triangulations.



Secondary cones in dimension 2

Triangulations corresponding to adjacent secondary cones differ by a (bi-stellar) flip.





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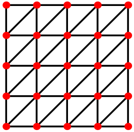

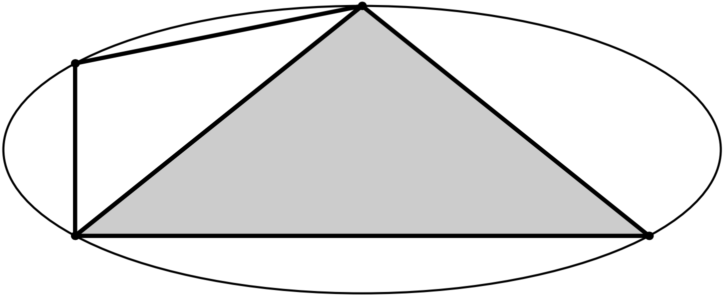

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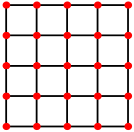
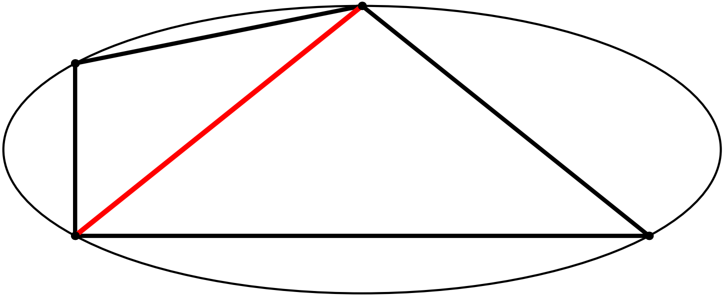


Cones of smaller dimensions are secondary cones of non-generic Delone decompositions.



Five-dimensional Voronoi parallelohedra

Theorem (Dutour-Sikirić, G., Schürmann, Waldmann, 2016)

There are 110244 *affine types of lattice Delone subdivisions in dimension* 5.

Additionally, all these classes correspond to combinatorially different Voronoi parallelohedra.

Proof of the Voronoi conjecture in $\mathbb{R}^5$

Let P be a five-dimensional parallelohedron.

- If P can be **extended**, then its extension has **combinatorics** of one of 110244 Voronoi parallelohedra in $\mathbb{R}^5$;
- In five-dimensional case, **global combinatorics** a Voronoi parallelohedron guarantees the geometric part of the Voronoi conjecture.
- **Local combinatorics** can be used to show that P can be extended.

Free directions

Definition

Let I be a segment. If $P + I$ and P are both parallelohedra, then I is called a **free** direction for P.

- If I is a free direction for P, then the Voronoi conjecture holds (or doesn't hold) for P and for $P + I$ simultaneously (Grishukhin, 2004; Végh, 2015; Magazinov, 2015).

Theorem (Erdahl, 1999)

The Voronoi conjecture is true for space-filling zonotopes.

Extensions of parallelohedra

Theorem (G., Magazinov)

Let P be a d-dimensional parallohedron. If I is a free direction for P and the projection of P along I satisfies the Voronoi conjecture, then $P + I$ has the combinatorics of a Voronoi parallelohedron.

Proof of the Voronoi conjecture in $\mathbb{R}^5$

Let P be a five-dimensional parallelohedron.

- ~~If P can be **extended**, then its extension has **combinatorics** of one of 110244 Voronoi parallelohedra in $\mathbb{R}^5$;~~ Done!

- In five-dimensional case, **global combinatorics** a Voronoi parallelohedron guarantees the geometric part of the Voronoi conjecture.

- **Local combinatorics** can be used to show that P can be extended.

Checking the Voronoi conjecture

For a given parallelohedron P, how can we check/prove the Voronoi conjecture for P?

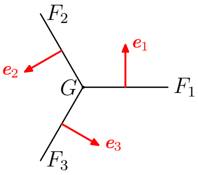
- We can try to construct "shells" above each copy of P tangent to **some** fixed paraboloid in $\mathbb{R}^{d+1}$ and then transform this paraboloid into $y = \mathbf{x}^t\mathbf{x}$.

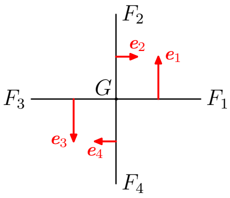
And here is a way to do it...

