ISCOs in AdS/CFT

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Gravitational waves

Binary black hole (BH) system with masses $M \gg \mu$ $(\eta = \mu/M \ll 1)$

- Approximate the motion of the secondary BH (μ) as a point particle inspiraling towards the primary BH (M) with outer horizon

$$r_+ = M\left(1 + \sqrt{1 - a^2}\right) \equiv M(1+\epsilon) \qquad a = \hat{a}/M$$

- Consider circular equatorial orbits (absence of radiation)

$$\tilde{E}(\tilde{r}, a) = \frac{E}{\mu} = \frac{1 - 2/\tilde{r} + a/\tilde{r}^{3/2}}{\sqrt{1 - 3/\tilde{r} + 2a/\tilde{r}^{3/2}}} \quad \text{with} \quad \tilde{r} = r/M$$

Classical stability in Kerr

Radial geodesic (equatorial plane) : defines $V_{\rm Kerr}(\tilde{E}, \tilde{L}, a, \tilde{r})$

$$\left(\tilde{r}^2 \frac{d\tilde{r}}{d\tilde{\tau}}\right)^2 = [\tilde{E}(\tilde{r}^2 + a^2) - a\tilde{L}]^2 - \Delta[(\tilde{L} - a\tilde{E})^2 - \tilde{r}^2]$$

Classical stability for equatorial circular orbits requires

$$V'_{\rm Kerr} = 0\,, \qquad V''_{\rm Kerr} > 0$$

Marginal stability defines the innermost stable circular orbit (ISCO)

$$V''_{\rm Kerr}(\tilde{r}_{\rm isco}) = 0 \Rightarrow \tilde{r}_{\rm isco} = 3 + Z_2 - [(3 - Z_1)(3 + Z_1 + 2Z_2)]^{1/2}$$
$$Z_1 = 1 + (1 - a^2)^{1/3}[(1 + a)^{1/3} + (1 - a)^{1/3}]$$
$$Z_2 = (3a^2 + Z_1^2)^{1/2}$$

[Bardeen,Press,Teukolsky]

EMRIs : Kerr picture

EMRI : extremal mass ratio inspiral ($M \gg \mu$)

- As secondary spins around primary, it generates gravitational waves
- Due to energy conservation ($E + E_{\mathrm{GW}} = \text{const}$)

$$\frac{dE}{dt} = \partial_r E \, \frac{dr}{dt} = -\dot{E}_{\mathrm{GW}}$$

($\dot{E}_{\mathrm{GW}}$ computable by solving Teukolsky's equation with secondary source)

- Secondary inspirals towards ISCO, where it will plunge into BH

Main questions today

Are there ISCOs in AdS ? (gravity)

- features/scaling of $r_{\rm isco}$, $E_{\rm isco}$, $L_{\rm isco}$?

What do they mean in the dual CFT ? (field theory)

- Can we identify a dual feature explaining their existence or absence ?
- If we view the localised bulk probes as CFT excitations, we should expect some relation with CFT bootstrap programme. Can we confirm this ? Similarities or differences ?

Point particle in AdS Schwarzschild BH

Background : (d+1)-global AdS Schwarzschild BH

$$ds^2 = -H(r)\,dt^2 + H^{-1}(r)\,dr^2 + r^2 d\Omega^2\,, \quad H(r) = 1 + \frac{r^2}{L^2} - \frac{2M}{r^{d-2}}$$

Action & conserved charges

Spherical symmetry reduces action to a 3d problem

$$S = \frac{1}{2}\int ds\,\left(-H\,\dot{t}^2 + H^{-1}\,\dot{r}^2 + r^2\dot{\phi}^2\right)$$

with constraint

$$-1 = -H\,\dot{t}^2 + H^{-1}\,\dot{r}^2 + r^2\dot{\phi}^2$$

and conserved charges (per unit mass)

$$e = H(r)\,\dot{t}, \quad \ell = r^2\dot{\phi}\,,$$

Classical stability

Effective potential $V(r)$

$$e^2 = \dot{r}^2 + V(r) \quad \text{with} \quad V(r) = \left(1 + \frac{\ell^2}{r^2}\right) H(r)$$

