On projectively quotient functors

T.F. ZHURAEV

Abstract. We introduce notions of projectively quotient, open, and closed functors. We give sufficient conditions for a functor to be projectively quotient. In particular, any finitary normal functor is projectively quotient. We prove that the sufficient conditions obtained are necessary for an arbitrary subfunctor $\mathcal{F}$ of the functor $\mathcal{P}$ of probability measures. At the same time, any "good" functor is neither projectively open nor projectively closed.

Introduction

All spaces are assumed to be Tychonoff and all mappings are continuous.

Recall that a covariant functor $\mathcal{F}\colon \mathrm{Comp} \to \mathrm{Comp}$ acting in the category of compact spaces is called *normal* if it has the following normality properties:

- *preserves the empty set and the singletons*, i.e., $\mathcal{F}(\emptyset) = \emptyset$ and $\mathcal{F}(\{1\}) = \{1\}$, where $\{k\}$ ($k \geq 0$) denotes the set $\{0, 1, \ldots, k-1\}$ of nonnegative integers smaller than k. In this notation, $0 = \{\emptyset\}$;
- is *monomorphic*, i.e., for any (topological) embedding $f: A \to X$, the mapping $\mathcal{F}(f): \mathcal{F}(A) \to \mathcal{F}(X)$ is also an embedding;
- is *epimorphic*, i.e., for any surjective mapping $f: X \to Y$, the mapping $\mathcal{F}(f): \mathcal{F}(X) \to \mathcal{F}(Y)$ is surjective;
- *preserves intersections*, i.e., for any family $\{A_\alpha : \alpha \in \mathcal{A}\}$ of closed subsets of a compact space X, the mapping $\mathcal{F}(i): \bigcap\{\mathcal{F}(A_\alpha) : \alpha \in \mathcal{A}\} \to \mathcal{F}(X)$ defined by $\mathcal{F}(i)(x) = \mathcal{F}(i_\alpha)(x)$, where $i_\alpha: A_\alpha \to X$ are the identity embeddings for all $\alpha \in \mathcal{A}$, is an embedding;
- *preserves preimages*, i.e., for any mapping $f: X \to Y$ and an arbitrary closed set $A \subset Y$, the mapping $\mathcal{F}(f \restriction f^{-1}(A))(f^{-1}(A)) \to \mathcal{F}(A)$ is a homeomorphism;
- *preserves weight*, i.e., $w(\mathcal{F}(X)) = w(X)$ for any infinite compact space X;
- is *continuous*, i.e., for any inverse spectrum $S = \{X_\alpha; \pi^\alpha_\beta : \alpha \in \mathcal{A}\}$ of compact spaces, the limit $f: \mathcal{F}(\lim S) \to \lim \mathcal{F}(S)$ of the mappings $\mathcal{F}(\pi_\alpha)$, where $\pi_\alpha: \lim S \to X_\alpha$ are the limiting projections of the spectrum S, is a homeomorphism.

In what follows, we assume that all functors under consideration are monomorphic and preserve intersections. We also assume that all functors preserve nonempty spaces. The latter assumption is not an essential limitation; the only functor it excludes from consideration is the empty functor, i.e., the functor $\mathcal{F}$ that maps any space to the empty set.

Indeed, suppose that $\mathcal{F}(X) = \emptyset$ for some nonempty compact space X. Then $\mathcal{F}(\emptyset) = \mathcal{F}(1) = \emptyset$, because $\mathcal{F}$ is monomorphic. Let Y be an arbitrary nonempty compact space. Consider the constant mapping $f: Y \to 1$. We have $\mathcal{F}(f)(\mathcal{F}(Y)) \subset \mathcal{F}(1) = \emptyset$; therefore, the space $\mathcal{F}(Y)$ is empty, because it is mapped to the empty set. Thus, we have proved that there exists a unique monomorphic functor that does not preserve nonempty spaces.

