Universal spectra of random Lindblad operators

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in collaboration with

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Spectral analysis belongs to the heart of quantum physics

H = system Hamiltonian

$$H^\dagger = H$$

$$H|\psi\rangle = E|\psi\rangle$$

energy spectrum $E \in \mathbb{R}$

von Neumann spectral theorem

$$A^\dagger A = A A^\dagger$$

$$A = \sum_k \lambda_k P_k$$

$$\lambda_k \in \mathbb{C}$$

General case

$$A \in \mathcal{M}_N$$

$$A = \sum_k (\lambda_k P_k + N_k)$$

$$\lambda_k \in \mathbb{C}$$

$$P_k P_l = \delta_{kl} P_l$$

$$N_k P_l = P_l N_k = \delta_{kl} N_k \ ; \quad N_k - \text{nilpotent}$$

A is **diagonalizable** if all $N_k = 0$.

- hydrogen atom
- harmonic oscillator
- other highly symmetric systems
- ...
- in general it is HARD (approximations, numerical methods,...)

What about the spectrum of a random matrix?

Gaussian random $N \times N$ matrices

$$A_{ij} \in \mathbb{C}$$

independent, identically distributed (i.i.d.) Gaussian variables

$$\mathcal{N}(0, \sigma^2) \ \ ; \ \sigma^2 = \frac{1}{N}$$

$$P(A) \ \propto \ \exp\left(-\frac{\beta}{2}\frac{\mathrm{Tr} A A^\dagger}{\sigma^2}\right) \ ; \ \ \beta > 0$$

- Wigner's approach to nuclear physics
- Dyson's statistical theory of spectra
- classical and quantum chaos
- 2D quantum gravity
- large N limit of gauge theory
- communication systems
- finances
- ...

RMT – Hermitian case

Dyson threefold way

$$P(A) \propto \exp\left(-\frac{\beta}{2}\frac{\mathrm{Tr} AA^\dagger}{\sigma^2}\right) \;\; ; \;\; \beta > 0$$

Hermitian over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Q}$

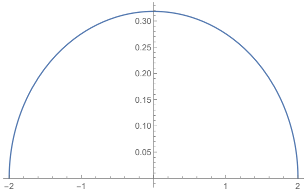
- $\beta = 1$ — Gaussian Orthogonal Ensemble (**GOE**)
- $\beta = 2$ — Gaussian Unitary Ensemble (**GUE**)
- $\beta = 4$ — Gaussian Symplectic Ensemble (**GSE**)

RMT – Hermitian case

in the large N limit

Wigner semicircle distribution

$$\rho(x) = \frac{1}{2\pi}\sqrt{4 - x^2}\ ; \qquad |x| \leq 2$$



- quantum channels
- Lindblad generators

- **random** quantum channels
- **random** Lindblad generators

- Random Matrix Model
- free probability

Quantum channel

$$\Phi : \mathcal{M}_N \to \mathcal{M}_N$$

- completely positive
- trace-preserving

$$\Phi(X) = \sum_\alpha K_\alpha X K_\alpha^\dagger$$

$$\sum_\alpha K_\alpha^\dagger K_\alpha = \mathbb{1}$$

$$\Phi \quad \to \quad \mathbf{C} = \sum_{i,j=1}^{N} |i\rangle\langle j| \otimes \Phi(|i\rangle\langle j|) \geq 0$$

Spectra of quantum channels

$$\Phi(X) = \lambda X$$

vectorisation: $X = \sum_{i,j} x_{ij} |i\rangle\langle j| \rightarrow \quad |X\rangle\!\rangle := \sum_{i,j} x_{ij} |i \otimes j\rangle$

$$\widehat{\Phi}|X\rangle\!\rangle = \lambda |X\rangle\!\rangle$$

$$\Phi(X) = \sum_{\alpha} K_{\alpha} X K_{\alpha}^{\dagger} \quad \rightarrow \quad \widehat{\Phi} = \sum_{\alpha} K_{\alpha} \otimes \overline{K_{\alpha}}$$

$$\boxed{\text{spectrum of } \Phi = \text{spectrum of } \widehat{\Phi}}$$

Perron-Frobenius theorem: stochastic matrices

S_{ij} – $N \times N$ stochastic matrix

$$S_{ij} \geq 0 \; ; \quad \sum_{i=1}^{N} S_{ij} = 1$$

Theorem

- $\lambda \in \mathrm{spec}(S) \;\Leftrightarrow\; \bar{\lambda} \in \mathrm{spec}(S)$
- *there is a leading eigenvalue* $\lambda_0 = 1$
- *the corresponding eigenvector defines probability distribution*
- $|\lambda_k| \leq 1$ *for* $k = 1, 2, \ldots, N-1$.

