Spectral form factor in double-scaled SYK

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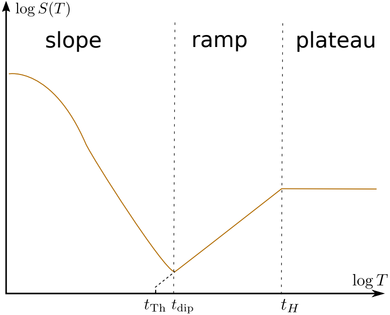
Introduction

The diagnostic of quantum chaos - spectral form factor

$$S(\beta, T) = \frac{Z(\beta + iT)Z(\beta - iT)}{Z(\beta)^2} = \frac{1}{Z(\beta)^2} \mathrm{Tr}\, \mathrm{e}^{-\beta H - iHT} \mathrm{Tr}\, \mathrm{e}^{-\beta H + iHT}$$

$$= \frac{1}{Z(\beta)^2} \sum_{n,m} \mathrm{e}^{-\beta(E_m + E_n)} \mathrm{e}^{iT(E_m - E_n)} .$$



Slope:

- Determined by spectral density
- Self-averaging in an ensemble

Ramp & plateau:

- RMT universality
- Non-self averaging

Numerical studies: Garcia-Garcia, Verbaarschot; Cotler et. al.; Saad, Shenker, Stanford; Sonner, Vielma; Gur-ari et. al., ...

Motivations

1. Study of the onset of RMT universality in an analytically tractable model of large-q SYK.

2. Physics behind nontrivial saddle points in the SYK path integral from the quantum chaos perspective.

Outline

1. Large q limits for the spectral form factor in SYK
2. Disconnected part: the slope in double scaling limit
   - Solutions of saddle point equations
   - One-loop correction
3. Connected part: the ramp and breakdown of double scaling limit
   - Solutions of saddle point equations, their existence and uniqueness
   - One-loop correction
   - Time scales of quantum chaos: Thouless time and the dip time

Spectral form factor in SYK

The SYK model (Sachdev, Ye; Kitaev):

$$H = (i)^{\frac{q}{2}} \sum_{i_1 < \cdots < i_q} j_{i_1 \ldots i_q} \chi_{i_1} \ldots \chi_{i_q} \,, \quad \langle j^2_{i_1 \ldots i_q} \rangle = \frac{2^{q-1}(q-1)! \mathcal{J}^2}{q N^{q-1}}$$

Spectral form factor at $\beta = 0$:

$$S(T) = 2^{-N} \int DG_{\alpha\beta} D\Sigma_{\alpha\beta} \mathrm{e}^{-NI[G,\Sigma]} \,,$$

where the action has the form (Saad, Shenker, Stanford):

$$I[G,\Sigma] = -\log \mathrm{Pf}[\delta_{\alpha\beta}\partial_t - \hat{\Sigma}_{\alpha\beta}] + \frac{1}{2}\int_0^T \int_0^T dt_1 dt_2 \left(\Sigma_{\alpha\beta}(t_1,t_2) G_{\alpha\beta}(t_1,t_2) - \frac{2^{q-1}\mathcal{J}^2}{q^2} s_{\alpha\beta} G_{\alpha\beta}(t_1,t_2)^q \right.$$

Here $\alpha, \beta = L, R$, $s_{LL} = s_{RR} = -1\,; \quad s_{LR} = s_{RL} = i^q\,.$

Large q (double scaling) limit (Maldacena, Stanford; Cotler et. al; Maldacena, Qi; Berkooz et. al; Streicher; Choi, Mezei, Sarosi, ...):

$$N \to \infty\,, \quad q \to \infty\,, \quad \lambda = \frac{q^2}{N} \text{ fixed}$$

Generalities of large q

Saddle point equations:

$$\begin{aligned}
&\partial_{t_1} G_{\alpha\beta}(t_1,t_2) - \int dt \Sigma_{\alpha\gamma}(t_1,t) G_{\gamma\beta}(t,t_2) = \delta(t_1-t_2)\delta_{\alpha\beta}, \\
&\Sigma_{\alpha\beta}(t_1,t_2) = s_{\alpha\beta} \frac{2^{q-1}\mathcal{J}^2}{q} G_{\alpha\beta}(t_1,t_2)^{q-1}.
\end{aligned} \tag{1}$$