By exp, we denote the well-known functor of hyperspace of closed subsets. This functor maps every (nonempty) compact space X to the set $\exp(X)$ of all its nonempty closed subsets endowed with the (finite) Vietoris topology (see [5]) and a continuous mapping $f: X \to Y$ to the mapping $\exp(f): \exp(X) \to \exp(Y)$ defined by $\exp(f)(A) = f(A)$.

In this paper, we introduce notions of projectively quotient, open, and closed functors. We give sufficient conditions for a functor to be projectively quotient (Theorem 1). In particular, any finitary normal functor is projectively quotient (Corollary 2). We prove that the sufficient conditions obtained are necessary for an arbitrary subfunctor $\mathcal{F}$ of the functor $\mathcal{P}$ of probability measures (Theorem 2). At the same time, any "good" functor is neither projectively open nor projectively closed (Theorems 3 and 4).

---

[5] Fedorchuk V.V., Filippov V.V., *General Topology: Basic Constructions*, Moscow, Mosk. Gos. Univ., 1988.

The main part

Let $\mathcal{F}\colon \mathrm{Comp} \to \mathrm{Comp}$ be a functor. By $C(X,Y)$, we denote the space of continuous mappings from X to Y with the compact-open topology.

In particular, $C(\{k\},Y)$ is naturally homeomorphic to the kth power Y^k of the space Y; the homeomorphism takes each mapping $\xi\colon \{k\} \to Y$ to the point $(\xi(0),\dots,\xi(k-1)) \in Y^k$.

For a functor $\mathcal{F}$, a compact space X, and a positive integer k, we define the mapping

$$\pi_{\mathcal{F},X,k}\colon C(\{k\},X) \times \mathcal{F}(\{k\}) \to \mathcal{F}(X)$$

by

$$\pi_{\mathcal{F},X,k}(\xi,a) = \mathcal{F}(\xi)(a) \quad \text{for} \quad \xi \in C(\{k\},X) \quad \text{and} \quad a \in \mathcal{F}(\{k\}).$$

When it is clear what functor $\mathcal{F}$ and what space X are meant, we omit the subscripts $\mathcal{F}$ and X and write $\pi_{X,k}$ or π_k instead of $\pi_{\mathcal{F},X,k}$.

According to a theorem of Shchepin ([1], Theorem 3.1), the mapping

$$\mathcal{F}\colon C(Z,Y) \to C(\mathcal{F}(Z),\mathcal{F}(Y))$$

is continuous for any *continuous* functor $\mathcal{F}$ and compact spaces Z and Y. This implies the following assertion.

[1] Shchepin E.V., *Functors and uncountable powers of compact spaces*, Uspekhi Mat. Nauk **36** (1981), no. 3, 3–62.

Proposition 1 ([2]). *If $\mathcal{F}$ is a continuous functor, X is a compact space, and k is a positive integer, then the mapping $\pi_{\mathcal{F},X,k}$ is continuous.*

Let $\mathcal{F}_k$ be the subfunctor of a functor $\mathcal{F}$ defined as follows. For a compact space X, $\mathcal{F}_k(X)$ is the image of the mapping $\pi_{\mathcal{F},X,k}$, and for a mapping $f\colon X \to Y$, $\mathcal{F}_k(f)$ is the restriction of $\mathcal{F}(f)$ to $\mathcal{F}_k(X)$. It is easy to verify that the diagram

$$
(1) \qquad
\begin{array}{ccc}
C(\{k\},X) \times \mathcal{F}(\{k\}) & \xrightarrow{\overline{f}\times \mathrm{id}} & C(\{k\},Y) \times \mathcal{F}(\{k\}) \\
\pi_{X,k}\big\downarrow & & \big\downarrow\pi_{Y,k} \\
\mathcal{F}(X) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(Y),
\end{array}
$$

where $\overline{f}(\xi) = f \circ \xi$, is commutative; therefore, $\mathcal{F}(f)(\mathcal{F}_k(X)) \subset \mathcal{F}_k(Y)$, and $\mathcal{F}_k$ is a functor.