Circular orbits $r = r_\mathrm{o}$

- Critical points $V'(r_\mathrm{o}) = 0$
- Classical stability requires $V''(r_\mathrm{o}) > 0$

ISCOs

Marginally stable : $V''(r_\mathrm{isco}) = 0$

- Any $r_\mathrm{p} < r_\mathrm{isco}$ has $V''(r_\mathrm{p}) < 0 \Rightarrow$ particle plunges into BH

Criticality

Solving $V'(r_\circ) = 0$

$$\ell^2 = \frac{r_\circ^4}{L^2} \frac{(r_\circ^d + (d-2)M\,L^2)}{r_\circ^d - d\,M r_\circ^2}$$

- $r_\circ \to \infty$ and $d > 2$, at fixed M, we recover global AdS ($M = 0$) scaling $r_\circ \simeq \sqrt{\ell\,L}$
- $d > 2$, $\ell^2\,L^2 > r_\circ^4$ (size shrinks at fixed ℓ, i.e. gravity attracts)
- $\ell^2 < 0$, $d = 2$ & $M > 1/2$: BTZ has no circular geodesic orbits [Cruz,Martinez,Pena]
- $r_\circ > r_{\min}$ where $r_{\min}$ corresponds to the light-ring of the BH

$$r_{\min}^d - dMr_{\min}^2 = 0 \quad \Rightarrow \quad r_{\min}^{d-2} = dM$$

Classical stability

$$r_\text{o}\, V''(r_\text{o}) = \frac{6\ell^2}{r_\text{o}^3} + \frac{2r_\text{o}}{L^2} - d(d+1)\frac{2M\,\ell^2}{r_\text{o}^{d+1}} - (d-2)(d-1)\frac{2M}{r_\text{o}^{d-1}} > 0$$

Criticality & marginality

$$8\ell_\text{isco}^2\, r_\text{isco}^{d-2} = d(d+2)\,2M\,\ell_\text{isco}^2 + d(d-2)\,2M\,r_\text{isco}^2$$

Exact d=4

$$r_\text{isco}^2 = 3\frac{2M\,\ell_\text{isco}^2}{\ell_\text{isco}^2 - 2M} \quad \Rightarrow \quad \ell_\text{isco}^2 > 2M$$

Together with criticality [Festuccia,Liu]

$$L^2(\ell_\text{isco}^2 - 2M)^3 = 27\,(2M\ell_\text{isco}^2)^2$$

General lesson

In $d > 2$, $\exists\ \ell_\text{isco}$ corresponding to r_isco such that for $\ell > \ell_\text{isco}$, $\exists$ classical stable circular orbits

Large mass limit

$r_{\rm o} > r_{\rm isco} > r_{\rm min} \sim M^{1/(d-2)}$: large mass limit M/r^{d-2} fixed

$$V'' \approx \frac{2}{L^2}\frac{1}{1-d\,\frac{\hat{M}}{\hat{r}^{d-2}}}\left[4-\frac{\hat{M}}{\hat{r}^{d-2}}\left(d(d+2)\right)\right] \qquad \hat{r}=\frac{r}{L}\,,\quad \hat{M}=\frac{M}{L^{d-2}}$$

ISCO size and angular momentum

$$\frac{r_{\rm isco}}{L} \approx \left(\frac{d(d+2)\hat{M}}{4}\right)^{1/(d-2)} \sim \left(\frac{r_{\rm h}}{L}\right)^{1+2/(d-2)}\,, \quad \frac{\ell_{\rm isco}}{L} \approx \sqrt{\frac{d+2}{d-2}}\,\hat{r}^2_{\rm isco}$$