Perron-Frobenius theorem: (C)PTP maps

$$\Phi : M_N \to M_N$$

- $\lambda \in \mathrm{spec}(\Phi) \;\Leftrightarrow\; \overline{\lambda} \in \mathrm{spec}(\Phi)$
- there is a leading eigenvalue $\lambda_0 = 1$
- the corresponding eigenvector defines a density operator
- $|\lambda_k| \leq 1$ for $k = 1, 2, \ldots, N^2 - 1$.

Markovian semigroup

$$\frac{d}{dt}\Lambda_t = \mathcal{L}\Lambda_t$$

Theorem (Gorini-Kossakowski-Sudarshan-Lindblad (1976))

$\Lambda_t = e^{t\mathcal{L}}$ *is CPTP if and only if*

$$\mathcal{L}(\rho) \;=\; -i[H,\rho] + \mathcal{L}_D(\rho)$$

$$\mathcal{L}_D(\rho) = \sum_k \gamma_k \left(V_k \rho V_k^\dagger - \frac{1}{2}\{V_k^\dagger V_k, \rho\} \right) \;;\; \gamma_k > 0$$

Spectra of generators

$$\mathcal{L} : \mathcal{M}_N \to \mathcal{M}_N$$

Theorem

- $\ell \in \mathrm{spec}(\mathcal{L}) \;\Leftrightarrow\; \bar{\ell} \in \mathrm{spec}(\mathcal{L})$
- *there is a leading eigenvalue* $\ell_0 = 0$
- *the corresponding eigenvector defines a density operator*
- $\mathrm{Re}\,\ell_k \leq 0$ *for* $k = 1, 2, \ldots, N^2 - 1.$

Spectra of generators

$$\mathcal{L} = \mathcal{L}_\phi + \sum_{\mathrm{Re}\ell_k<0} (\ell_k \mathcal{P}_k + \mathcal{N}_k)$$

$$\mathcal{L}_\phi = \sum_{\mathrm{Re}\ell_k=0} \ell_k \mathcal{P}_k$$ – “peripheral” part

$$\mathcal{P}_\phi := \sum_{\mathrm{Re}\ell_k=0} \mathcal{P}_k$$ – CPTP projector

Random channels and generators

$$\Phi \quad \longleftrightarrow \quad \mathbf{C} = \sum_{i,j} |i\rangle\langle j| \otimes \Phi(|i\rangle\langle j|)$$

$$\mathbf{C} \geq 0 \quad ; \quad \mathrm{Tr}_2 \mathbf{C} = \mathbb{1}$$

$$\mathbf{G} \in \mathcal{M}_{N^2} \quad ; \quad \mathbf{G}_1 = \mathrm{Tr}_2 \mathbf{G}\mathbf{G}^\dagger$$

$$\mathbf{C} = (\mathbf{G}_1^{-1/2} \otimes \mathbb{1})\, \mathbf{G}\mathbf{G}^\dagger\, (\mathbf{G}_1^{-1/2} \otimes \mathbb{1})$$

random $\Phi \quad \longleftrightarrow \quad$ **random** G

random Φ $\longleftrightarrow$ random $\mathbf{G}$

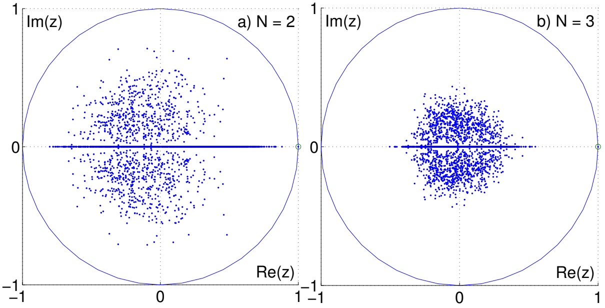
$\mathbf{G} \in N^2 \times N^2$ complex Ginibre ensamble