A functor $\mathcal{F}$ is called a *functor of degree n* if $\mathcal{F}_n(X) = \mathcal{F}(X)$ for any compact space X but $\mathcal{F}_{n-1}(X) \neq \mathcal{F}(X)$ for some X.

For a functor $\mathcal{F}$ and an element $a \in \mathcal{F}(X)$, the *support* of a is defined as the intersection of all closed sets $A \subset X$ such that $a \in \mathcal{F}(A)$; it is denoted by $\operatorname{supp}_{\mathcal{F}(X)}(a)$. When it is clear what functor and space are meant, we denote the support of a merely by $\operatorname{supp}(a)$.

---

[2] Basmanov V.N., *Covariant functors, retracts and dimension*, Dokl. Akad. Nauk USSR **271** (1983), 1033–1036.

By definition,

$$f(\operatorname{supp}(a)) \supset \operatorname{supp}(\mathcal{F}(f)(a)) \tag{2}$$

for a continuous mapping $f\colon X \to Y$ and $a \in \mathcal{F}(X)$. Clearly,

$$a \in \mathcal{F}(\operatorname{supp}(a)). \tag{3}$$

If a functor $\mathcal{F}$ *preserves preimages*, then $\mathcal{F}$ preserves supports, i.e.,

$$f(\operatorname{supp}(a)) = \operatorname{supp}(\mathcal{F}(f)(a)). \tag{4}$$

Proposition 2. *For any functor $\mathcal{F}$ and compact space X,*

$$\mathcal{F}_k(X) = \{a \in \mathcal{F}(X) : |\operatorname{supp}(a)| \leq k\}.$$

The definition of support and property (3) imply the following assertion.

Proposition 3. *For a functor $\mathcal{F}$, a compact space X, and a closed subset A of X,*

$$\mathcal{F}(A) = \{a \in \mathcal{F}(X) : \operatorname{supp} a \subset A\}.$$

Chigogidze [3] extended an arbitrary intersection-preserving monomorphic functor $\mathcal{F}\colon \mathrm{Comp} \to \mathrm{Comp}$ over the category Tych of Tychonoff spaces by setting

$$\mathcal{F}_\beta(X) = \{a \in \mathcal{F}(\beta X) : \operatorname{supp}(a) \subset X\}$$

for any Tychonoff space X. If $f\colon X \to Y$ is a continuous mapping of Tychonoff spaces and $\beta f\colon \beta X \to \beta Y$ is the (unique) extension of f over their Stone-Čech compactifications, then (2) implies that

[3] Chigogidze A.Ch., *Extension of normal functors*, Vestnik Mosk. Univ. Ser. I Mat. Mekh. **6** (1984), 40–42.

$$\mathcal{F}(\beta f)(\mathcal{F}_\beta(X)) \subset \mathcal{F}_\beta(Y).$$

Therefore, we can define $\mathcal{F}_\beta(f) = \mathcal{F}(\beta f) \restriction X$, which makes $\mathcal{F}_\beta$ a functor.

Chigogidze proved [3] that, if a functor $\mathcal{F}$ has some normality property, then $\mathcal{F}_\beta$ also has this property (modified when necessary). The definition of the functor $\mathcal{F}_\beta$ implies, in particular, that

$$f(\operatorname{supp}_{\mathcal{F}_\beta(X)}(a)) = \operatorname{supp}_{\mathcal{F}_\beta(Y)} \mathcal{F}_\beta(f)(a) \tag{5}$$

for any preimage-preserving functor $\mathcal{F}\colon \mathrm{Comp} \to \mathrm{Comp}$, continuous mapping $f\colon X \to Y$, and $a \in \mathcal{F}_\beta(X)$. In what follows, we denote both functor $\mathcal{F}\colon \mathrm{Comp} \to \mathrm{Comp}$ and its extension $\mathcal{F}_\beta\colon \mathrm{Tych} \to \mathrm{Tych}$ over the category of Tychonoff spaces by the same symbol $\mathcal{F}$.