Scaling with temperature T

$$\frac{r_{\rm h}}{L} \approx \left(\frac{2M}{L^{d-2}}\right)^{1/d}\,, \quad T\,L \approx \frac{d}{2}\frac{r_{\rm h}}{L}$$

e and ℓ scale like

$$e\,,\frac{\ell}{L} \simeq (T\,L)^{2d/(d-2)} > (T\,L)$$

Near-circular orbits

At fixed ℓ : $r = r_\circ + \delta r$ and $e = e_\circ + \delta e$

$$2e_\circ\,\delta e = \left(\frac{d\delta r}{dt}\right)^2\,\dot{t}^2 + \frac{1}{2}V''(r_\circ)\delta r^2$$

Harmonic oscillator with frequency (large M)

$$\omega_{\rm r}\,L \simeq 2 - \frac{d(d+2)}{4}\frac{M}{(\ell\,L)^{d/2-1}} \qquad (d \geq 3)$$

Exact d=2

$$\omega_{\rm r}\,L = 2\sqrt{1-2M}$$

- ℓ independent, vanishes at the threshold of black hole formation, i.e. $M = 1/2$ and imaginary for $M > 1/2$ (unstable)

General CFT remarks

Standard dictionary of quantum numbers (CFT $\leftrightarrow$ bulk)

$$E = \Delta e\,, \quad J = \Delta \frac{\ell}{L}$$

Meaning of ISCO

- $r_{\rm isco}$ separates between plunging & classically stable orbits
- Plunging $\sim$ dynamics of thermalisation
  - See scrambling, CFT quenches, butterfly effect ...
- Semiclassically, bulk excitations can tunnel past potential barrier
  - amplitude $A_{\rm tun} \simeq \exp(-\Delta S)$

Bulk excitations $\sim$ long-lived meta-stable CFT states [Festuccia,Liu]

CFT origin

Proposal : non-perturbative curvature effect on the CFT

Planar AdS black hole

Characterised by a single scale

$$ds^2 = -d\tilde{t}^2(\rho^2 - 2M_0\,\rho^{2-d}) + d\rho^2(\rho^2 - 2M_0\,\rho^{2-d})^{-1} + \rho^2 d\vec{x}_\perp^2$$

Horizon $\rho_{\mathrm{h}}^d = 2M_0 \equiv \gamma$, determines the temperature $T \sim \gamma^{-1/d}$
Energy and linear momentum of bulk particles

$$E \simeq \Delta T \quad J \simeq \Delta T$$

Compared linear scaling with ISCO scaling

$$e\,,\frac{\ell}{L} \simeq (T\,L)^{\frac{2d}{d-2}}$$

CFT origin

CFT perspective

global AdS BHs $\Leftrightarrow$ CFT on sphere (L) at finite T (2 scales)

1. High temperature (large M limit) : physics of flat space $\simeq$ sphere, up to finite volume corrections
2. When zooming into regions of size $\simeq T$: $E \simeq T$
   - as in the planar AdS BH (double scaled limit of the global AdS BH)
   - ISCO size is pushed out of the region captured by planar AdS BH
   - $\nexists$ ISCOs in planar AdS BHs
3. Finite volume is not enough !!
   - BTZ BHs are dual to CFT on a circle : finite volume, no curvature

CFT origin

Previous arguments $\Rightarrow$ curvature is necessary

1. Bulk physics $\Rightarrow$ $\exists$ tunneling with amplitude

$$A_{\rm tun} \simeq \exp(-\Delta S)$$

   S action for a unit mass point particle to overcome the potential barrier

2. Using AdS/CFT, bulk excitations $\Rightarrow$ metastable CFT states
3. Using semiclassical gravity computation, width of the wave function is exponentially suppressed in T
   - Perturbative Feynman diagrams can only reproduce power law behaviour in E, J (T)

Hence, we expect this to be a non-perturbative effect in the curvature

Comparison with Bootstrap (I)