We look for solutions in the form:

$$\begin{aligned}
G_{\alpha\beta} &= G^{(0)}_{\alpha\beta} + \frac{1}{q} G^{(1)}_{\alpha\beta} + \frac{1}{q^2} G^{(2)}_{\alpha\beta} + \ldots, \\
\Sigma_{\alpha\beta} &= \Sigma^{(0)}_{\alpha\beta} + \frac{1}{q} \Sigma^{(1)}_{\alpha\beta} + \frac{1}{q^2} \Sigma^{(2)}_{\alpha\beta} + \ldots.
\end{aligned}$$

1. Find the solutions of saddle point equations (1) for times small enough where the $1/q$-expansion is applicable
2. Find an approximate solutions in the late time regime
3. Glue the two regimes together, use the smoothness conditions to establish global existence of solutions.

Generalities of large q: order q^0

EOM:

$$\partial_{t_1} G_{LL}^{(0)} = \partial_{t_1} G_{RR}^{(0)} = \delta(t_1 - t_2)\,;$$
$$\partial_{t_1} G_{LR}^{(0)} = \partial_{t_1} G_{RL}^{(0)} = 0\,;$$
$$\Sigma_{\alpha\beta}^{(0)} = 0\,.$$

Solutions:

$$\begin{aligned} G_{LL}^{(0)} &= G_{RR}^{(0)} = G_f(t_1 - t_2) = \frac{1}{2}\mathrm{sgn}(t_1 - t_2)\,; \\ G_{LR}^{(0)} &= -G_{RL}^{(0)} = C\,. \end{aligned}$$

- Disconnected part of SFF: replica-diagonal solution $C = 0$
- Connected part of SFF: replica-nondiagonal solution $C = \frac{i}{2}$.

Generalities of large q: order q^{-1}

Disconnected part of SFF. Define a new variable $g_{\alpha\beta}(t_1, t_2)$:

$$g_{\alpha\alpha}(t_1, t_2) := \frac{G^{(1)}_{\alpha\alpha}(t_1, t_2)}{G^{(0)}_{\alpha\alpha}(t_1, t_2)}\,; \quad g_{\alpha\beta}(t_1, t_2) := G^{(1)}_{\alpha\beta}(t_1, t_2)\,, \quad \alpha \neq \beta\,.$$

In this case the saddle point equations can be written as

$$\partial_{t_1}\partial_{t_2}\left(\mathrm{sgn}(t_1 - t_2) g_{\alpha\alpha}(t_1, t_2)\right) = 2\mathcal{J}^2 \mathrm{sgn}(t_1 - t_2) \mathrm{e}^{g_{\alpha\alpha}(t_1, t_2)}\,; \tag{2}$$

$$\partial_{t_1}\partial_{t_2} g_{\alpha\beta}(t_1, t_2) = 0,\ \alpha \neq \beta\,. \tag{3}$$

Connected part of SFF.

$$g_{\alpha\beta}(t_1, t_2) := \frac{G^{(1)}_{\alpha\beta}(t_1, t_2)}{G^{(0)}_{\alpha\beta}(t_1, t_2)}\,.$$

Saddle point equation:

$$\partial_{t_1}\partial_{t_2}\left[G^{(0)}_{\alpha\beta}(t_1, t_2) g_{\alpha\beta}(t_1, t_2)\right] = -s_{\alpha\beta}\mathcal{J}^2 \left(2G^{(0)}_{\alpha\beta}\right)^{q-1} \mathrm{e}^{g_{\alpha\beta}(t_1, t_2)}\,. \tag{4}$$

Slope solution at early times

Large-q ansatz:

$$G_{\alpha\alpha}(t_1,t_2) = \frac{1}{2}\mathrm{sgn}(t_1-t_2)\left(1+\frac{g_{\alpha\alpha}(t_1,t_2)}{q}+o\left(\frac{1}{q}\right)\right),$$
$$G_{\alpha\beta}(t_1,t_2) = \frac{g_{\alpha\beta}(t_1-t_2)}{q}+o\left(\frac{1}{q}\right), \quad \alpha\neq\beta$$

General translation-invariant solution:

$$\begin{aligned} e^{g_{\alpha\alpha}(t)} &= \frac{a_\alpha^2}{\mathcal{J}^2\cosh^2(a_\alpha|t|+b_\alpha)}; \\ g_{LR}(t) &= g_{RL}(t) = d+ct\,. \end{aligned}$$