Canonical scaling

Definition

A (positive) real-valued function $n(F)$ defined on set of all facets of the parallelohedral tiling is called a **canonical scaling**, if it satisfies the following conditions for facets F_i that contain arbitrary $(d-2)$-face G:



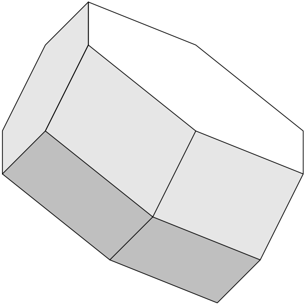


$$\sum \pm n(F_i)\mathbf{e}_i = \mathbf{0}$$

Belts of parallelohedra

Definition

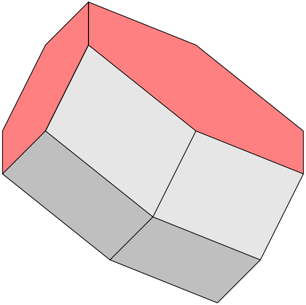
The set of facets parallel to a given $(d-2)$-face is called **belt**. These facets are projected onto sides of a parallelogram or a hexagon.



Belts of parallelohedra

Definition

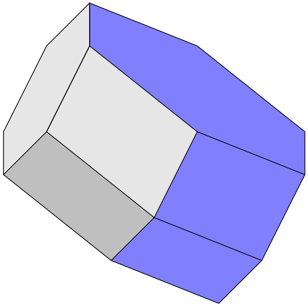
The set of facets parallel to a given $(d-2)$-face is called **belt**. These facets are projected onto sides of a parallelogram or a hexagon. There are 4-belts



Belts of parallelohedra

Definition

The set of facets parallel to a given $(d-2)$-face is called **belt**. These facets are projected onto sides of a parallelogram or a hexagon. There are 4-belts and 6-belts.

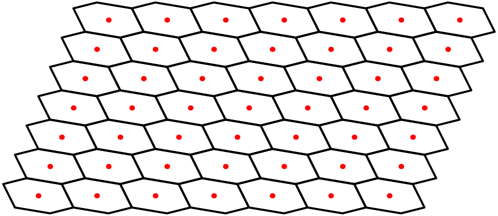


Constructing canonical scaling

How to construct a canonical scaling for a given tiling?

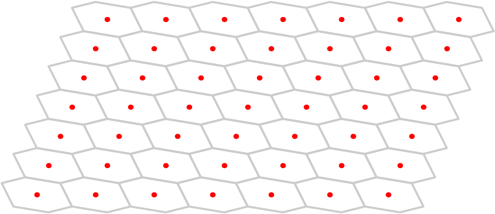
- If two facets F_1 and F_2 of the tiling have a common $(d-2)$-face from 6-belt, then the value of canonical scaling on F_1 uniquely defines the value on F_2 and vice versa.
- If facets F_1 and F_2 have a common $(d-2)$-face from 4-belt then the only condition is that if these facets are opposite then values of canonical scaling on F_1 and F_2 are equal.
- If facets F_1 and F_2 are opposite in one parallelohedron then values of canonical scaling on F_1 and F_2 are equal.

Voronoi's generatrix



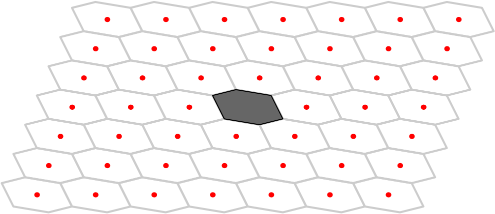
Consider we have a canonical scaling defined on the tiling with copies of P.

Voronoi's generatrix



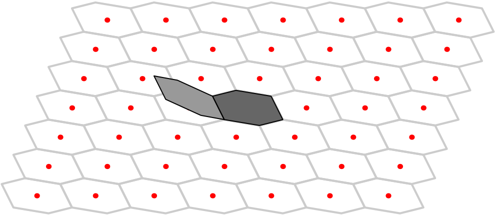
We will construct a piecewise linear generatrix function $\mathcal{G} : \mathbb{R}^d \longrightarrow \mathbb{R}$.