For a Tychonoff space X, a functor $\mathcal{F}\colon \mathrm{Comp} \to \mathrm{Comp}$, and a positive integer k, we put

$$\mathcal{F}_k(X) = \pi_{\mathcal{F},\beta X,k}(C(\{k\}, X) \times \mathcal{F}(k))$$

and denote the restriction of $\pi_{\mathcal{F},\beta X,k}$ to $C(\{k\}, X) \times \mathcal{F}(\{k\})$ by $\pi_{\mathcal{F},X,k}$. If $f\colon X \to Y$ is a continuous mapping, then $\mathcal{F}(\beta f)(\mathcal{F}_k(X)) \subset \mathcal{F}_k(Y)$; this is implied by the

[3] Chigogidze A.Ch., *Extension of normal functors*, Vestnik Mosk. Univ. Ser. I Mat. Mekh. **6** (1984), 40–42.

commutativity of diagram (1) for the mapping βf. Therefore, setting $\mathcal{F}_k(f) = \mathcal{F}(\beta f|\mathcal{F}(X))$, we obtain a mapping

$$\mathcal{F}_k(f)\colon \mathcal{F}_k(X) \to \mathcal{F}_k(Y).$$

Thus, we have defined a covariant functor $\mathcal{F}_k\colon \mathrm{Tych} \to \mathrm{Tych}$ that extends $\mathcal{F}_k\colon \mathrm{Comp} \to \mathrm{Comp}$. Proposition 2 implies the following assertion.

Proposition 4. *If $\mathcal{F}\colon \mathrm{Comp} \to \mathrm{Comp}$ is a functor, then $\mathcal{F}_k\colon \mathrm{Tych} \to \mathrm{Tych}$ is a subfunctor of the functor $\mathcal{F}_\beta$, and*

$$\mathcal{F}_k(X) = \mathcal{F}_\beta(X) \cap \mathcal{F}_k(\beta X) \tag{6}$$

for any Tychonoff space X.

Proposition 5 ([1, Proposition 3.11]). *For any compact space X and functor $\mathcal{F}$, the mapping*

$$\mathrm{supp}_{\mathcal{F}(X)} : \mathcal{F}(X) \to \exp X$$

is lower semicontinuous.

Let $U \subset X$ be an open set. Put

$$\mathcal{F}_+(U) = \{a \in \mathcal{F}(X) : \mathrm{supp}(a) \cap U \neq \emptyset\}.$$

---

[1] Shchepin E.V., *Functors and uncountable powers of compact spaces*, Uspekhi Mat. Nauk **36** (1981), no. 3, 3–62.

Proposition 5 is equivalent to the assertion that the set $\mathcal{F}_+(U)$ is open for any open $U \subset X$.

Proposition 6. *For a compact space X, a functor $\mathcal{F}$, and a positive integer k, the set $\mathcal{F}_k(X)$ is closed in $\mathcal{F}(X)$.*

Remark 1. The definition of the set $\mathcal{F}_k(X)$ and Proposition 1 imply that $\mathcal{F}_k(X)$ is closed in $\mathcal{F}(X)$ for any continuous functor $\mathcal{F}$.

Propositions 4 and 6 imply the following assertion.

Proposition 7. *For a Tychonoff space X, a functor $\mathcal{F}$, and a positive integer k, the set $\mathcal{F}_k(X)$ is closed in $\mathcal{F}_\beta(X)$.*

Let us mention several simple but important facts.[6].

Proposition 8. *If $f: X \to Y$ is a closed mapping and $Z \subset Y$, then the mapping*

$$f \restriction f^{-1}(Z): f^{-1}(Z) \to Z$$

is also closed.