The general solution is not periodic in T. We solve on the segment $t\in[0,T/2]$ and then continue the solution to the segment $[T/2,T]$ using the condition

$$G_{\alpha\beta}(t) = G_{\beta\alpha}(T-t)\,,$$

Slope solution at early times

Boundary conditions for $t \in \left[0, \frac{T}{2}\right]$:

$$g_{\alpha\alpha}(0) = 0\,, \quad g'_{\alpha\alpha}\left(\frac{T}{2}\right) = 0\,;$$

$$g_{LR}\left(\frac{T}{2}\right) = 0\,.$$

Taking them into account, the solution reads

$$\begin{aligned} \mathrm{e}^{g_{\alpha\alpha}(t)} &= \left\{ \frac{\cosh \frac{\tilde{a}_\alpha}{2}}{\cosh\left[\tilde{a}_\alpha\left(\frac{1}{2} - \frac{t}{T}\right)\right]} \right\}^2 ; \\ g_{LR}(t) &= -g_{RL}(t) = c\left(t - \frac{T}{2}\right) , \end{aligned} \tag{5}$$

where $\tilde{a}_\alpha := a_\alpha T$ has to solve a constraint

$$\tilde{a}_\alpha = \mathcal{J}\, T \cosh \frac{\tilde{a}_\alpha}{2}\,.$$

Slope solution at late times

From EOM $\Sigma_{\alpha\beta}$ varies q times faster than $G_{\alpha\beta}$ (Maldacena, Qi), and we have $\Sigma_{LR} = \Sigma_{RL} = 0$. $\Sigma_{\alpha\alpha}$ are odd functions $\Rightarrow$ at late times we can approximate

$$\Sigma_{\alpha\alpha}(t) = \mu_\alpha \delta'(t)$$

EOM are written as

$$(1 - \mu_{L,R})\partial_t G_{\alpha\beta} = 0\,.$$

The solutions:

$$G_{\alpha\alpha}(t) = A_\alpha\,; \qquad G_{LR}(t) = -G_{RL}(t) = C\,.$$

Gluing smoothly to the early-time solution (5)-(6), we get that

- The $1/q$-expansion for the disconnected part of the spectral form factor is valid at all times.
- The ansatz with $G_{LR}^{(0)} = 0$ on the saddle point equations yields $G_{LR} = G_{RL} = 0$.

Action on the slope

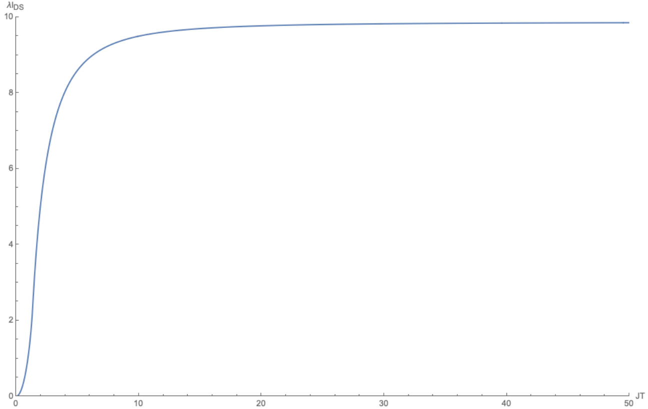
Double-scaled effective action:

$$I_{\mathrm{DS}}[g] = \frac{1}{4\lambda}\int_0^T\int_0^T dt_1 dt_2 \Bigg(\sum_{\alpha\neq\beta} \partial_{t_1} g_{\alpha\beta}(t_1,t_2)\partial_{t_2} g_{\alpha\beta}(t_1,t_2) + \mathcal{J}^2 \mathrm{e}^{g_{\alpha\alpha}(t_1,t_2)} + \frac{1}{4}\sum_{\alpha} \partial_{t_1}\left(\mathrm{sgn}(t_1-t_2) g_{\alpha\alpha}(t_1,t_2)\right)\partial_{t_2}\left(\mathrm{sgn}(t_1-t_2) g_{\alpha\alpha}(t_1,t_2)\right)\Bigg).$$

On the solutions $\lambda I_{\mathrm{DS}} = \sum_{\alpha=L,R}\left(2\tilde{a}_\alpha \tanh\left(\frac{\tilde{a}_\alpha}{2}\right) - \frac{\tilde{a}_\alpha^2}{2}\right)$.

The saddle points with the smallest Re $\tilde{a}$ give the leading contribution.