Voronoi's generatrix



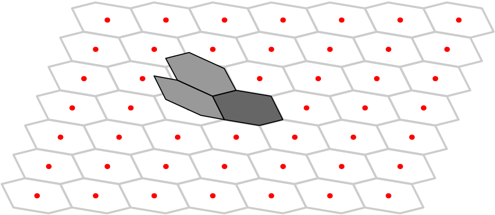
Step 1: Put $\mathcal{G}$ equal to 0 on one of the tiles.

Voronoi's generatrix



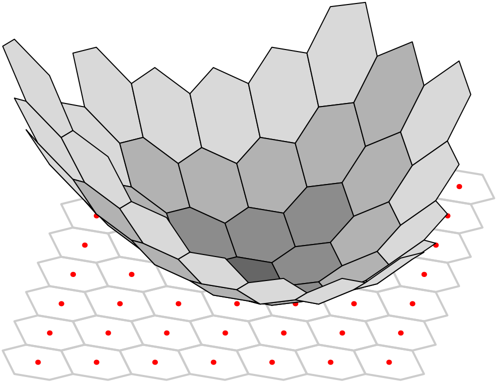
Step 2: When we pass across one facet of the tiling, the gradient of $\mathcal{G}$ changes according to the canonical scaling.

VORONOI'S GENERATRIX



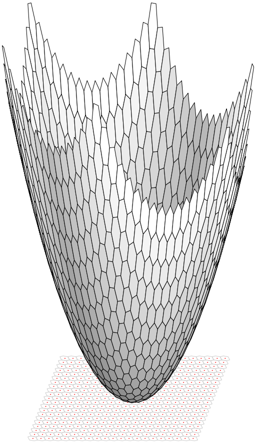
Step 2: Namely, if we pass a facet F with the normal vector $\mathbf{e}$, then we add the vector $n(F)\mathbf{e}$ to the gradient.

Voronoi's generatrix



We obtain the graph of the generatrix function $\mathcal{G}$.

Voronoi's generatrix II



PROPERTIES OF GENERATRIX

- The graph of generatrix $\mathcal{G}$ looks like a "piecewise linear" paraboloid.
- And actually there is a paraboloid $y = \mathbf{x}^t Q\mathbf{x}$ for some positive definite quadratic form Q tangent to generatrix in the centers of its shells.
- Moreover, if we consider an affine transformation $\mathcal{A}$ of this paraboloid into paraboloid $y = \mathbf{x}^t\mathbf{x}$ then the tiling by copies of P will transform into the Voronoi tiling for some lattice.

So to prove the Voronoi conjecture for P it is sufficient (and necessary) to construct a canonical scaling on the tiling by copies of P.

Primitive parallelohedra

Definition

A d-dimensional parallelohedron P is called **primitive**, if every vertex of the corresponding tiling belongs to exactly $d+1$ copies of P.

Primitive parallelohedra appear exactly as dual to Delone triangulations (not arbitrary Delone decompositions).

Theorem (Voronoi, 1909)

The Voronoi conjecture is true for primitive parallelohedra.

Primitive parallelohedra II

Definition

A d-dimensional parallelohedron P is called **k-primitive** if every k-face of the corresponding tiling belongs to exactly $d+1-k$ copies of P.

Theorem (Zhitomirskii, 1929)

The Voronoi conjecture is true for $(d-2)$-primitive d-dimensional parallelohedra. Or the same, it is true for parallelohedra without belts of length 4.

Dual cells

Definition

The **dual cell** of a face F of given parallelohedral tiling is the set of all centers of parallelohedra that share F.
If F is $(d-k)$-dimensional then the corresponding cell is called **k-cell**.

The set of all dual cells of the tiling with corresponding incidence relation determines a structure of a cell complex.

Conjecture (Dimension conjecture)

The dimension of a dual k-cell is equal to k.

The dimension conjecture is necessary for the Voronoi conjecture.

Dual 3-cells and 4-dimensional parallelohedra

Lemma (Delone, 1929)

There are five types of three-dimensional dual cells: tetrahedron, octahedron, quadrangular pyramid, triangular prism and cube.

Theorem (Ordine, 2005)

The Voronoi conjecture is true for parallelohedra without cubical or prismatic dual 3-cells.

Topology meets canonical scaling

We know how canonical scaling should change when we cross a primitive $(d-2)$-face of F.

Question

Are there any **topological** *reasons that will prevent us to assign values of canonical scaling to all facets using such local guidance?*

Definition

Let P_π, the **π-surface** of P, be the manifold obtained from the surface of P by removing non-primitive $(d-2)$-faces and identifying opposite points.