Proposition 9. *Let $f: X \to Y$ be a continuous surjective mapping, and let X_1, ..., X_n be closed subsets of X such that*

1. *$X = X_1 \cup \cdots \cup X_n$;*
2. *all $f(X_i)$ are closed in Y;*
3. *all $f \restriction X_i: X_i \to f(X_i)$ are quotient mappings.*

Then the mapping f is quotient.

Proposition 10. *If $f: X \to Y$ is a continuous mapping, X_0 is a subset of $X: f(X_0) = Y$, and $f \restriction X_0$ is quotient, then f is also quotient.*

We say that a functor $\mathcal{F}$ is *finitely open* if the set $\mathcal{F}_k(\{k+1\})$ is open in $\mathcal{F}(\{k+1\})$ for any positive integer k. For example, the *finitary* functors, i.e., the functors $\mathcal{F}$ such that $\mathcal{F}(\{k\})$ are finite for all positive integers k, are finitely open. We say that a functor $\mathcal{F}$ is *projectively quotient* if, for any Tychonoff space X and any positive integer k, the mapping

$$\pi_{\mathcal{F},X,k}: C(\{k\}, X) \times \mathcal{F}(\{k\}) \to \mathcal{F}_k(X)$$

is quotient.

[6] Engelking R., *General Topology*, Warszawa, PWN, 1977.

Theorem 1. *Any continuous finitely open functor $\mathcal{F}\colon \mathrm{Comp} \to \mathrm{Comp}$ preserving the empty set and preimages is projectively quotient.*

Corollary 1. *Any finitary continuous functor* $\mathcal{F}\colon \mathrm{Comp} \to \mathrm{Comp}$ *preserving the empty set and preimages is projectively quotient.*

Corollary 2. *Any finitary normal functor, in particular, the hyperspace functor* exp, *is projectively quotient.*

In relation to Theorem 1 and its corollaries, several questions arise.

The first question is as follows:

Question 1. Is Theorem 1 valid without the assumption that the functor $\mathcal{F}$ is finitely open?

This question is especially important because not all normal functors are finitely open. In particular, the functor $\mathcal{P}$ of probability measures (which is the most interesting normal functor) is not finitely open. Theorem 2 proved below not only gives a negative answer to Question 1, but also characterizes the quotient normal subfunctors of the functor $\mathcal{P}$.

Arbitrary normal subfunctors of the functor $\mathcal{P}$ are described in [4], [5].

Theorem 2. *A normal subfunctor* $\mathcal{F}$ *of the functor* $\mathcal{P}$ *is projectively quotient if and only if* $\mathcal{F}$ *is finitely open.*

---

[4] Fedorchuk V.V., *Probability measures in topology*, Uspekhi Mat. Nauk **46** (1991), no. 1, 41–80.

[5] Fedorchuk V.V., Filippov V.V., *General Topology: Basic Constructions*, Moscow, Mosk. Gos. Univ., 1988.

The second question is also related to Theorem 1:

Question 2. Is Theorem 1 valid without the assumption that $\mathcal{F}$ preserves the empty set?

A negative answer to this question is given by the *metrizable cone functor* $\mathrm{Cone}_{\mathrm{m}}$. Recall that $\mathrm{Cone}_{\mathrm{m}}(X)$ is the cone over the space X such that its vertex

v_x has a countable neighborhood base. For any metrizable space X, $\mathrm{Cone_m}(X)$ is metrizable. Hence, for example, the mapping $\pi_{\mathbb{R},1} : \mathbb{R} \times [0,1] \to \mathrm{Cone}_m(\mathbb{R})$ is not quotient.

The third question is as follows:

Question 3. Is Theorem 1 valid without the assumption that $\mathcal{F}$ preserves preimages?

I do not know the answer to this question. The best known non-preimage-preserving functor is the superextension functor λ, which has all the normality properties except this one.

Question 4. Is the functor λ projectively quotient?