Action on the slope



Complex saddles

$\tilde{a}_\alpha = \mathcal{J} T \cosh \frac{\tilde{a}_\alpha}{2}\,, \quad \text{Re } \tilde{a} > 0$

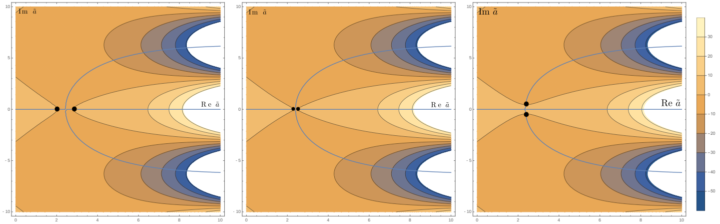


Figure: Transition from real to complex leading solutions at $T_{\rm cr}$. Shown are the contour plots of the real part of the constraint (6) together with the contour line given by Im(consraint)$= 0$.

- $T < T_{\rm cr}$: Two real positive solutions $\tilde{a}_1$ and $\tilde{a}_2$, with $\tilde{a}_1 < \tilde{a}_2$. At $T = T_{\rm cr}$ they coalesce into a unique solution. Also an infinite family of complex solutions with Re $\tilde{a} > \tilde{a}_2$.
- $T > T_{\rm cr}$: Two complex-valued solutions, $\tilde{a}_1$ and $\tilde{a}_2 = \tilde{a}_1^*$. Other complex solutions have Re $\tilde{a} >$ Re $\tilde{a}_1$.

Complex saddles at late times

For $\mathcal{J} T \gg 1$ we can solve the constraint explicitly:

$$\tilde{a} = 2\frac{(2n+1)\pi}{\mathcal{J} T} \pm \left((2n+1)\pi - 4\frac{(2n+1)\pi}{(\mathcal{J} T)^2} \right) i + o\left(\frac{1}{(\mathcal{J} T)^2} \right) , \quad n \in \mathbb{Z}_+ .$$

The on-shell action reads

$$I_{\rm DS} = \frac{\pi^2}{2} \left((2n+1)^2 + (2m+1)^2 \right) + O\left(\frac{1}{(\mathcal{J} T)^2} \right) , \quad n, m \in \mathbb{Z}_+ ;$$

$$\text{Im}\, \lambda I_{\rm DS} = \pm \mathcal{J} T \pm \mathcal{J} T + O\left(\frac{1}{\mathcal{J} T} \right) ,$$

The leading contribution is given by the $n = m = 0$ saddles.

Phase transition and replica symmetry breaking on the slope

Let us denote as $I[\tilde{a}_L, \tilde{a}_R]$ the on-shell action for a generic solution.

$$\begin{aligned} X[\tilde{a}_1] := \text{Re } I[\tilde{a}_1, \tilde{a}_1^*] &= \text{Re } I[\tilde{a}_1^*, \tilde{a}_1] = \text{Re } I[\tilde{a}_1, \tilde{a}_1] = \text{Re } I[\tilde{a}_1^*, \tilde{a}_1^*]\,; \\ Y[\tilde{a}_1] := \text{Im } I[\tilde{a}_1, \tilde{a}_1] &= -\text{Im } I[\tilde{a}_1^*, \tilde{a}_1^*]\,. \end{aligned}$$

A pair of leading complex-conjugated solutions parametrized by $\tilde{a}_1$ will give the contribution to the spectral form factor

$$\begin{aligned} S(T) &\to \frac{1}{4}\left[\mathrm{e}^{-I[\tilde{a}_1,\tilde{a}_1^*]} + \mathrm{e}^{-I[\tilde{a}_1,\tilde{a}_1^*]} + \mathrm{e}^{-I[\tilde{a}_1,\tilde{a}_1]} + \mathrm{e}^{-I[\tilde{a}_1^*,\tilde{a}_1^*]}\right] = \\ &= \mathrm{e}^{-X[\tilde{a}_1]} \cos^2 \frac{Y[\tilde{a}_1]}{2}\,. \end{aligned}$$

AT $T = T_{\text{cr}}$ a phase transition happens with $Y[\tilde{a}_1] = 0$.
At late times we get the semiclasscal result:

$$|\langle Z(iT)\rangle|^2 \simeq \mathrm{e}^{-\frac{\pi^2}{\lambda}} \cos^2 \frac{\mathcal{J} T}{\lambda}\,.$$

Similar to GUE slope (Cotler et. al., Gharbyan et. al.)