Stable circular orbits correspond to CFT excitations with energy

$$E = \Delta + J - \Delta \frac{M}{L^{d-2}} \left(\frac{\Delta}{J}\right)^{d/2-1} \qquad \text{fix M, large } \ell$$

- Last term reproduces binding energy in [Fitzpatrick,Kaplan,Walters]

Anomalous dimensions of $[\mathcal{O}_1 \mathcal{O}_2]_\ell$

Bootstrap requires operators with conformal dimension

$$\Delta_{BH} + \Delta + J + \mathcal{O}(1/J^\tau) \qquad \tau = d - 2$$

Our corrections are $\mathcal{O}(1/J^{\tau/2})$

- Bootstrap requires $\ell \gg \Delta_1 + \Delta_2$, whereas $\Delta_{BH} \gg J \sim \ell$ for us
- Also, our excitations are meta-stable, not exact conformal dimensions of primary operators

Comparison with Bootstrap (II)

4-pt functions : 2 heavy & 2 light operators

Bulk calculations determining the phase shift [Kulaxizi,Parnachev,..]

To interpret this calculation in terms of conformal dimensions of composite operators, we must transform t-channel into s-channel $\Rightarrow$ corrections $\mathcal{O}(1/J^{\tau/2})$

- Bootstrap assumes double trace operator $\sim$ generalised free field
- Hence, no decay can be observed (by assumption)
- Expectation : to observe the decay one requires to know a very fine-grained density of states capturing the asymptotics of OPE coefficients.
    - Bootstrap estimates an average of these OPE coefficients convoluted with the density of states with the right intermediate quantum numbers

Nearly circular orbits

$$E = \Delta + J + 2k - \Delta \left(1 + \frac{k}{\Delta}\frac{d(d+2)}{4}\right) \frac{M}{L^{d-2}} \left(\frac{\Delta}{J}\right)^{d/2-1} \qquad (d \geq 3)$$

- $M = 0 \;\Rightarrow\; E = \Delta + J + 2k$ matching descendants $\partial_\mu^J \Box^k \mathcal{O}_\Delta$ of the operator $\mathcal{O}_\Delta$
- $M \neq 0$, no descendants (no integer valued) : center of mass motion of the BH describing the Goldstone modes associated with breaking of conformal symmetry by BH

Matching d=2 results on conical defects

$$E = \Delta \sqrt{1-2M} \left(1 + 2\frac{k}{\Delta}\right) + J$$

- Density of states is much lower and bootstrap double trace approximation is better justified

How do we generate these states ?

Superficially,

- Bulk particles following geodesics $\leftrightarrow$ WKB approximations to the wave function of some bulk field $\Phi(\rho, t, \Omega_i)$
- Using AdS/CFT : $\Phi(\rho, t, \Omega_i) \ \leftrightarrow \ \mathcal{O}_\Delta(t, \Omega_i)$

However, the insertion of $\mathcal{O}_\Delta(t, \Omega_i)$ generates an infinite amount of energy ...making the bulk AdS state non-normalizable

HKLL construction

Standard perturbative solution where boundary operator is integrated against some kernel

- High energy modes interfere between different times $\Rightarrow$ not excited
- Integration over boundary time makes them non-local in time
  - Less useful for scattering processes in AdS $\Rightarrow$ Mellin transformation

Alternative regularisation [Takayanagi]