Another group of problems is related to the potential possibility of obtaining stronger properties of the mapping $\pi_{\mathcal{F},X}$ under certain constraints on the functor $\mathcal{F}$. We say that a functor $\mathcal{F}$ is *projectively open* (*closed*) if the mapping $\pi_{\mathcal{F},X,k}$ is open (closed) for any Tychonoff space X and a positive integer k. A functor $\mathcal{F}$ is said to be *finitely nondegenerate* if the set $\mathcal{F}(\{k+1\}) \setminus \mathcal{F}_k(\{k+1\})$ is nonempty for some positive integer k.

Proposition 11. *If $\mathcal{F}$ is a finitely nondegenerate functor preserving preimages, then*

$$\mathcal{F}(\{2\}) \setminus \mathcal{F}_1(\{2\}) \neq \emptyset.$$

Proposition 12. *If $\mathcal{F}$ is a finitely nondegenerate continuous functor preserving preimages and singletons, then $\mathcal{F}_1(X)$ is nowhere dense in $\mathcal{F}_2(X)$ for any nonempty first countable compact space X without isolated points.*

Remark 2. In Theorem 3, the assumption that $\mathcal{F}$ is finitely nondegenerate is essential. As an example, we can take the continuum exponent functor $\exp^{\mathfrak{c}}$ (of hyperspace of subcontinua). It is a finitary functor satisfying all the normality conditions, except it is not epimorphic. For any Tychonoff space X and positive integer k, we have $(\exp^{\mathfrak{c}})_k(X) = (\exp^{\mathfrak{c}})_1(X) = X$ and $C((\{k\}, X) \times \exp^{\mathfrak{c}}(\{k\})) = X^k \times \{k\}$. The mapping $\pi_k = \pi_{\exp^c, X, k}$ is open, because it is the sum of the mappings $\pi_k \restriction X^k \times \{i\}$ of open subspaces $X^k \times \{i\}$, where each $\pi_k \restriction X^k \times \{i\}$ coincides with the projection $X^k \to X$ onto the ith coordinate.

Theorem 4 proved below shows that no "good" functors $\mathcal{F}$ can be projectively closed. As previously, we start with auxiliary statements. Recall that a functor $\mathcal{F}\colon \mathrm{Comp} \to \mathrm{Comp}$ is called a *functor with continuous supports* if, for any compact space X, the mapping

$$\mathrm{supp}_{\mathcal{F}(X)}\colon \mathcal{F}(X) \to \exp X$$

is continuous. Note that, for any Tychonoff space X, the mapping

$$\mathrm{supp}_{\mathcal{F}_\beta(X)}\colon \mathcal{F}_\beta(X) \to \exp X$$

is continuous as the restriction of the continuous mapping $\mathrm{supp}_{\mathcal{F}(\beta X)}$ to $\mathcal{F}_\beta(X)$. In what follows, we denote the mapping $\mathrm{supp}_{\mathcal{F}_\beta(X)}$ by $\mathrm{supp}_{\mathcal{F}(X)}$.

Proposition 13. *For a functor $\mathcal{F}$ with continuous supports, the mapping*

$$\operatorname{supp}_{\mathcal{F}(X)}\colon \mathcal{F}(X) \to \exp X$$

is closed as a mapping onto its image.

Indeed, it suffices to show that $\operatorname{supp}_{\mathcal{F}(X)}$ is the restriction of the closed mapping $\operatorname{supp}_{\mathcal{F}(\beta X)}$ to its full preimage ([6]), i.e.,

$$\operatorname{supp}^{-1}_{\mathcal{F}(X)}(K) = \operatorname{supp}^{-1}_{\mathcal{F}(\beta X)}(K)$$

for any compact $K \subset X$. But this follows from the definition of the set

$$\mathcal{F}(X) = \mathcal{F}_\beta(X) \subset \mathcal{F}(\beta X).$$

Theorem 4. *No preimage-preserving continuous functor $\mathcal{F}$ with continuous supports is projectively closed.*

[6] Engelking R., *General Topology*, Warszawa, PWN, 1977.

thank you for your attention