Example solutions for the slope

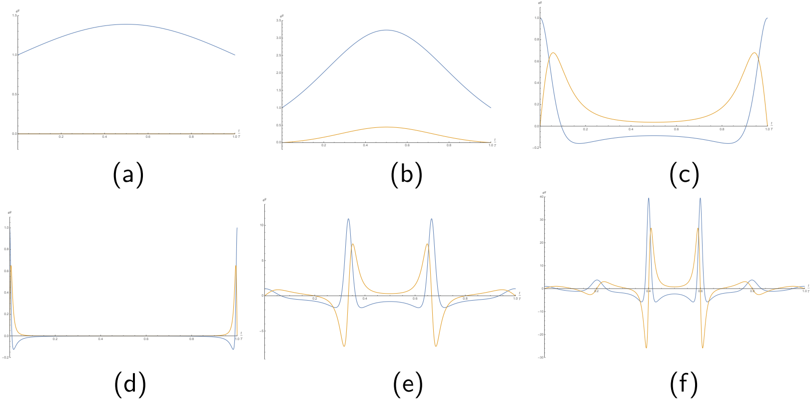


Figure: The solution (5) for different times and different solutions of the constraint (6). (a) $\mathcal{J}T = 1$, $\tilde{a} \simeq 1.18$; (b) $\mathcal{J}T = 1.33$, $\tilde{a} \simeq 2.4 + 0.17i$; (c) $\mathcal{J}T = 10$, $\tilde{a} \simeq 0.60 + 3.03i$; (d) $\mathcal{J}T = 100$, $\tilde{a} \simeq 0.06 + 3.14i$; (e) $\mathcal{J}T = 10$, $\tilde{a} \simeq 1.65 + 9.18i$; (f) $\mathcal{J}T = 10$, $\tilde{a} \simeq 2.45 + 15.44i$

1-loop correction to the slope

We split $g_{\alpha\beta}$ into classical and quantum parts: $g_{\alpha\beta} = g^{cl}_{\alpha\beta} + \mathfrak{g}_{\alpha\beta}$.
New time variables $u = \mathcal{J}(t_1 - t_2)$ and $v = \mathcal{J}\frac{t_1+t_2}{2}$ The one-loop correction is given by the determinant

$$\frac{1}{\det\left(\partial_u^2 - \frac{1}{4}\partial_v^2\right)} \prod_{\alpha=L,R} \left[\det\left(\mathcal{L}_\alpha\right)\right]^{-\frac{1}{2}},$$

where the differential operator is given by

$$\mathcal{L}_\alpha = \frac{1}{2}\mathrm{sgn}(u)\partial_u^2\mathrm{sgn}(u) - \frac{1}{8}\mathrm{sgn}(u)\partial_v^2\mathrm{sgn}(v) + \mathcal{J}^2 \mathrm{e}^{g^{cl}_{\alpha\alpha}(u)}$$

We can write down the determinant as the specific product over eigenvalues:

$$\frac{1}{\det(\mathcal{L}_\alpha)} = \prod_{\substack{n\in 2\mathbb{Z} \\ m_\alpha \in \mathcal{M}^e}} \prod_{\substack{n\in 2\mathbb{Z}+1 \\ m_\alpha \in \mathcal{M}^o}} \frac{2}{\left(\frac{\pi n}{\mathcal{J}T}\right)^2 + \frac{m_\alpha^2}{\mathcal{J}^2}}.$$

1-loop correction to the slope

At late times we can find m_α explicitly:

$$\mathcal{M}^e : m_\alpha \simeq \frac{i\pi 2k}{T}, \quad k \in \mathbb{Z}_+$$
$$\mathcal{M}^o : m_\alpha \simeq \frac{i\pi(2k-1)}{T}, \quad k \in \mathbb{Z}_+,$$

Then for the one-loop correction we get (using zeta-function regularization)

$$\frac{1}{\det(\mathcal{L}_L \mathcal{L}_R)} = \prod_{\substack{n\in 2\mathbb{Z} \\ k\in 2\mathbb{Z}_+}} \prod_{\substack{n\in 2\mathbb{Z}+1 \\ k\in (2\mathbb{Z}+1)_+}} \left(\frac{\mathcal{J}T}{\pi}\right)^2 \frac{2}{n^2-k^2} = \frac{\text{const}}{(\mathcal{J}T)^3}$$