Replace the operator insertion by evolving in Euclidean time ϵ

$$\mathcal{O}_\epsilon(t) = \exp(-\epsilon H)\,\mathcal{O}(t)\exp(\epsilon H)$$

Brief reminder of the issue

$\int d\Omega\,\mathcal{O}_\Delta(\theta,t)|0\rangle$ generates $|\mathcal{O}_\Delta\rangle$ and $\left|(\partial_\mu\partial^\mu)^k\mathcal{O}_\Delta\right\rangle$ $(|\Delta+2k\rangle)$ with amplitudes

$$A_{\Delta+2k} \simeq \langle\Delta+2k|\int d\Omega\,\mathcal{O}_\Delta(\theta,t)|0\rangle$$

$$|A_{\Delta+2k}|^2 \propto \exp((2\Delta-d)\log(k)+\mathcal{O}(1))$$

Individual amplitudes are finite, but convergence of the sum of amplitudes squared requires

$$2\Delta < d-1$$

However, CFT unitarity requires $\Delta \geq \frac{d}{2}-1$

Regularisation in action

$\int d\Omega\, \mathcal{O}_{\Delta,\epsilon}(\theta, t)|0\rangle$ has amplitude $A_{\Delta+2k,\epsilon} \simeq A_{\Delta+2k}\exp(-\epsilon(\Delta+2k))$

Upshot

For $\Delta \gg 1$ and $\epsilon \ll 1$

- $|A_{2k+\Delta,\epsilon}|^2$ is maximised at $k_{\rm max} = \frac{2\Delta - d}{4\epsilon}$
- Regularized operators $\simeq$ states of finite energy $E \simeq \frac{\Delta}{\epsilon}$ and small fluctuations since $\frac{|\delta E|^2}{E^2} \simeq \frac{1}{2\Delta - d}$

These operators are candidates to produce an initial condition for a state at fixed energy, like scattering states

- next : we discuss semiclassical AdS picture and associate a bulk particle in a geodesic
- we discuss global AdS, for simplicity, but similar ideas can be pushed for our circular orbits in global AdS BHs

AdS interpretation

Scalar primary operators of conformal dimension Δ (single trace) [Witten]

$$m = \sqrt{\Delta(\Delta - d)} \simeq \Delta - \frac{d}{2} + \mathcal{O}(\Delta^{-1}) \qquad (L = 1)$$

In some semiclassical regime $\Rightarrow$ AdS point particle of mass m

What are the initial conditions ?

Inserting $\int d\Omega\, \mathcal{O}_{\Delta,\epsilon}(\theta, 0)|0\rangle$ corresponds to

1. Particle must be at rest
   - All amplitudes are simultaneously real $\Rightarrow$ initial state has time reversal symmetry
   - Classically $\Rightarrow$ vanishing velocity
2. s-wave operator $\Rightarrow$ radial infalling geodesic as time evolves

Bulk matching

Classical point particle in global AdS

$$S = m \int d\tau \sqrt{\cosh^2 \rho - \dot{\rho}^2}$$

carries energy

$$E = m \frac{\cosh^2 \rho}{\sqrt{\cosh^2 \rho - \dot{\rho}^2}}$$

At the turning point $\rho = \rho_\star$ ($\dot{\rho} = 0$), and matching CFT energy

$$E = m \cosh \rho_\star \quad \Rightarrow \quad \cosh \rho_\star = \frac{1}{\epsilon} + 1$$

- $\epsilon \simeq$ radial position of the bulk excitation at the turning point

Further physical remarks

- CFT state fluctuations within bulk Compton wavelength λ_c

$$\delta\rho_\star \sinh\rho_\star = \frac{\delta E}{m} \simeq \frac{1}{m}\frac{E}{\sqrt{m}} \Rightarrow 1 \gg \delta\rho_\star \simeq \frac{1}{\sqrt{m}} \gg \lambda_c \simeq \frac{1}{m}$$

  - Bulk localisation in parametrically larger region than λ_c $\Rightarrow$ particle at rest !! $(\Delta \gg 1)$

- CFT amplitude evaluated at $k_{\rm max}$

$$A_{2k_{max}+\Delta} \simeq \exp(m\rho_\star) \simeq \exp(-m\tilde{L}_\star)$$
$$\tilde{L}_\star = \lim_{\rho_\infty\to\infty}\left[\int_{\rho_\star}^{\rho_\infty} ds\,\sqrt{\dot{\rho}^2} - \rho_\infty\right] = -\rho_\star$$