- We can assign values of canonical scaling along every curve on P_π and the canonical scaling exists if and only if we can assign values **consistently** along every closed curve on P_π.

GGM condition

- Any **half-belt cycle** which starts at the center of a facet and ends at the center of the opposite facet crossing only three parallel primitive $(d-2)$-faces gives consistent values for canonical scaling.

Theorem (G., Gavrilyuk, Magazinov, 2015)

If the group of one-dimensional homologies $H_1(P_\pi, \mathbb{Q})$ of the π-surface of a parallelohedron P is generated by the half-belt cycles then the Voronoi conjecture is true for P.

How many parallelohedra satisfy the GGM condition?

- All 5 parallelohedra in $\mathbb{R}^3$.
- All 52 parallelohedra in $\mathbb{R}^4$.
- All 110244 **Voronoi** parallelohedra in $\mathbb{R}^5$ (Dutour-Sikirić, G., and Magazinov, 2020).

Corollary

If a 5-dimensional parallelohedron P has a free direction, then P satisfies the Voronoi conjecture.

Proof of the Voronoi conjecture in $\mathbb{R}^5$

Let P be a five-dimensional parallelohedron.

- ~~If P can be **extended**, then its extension has **combinatorics** of one of 110244 Voronoi parallelohedra in $\mathbb{R}^5$;~~ Done!

- ~~In five-dimensional case, **global combinatorics** a Voronoi parallelohedron guarantees the geometric part of the Voronoi conjecture.~~ Done!

- **Local combinatorics** can be used to show that P can be extended.

Proof of the Voronoi conjecture in $\mathbb{R}^5$

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- ~~In five-dimensional case, **global combinatorics** a Voronoi parallelohedron guarantees the geometric part of the Voronoi conjecture.~~ Done!

- ~~**Local combinatorics** can be used to show that P can be extended.~~ Analysis of dual 3-cells and dual 4-cells to prove existence of a free direction for P.

Proof. Dual 3-cells

What are possible dual 3-cells of a five-dimensional parallelohedron P?

Proof. Dual 3-cells

What are possible dual 3-cells of a five-dimensional parallelohedron P?

- If all dual 3-cells are either tetrahedra, octahedra, or pyramids, then P satisfies the Voronoi conjecture (Ordine's theorem).
- If P has a cubical dual 3-cell, then it has a free direction, and hence satisfies the Voronoi conjecture (proof on the next slide).
- If two-dimensional face F of P has prismatic dual cell, then either an edge of F gives a free direction of P, or F is a triangle.

The main tool used is a careful inspection of 32 parity classes of lattice points and all half-lattice points. Central symmetry in each half-lattice point preserves the tiling $\mathcal{T}(P)$, and lattice equivalent points must carry the same local combinatorics.

Proof. Cubic dual 3-cell

Lemma (Grishukhin, Magazinov)

A direction I is free for P if and only if every 6-belt of P has at least one facet parallel to I.

- The space of half-lattice points is isomorphic to a five-dimensional space over $\mathbb{F}_2$.
- Let F have a cubical dual cell. An edge e of F has an additional point in its dual cell. Set of all midpoints between these nine points give a 4-dimensional subspace of the half-lattice space.
- The centers of facets of a 6-belt B give a two-dimensional subspace of the half-lattice space.
- 4- and 2-dimensional subspaces of 5-dimensional space intersect non-trivially, so there is a facet in B parallel to e.

Proof. Dual 4-cells

For a triangular face F of P with prismatic dual 3-cells, the edges may have only two types of dual 4-cells (or there is a free direction for P).

- Pyramid over triangular prism.
- Prism over tetrahedron.

In all four possible choices for dual cells of edges of F we were able to prove that either P has a free direction, or it admits a canonical scaling.

Again, using a lot of local combinatorics and in most cases exhaustively analyzing all 32 parity classes of lattice points.

Proof. Prism-Prism-Pyramid case

What about $\mathbb{R}^6$?

Challenges in six-dimensional case.

- There is a significant jump in the number of parallelohedra. Baburin and Engel (2013) reported about half a billion of different Delone triangulations in $\mathbb{R}^6$.

- The classification of dual 4-cells is not known and dual 3-cells might be not enough.

THANK YOU!