Matches the Schwarzian result. (Cotler et. al., Stanford, Witten)

Spectral form factor on the slope at late times

$$S(T)_{\text{slope}} \sim \frac{1}{(\mathcal{J}T)^3} \cos^2 \frac{\mathcal{J}T}{\lambda} \mathrm{e}^{-\frac{\pi^2}{\lambda}} \, .$$

Ramp solution at early times

Large-q ansatz:

$$G_{\alpha\alpha}(t_1,t_2) = \frac{1}{2}\mathrm{sgn}(t_1-t_2)\left(1+\frac{g_{\alpha\alpha}(t_1,t_2)}{q}+o\left(\frac{1}{q}\right)\right),$$
$$G_{LR}(t_1,t_2) = \frac{i}{2}\left(1+\frac{g_{LR}(t_1,t_2)}{q}+o\left(\frac{1}{q}\right)\right),$$
$$G_{RL}(t_1,t_2) = -\frac{i}{2}\left(1+\frac{g_{RL}(t_1,t_2)}{q}+o\left(\frac{1}{q}\right)\right).$$

EOM:

$$\partial_t^2\left(\mathrm{sgn}(t)g_{\alpha\alpha}(t)\right) = -2\mathcal{J}^2\mathrm{sgn}(t)\mathrm{e}^{g_{\alpha\alpha}(t)},$$
$$\partial_t^2 g_{\alpha\beta}(t) = -2\mathcal{J}^2\mathrm{e}^{g_{\alpha\beta}(t)}, \quad \alpha\neq\beta.$$

The general solution:

$$e^{g_{LL}(t)} = \frac{a_{LL}^2}{\mathcal{J}^2\cosh^2(a_{LL}|t|+b_{LL})}; \quad e^{g_{RR}(t)} = \frac{a_{RR}^2}{\mathcal{J}^2\cosh^2(a_{RR}|t|+b_{RR})}; \tag{6}$$

$$e^{g_{LR}(t)} = \frac{a_{LR}^2}{\mathcal{J}^2\cosh^2(a_{LR}t+b_{LR})}; \quad e^{g_{RL}(t)} = \frac{a_{RL}^2}{\mathcal{J}^2\cosh^2(a_{RL}t+b_{RL})}. \tag{7}$$

Ramp solution at early times

We again impose the Dirichlet condition on the diagonal components

$$g_{\alpha\alpha}(0) = 0\,.$$

It implies the constraint

$$a_{\alpha\alpha} = \mathcal{J} \cosh b_{\alpha\alpha}.$$

The solution of the form (6)-(7) on its own cannot be smoothly continued to the entire segment $[0, T]$.
$\Rightarrow$ Breakdown of the double-scaled limit

Ramp solution at late times

Assume that at late times

$$\Sigma_{LR}(t) = -\Sigma_{RL}(-t) \simeq -i\nu\delta(t), \quad \nu \equiv i\int_{-\infty}^{\infty} dt \Sigma_{LR} = \frac{2a_{LR}}{q}\text{sgn}(\text{Re}\, a_{LR}).$$

Then EOM reduce to

$$\begin{aligned}
\partial_t G_{LL} + i\nu G_{RL} = 0\,; \\
\partial_t G_{LR} + i\nu G_{RR} = 0\,; \\
\partial_t G_{RR} - i\nu G_{LR} = 0\,; \\
\partial_t G_{RL} - i\nu G_{LL} = 0\,.
\end{aligned}$$

The solution:

$$G_{LL} = G_{RR} = A\cosh[\nu(T/2 - t)], \quad G_{RL} = -G_{LR} = -iA\sinh[\nu(T/2 - t)]. \tag{8}$$

Extrapolating the $1/q$-expansion to late times

Expanding (6), (7) at late times and (8) at early times gives

$$G_{LL} \sim \frac{1}{2} - \frac{1}{q}\left(\frac{1}{2}\log\left(\frac{\mathcal{J}}{2a_{LL}}\right)^2 + b_{LL} + a_{LL}t\right) = A\cosh\frac{\nu T}{2} - \nu t A\sinh\frac{\nu T}{2},$$

$$iG_{RL} \sim \frac{1}{2} - \frac{1}{q}\left(\frac{1}{2}\log\left(\frac{\mathcal{J}}{2a_{RL}}\right)^2 + b_{RL} + a_{RL}t\right) = A\sinh\frac{\nu T}{2} - \nu t A\cosh\frac{\nu T}{2},$$

which introduces a set of relations between the parameters. Resolving them gives the new condition:

$$\sigma = qe^{-\nu T} = \text{const}\,.$$

Therefore **we have to scale the time as $T \sim q\log q$ in order for the smooth replica-nondiagonal solution to exist**.