$\rho_\star$ (spacelike geodesic length from the turning point to the origin) $\leftrightarrow$ the negative of a regularized spatial geodesic length from $\rho_\star$ to the AdS boundary at a fixed time $(\dot{\tau} = 0)$

Tunneling interpretation

This requires

1. $A_{2k_{\max}+\Delta} \simeq \exp(-S_{\text{ec}})$
2. Euclidean time evolution ϵ matches the bulk euclidean time between the boundary and the turning point $\rho_\star$

Euclidean action principle

$$S_{\text{ec}} = m \int \sqrt{\dot\rho^2 + \cosh^2\rho\,\dot\tau^2}\,ds$$

Time check

Using gauge $\dot\tau = 1$ and $\rho_\star \gg 1$

$$\int_0^\tau d\tau = \lim_{\rho_\infty\to\infty} \int_{\rho_\star}^{\rho_\infty} \frac{\cosh\rho_\star\,d\rho}{\cosh\rho\sqrt{\cosh^2\rho - \cosh^2\rho_\star}}$$

$$\tau \simeq 2\,e^{-\rho_\star} \lim_{\rho_\infty\to\infty} \int_{\rho_\star}^{\rho_\infty} \frac{e^{2(\rho_\star-\rho)}\,d\rho}{\sqrt{1-e^{2(\rho_\star-\rho)}}} = 2\,e^{-\rho_\star} \simeq \frac{1}{\cosh\rho_\star} = \epsilon$$

Euclidean action

On-shell evaluation requires regularisation

- Cut it off at ρ_∞, subtract the length of the spacelike geodesic from the AdS origin till ρ_∞ at constant $\tau = 0$ and take the limit ρ_∞ afterwards

$$S_{\text{ec}} = \lim_{\rho_\infty \to \infty} \left[\int_{\rho_\star}^{\rho_\infty} d\rho \frac{\cosh(\rho)}{\sqrt{\cosh^2 \rho - \cosh^2 \rho_\star}} - \rho_\infty \right]$$

$$S_{\text{ec}} = -\log(\sinh \rho_\star) \simeq -\rho_\star \equiv \tilde{L}_\star$$

$$\Rightarrow A_{2k_{\text{max}}+\Delta} \simeq \exp(-S_{\text{ec}})$$

Extensions

- States with fixed ℓ

$$\int d\Omega \, Y_\ell(\Omega)\mathcal{O}_{\Delta,\epsilon}(\Omega, t)|0\rangle$$

  - $Y_\ell(\Omega)$ highest weight spherical harmonic (to get a semiclassical state)
  - Bulk initial condition : turning point corresponds to the aphelion (farthest away from the origin)
  - Similar bulk localisation properties
  - Tunneling calculation involves complex geodesics, but it is still related to the WKB wave function

- Thermal case, i.e. black hole background, involves elliptical integrals

Summary

1. Existence & main features of ISCOs in AdS Schwarzschild BHs
2. CFT origin : non-perturbative effect on the curvature of the boundary theory
3. Matched binding energy results + radial binding energy
   - Comparison with bootstrap
4. Operational generation of localised bulk excitations using normalizable CFT operator insertions

Outlook

Existence of ISCOs for more general BHs

1. Rotating and/or charged BHs : near-extremal regime
   - Co-rotating near-extremal Kerr

$$r_{\rm isco} - r_{\rm ext} = 2^{1/3}\epsilon^{2/3} + \dots$$

   Same scaling in RN : universality, physical meaning ?

2. Existence of ISCOs for RN depends on the ratio m/q of the probe
   - Relevance/consequences of the weak gravity conjecture
3. Comparison/lessons/matching with bootstrap programme.
4. New features for dilatonic charged BHs ?
5. Relation between bulk localisation prescription and entanglement wedge zero modes ?