Full ramp solution

Early times:

$$
\begin{aligned}
e^{g_{RR}(t)} &= e^{g_{LL}(t)} = \frac{\cosh^2 b}{\cosh^2(\mathcal{J}(\cosh b)|t| + b)}\,; \\
e^{g_{LR}(t)} &= e^{g_{RL}(t)} = \frac{\cosh^2 b}{\cosh^2(\mathcal{J}(\cosh b)t + b + \sigma)},
\end{aligned}
$$

Late times:

$$
\begin{aligned}
G_{RR} &= G_{LL} = \mathrm{e}^{-\frac{\nu T}{2}} \cosh[\nu(T/2 - t)]\,; \\
G_{RL} &= -G_{LR} = -i\mathrm{e}^{-\frac{\nu T}{2}} \sinh[\nu(T/2 - t)]\,; \quad \nu = \frac{2\mathcal{J}\cosh b}{q}.
\end{aligned}
$$

Free parameters: $b \in \mathbb{R}$, $\sigma \in \mathbb{R}$.

The solution only exists for times $T \sim q \log q$ or later.

Action on the ramp

We evaluate the full action under assumption of the late time solutions

$$\frac{I[G,\Sigma]}{N} =$$

$$-\log \mathrm{Pf}[\delta_{\alpha\beta}\partial_t - \hat{\Sigma}_{\alpha\beta}] + \frac{T}{2}\int_{-T}^{T} dt \left(\Sigma_{\alpha\beta}(t)G_{\alpha\beta}(t) - \frac{2^{q-1}\mathcal{J}^2}{q^2} s_{\alpha\beta} G_{\alpha\beta}(t)^q\right),$$

The Pfaffian term:

$$-\frac{1}{2}\mathrm{Tr}\log[\delta_{\alpha\beta}\partial_t - \hat{\Sigma}_{\alpha\beta}] = -\frac{\nu T}{2} + O(\mathrm{e}^{-\nu T}).$$

The polynomial term:

$$\frac{T}{2}\left(1-\frac{1}{q}\right)\int_{-T}^{T} dt\, \Sigma_{\alpha\beta}(t)G_{\alpha\beta}(t) = \frac{\nu T}{2}\left(1-\frac{1}{q}\right) = \frac{\nu T}{2}\left(1-\frac{1}{\sigma}\mathrm{e}^{-\nu T}\right).$$

Thus the full action is zero up to exponentially small corrections:

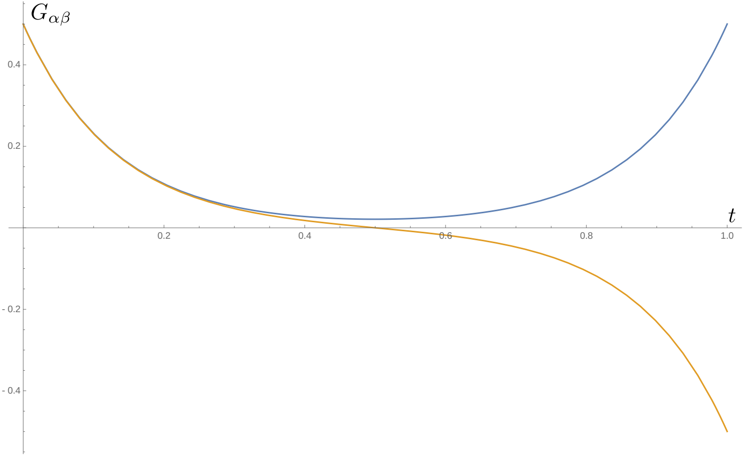
$$I_{\mathrm{DS}} = 0 + O(\nu T\, \mathrm{e}^{-\nu T}).$$

Spontaneously broken symmetries of the ramp solution

$$\begin{aligned} e^{g_{RR}(t)} &= e^{g_{LL}(t)} = \frac{\cosh^2 b}{\cosh^2(\mathcal{J}(\cosh b)|t| + b)}\,; \\ e^{g_{LR}(t)} &= e^{g_{RL}(t)} = \frac{\cosh^2 b}{\cosh^2(\mathcal{J}(\cosh b)t + b + \sigma)}, \end{aligned}$$

- $b \to -b$
- Parity: $G^{(0)}_{\alpha\beta} = \pm\frac{i}{2} \to \mp\frac{i}{2}$
- Time translations: $G_{LR}(t) \to G_{LR}(t+\Delta)$.

Example of the ramp solution



1-loop correction to the ramp

One-loop correction to the replica-nondiagonal saddle:

$$S(T) = \mathrm{e}^{-I_{\rm DS}} \int \mathcal{D}\mathfrak{G}_{\alpha\beta} \exp\left\{ \frac{N}{4}\left[\frac{1}{2}\int d\tau dT' \, \mathfrak{G}_{\alpha\beta}(\tau, T') \left(\frac{1}{4}\partial^2_{T'} - \partial^2_\tau\right) \mathfrak{G}_{\alpha\beta}(\tau, T') + \right.\right.$$
$$\left.\left. + \mathcal{J}^2 \int d\tau dT' \, \left[2G^{\rm cl}_{\alpha\beta}(\tau)\right]^q \left(G^{\rm cl}_{\alpha\beta}(\tau)\right)^{-2} s_{\alpha\beta}\mathfrak{G}^2_{\alpha\beta}(\tau, T')\right]\right\}.$$

The eigenvalues are determined by the Dirichlet boundary condition $\Rightarrow$ we can approximate the differential operator by the early-time solution. The determinant is given by

$$\frac{1}{\det(\mathcal{L})} = \prod_{\substack{n\in 2\mathbb{Z} \\ m\in\mathcal{M}}} \prod_{\substack{n\in 2\mathbb{Z}+1 \\ m\in\mathcal{M}}} \frac{2}{\left(\frac{\pi n}{\mathcal{J} T}\right)^2 + \frac{m^2}{\mathcal{J}^2}}$$

One-loop correction to the ramp

The eigenvalues m are solutions of the equation

$$m \tanh\left(\frac{mb}{\mathcal{J}\cosh b}\right) - \mathcal{J}\cosh b \tanh(b) = 0.$$

Note that there is no time dependence. We can estimate

$$\frac{1}{\det(\mathcal{L})} = \prod_{n\in\mathbb{Z}}^{n_0} \frac{2}{\left(\frac{\pi n}{\mathcal{J} T}\right)^2 + h^2(b)} \simeq \prod_{n\in\mathbb{Z}}^{n_0} \frac{2}{h^2(b)} = \text{const} \cdot h^2(b),$$

(assuming the consistency of the cutoff regularization with large q limit).

Result for the ramp

$$S(T)_{\text{ramp}} = 2\int_0^T d\Delta \int_{-\infty}^{+\infty} \mu(b) db \times 2^{-N} = 2 \times 2^{-N} T \int_{-\infty}^{+\infty} \mu(b) db\,.$$

Time scales of chaos at large q

Thouless time:

$$t_{Th} \sim \frac{q}{2\cosh b} \log \frac{q}{\Delta \cosh b} \sim \sqrt{N} \log N \,.$$

Dip time: $S(T)_{\text{ramp}} = S(T)_{\text{slope}}$

$$2^{-N} \mathcal{J} t_d \sim \mathrm{e}^{-\frac{\pi^2}{\lambda}} \cos^2 \frac{\mathcal{J} t_d}{\lambda} \frac{1}{(\mathcal{J} t_d)^3} \Rightarrow t_d \sim \mathrm{e}^{\alpha N}, \quad \alpha > 0.$$

Conclusions

- We have constructed analytic solutions in the large q SYK which correspond to saddle points contributing to the slope and the ramp regions of the spectral form factor, and estimated 1-loop corrections at late times.
- Slope solutions exist at all times and are always valid in the double-scaled limit.
- Ramp solutions only exist for times of order $q \log q$ or later, and we need to go beyond the perturbative $1/q$-expansion to study them.
- Slope region has subleading saddle points. In the ramp regime, all existing replica-nondiagona.l saddles are leading (assuming time translation invariance).
- There is a phase transition on the slope accompanied by the replica symmetry breaking, which generates the RMT-like mild oscillations at intermediate times.
- Obtained hierarchy between the Thouless time and the dip time. The latter is the same as for finite-q SYK, but the former is different.