$$G_{ij} \text{ i.i.d. } \mathcal{N}(0, 1/N^2)$$

$\mathbf{G}\mathbf{G}^\dagger$ – Wishart matrix

$\mathbf{G} \longrightarrow \mathbf{C}$ – random Choi matrix

Spectra of random quantum channels



i) the leading eigenvalue $z_1 = 1$ corresponding to the inv. state,
ii) real eigenvalues,
iii) symmetric complex eigenvalues inside the disk $r = |z_2| < 1$.

Bruzda et al Phys. Lett. A 2009

How to explain the spectrum by Random Matrix Theory?

Bloch vector representation of any state ρ

$$\rho = \frac{1}{N}\mathbb{1} + \sum_{\alpha=1}^{N^2-1} x_\alpha \, \sigma_\alpha$$

where σ_α are generators of SU(N) such that $\mathrm{tr}\,(\sigma_\alpha \sigma_\beta) = \delta_{\alpha\beta}$

Generalized Bloch vector $\mathbf{x} = [x_1, \ldots, x_{N^2-1}] \in \mathbb{R}^{N^2-1}$

Quantum channel Φ in the Bloch representation

$$\rho' = \Phi(\rho) \quad \to \quad \mathbf{x}' = \mathbf{T}\mathbf{x} + \mathbf{r},$$

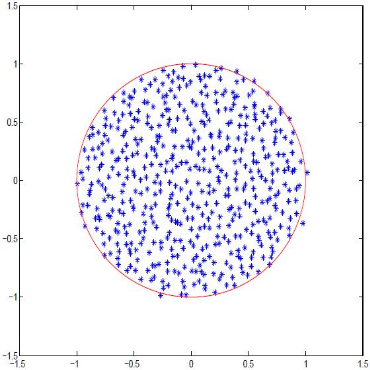
that is,

Φ is represented by $N^2 \times N^2$ real matrix $\left(\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{r} & \mathbf{T} \end{array}\right)$

$$\mathrm{spec}(\Phi) \; = \; \mathrm{spec}(\mathbf{T}) \cup \{1\}$$

Random Matrix Model for $\mathbb{T}$

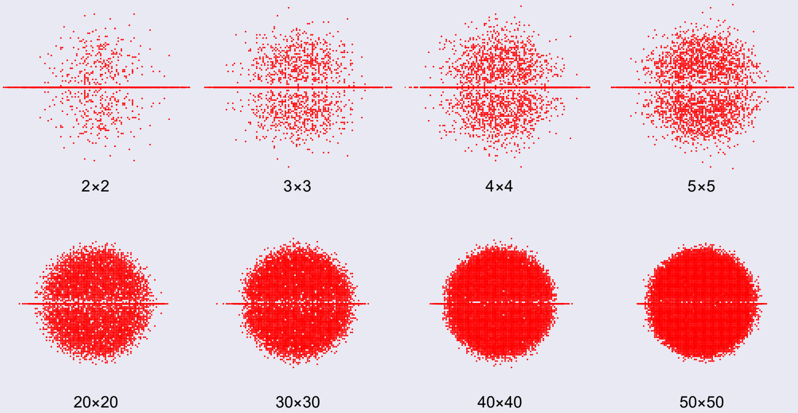
$G \in$ real $n \times n$ Ginibre ensamble ; G_{ij} iid $\mathcal{N}(0, 1/n)$



Tulino, Verdu, "Random matrix theory"

Girko disc; Girko circular law

Girko circular law



from WolframMathWorld

From channels to generators

Markovian semigroup

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$$\mathcal{L}(\rho) \;=\; -i[H,\rho] + \mathcal{L}_D(\rho)$$

$$\mathcal{L}_D(\rho) = \sum_{k,l=1}^{N^2-1} K_{kl}\left(F_k\rho F_l^\dagger - \frac{1}{2}\{F_l^\dagger F_k, \rho\}\right)$$

$$[K_{kl}] \geq 0\ , \quad \mathrm{Tr}(F_k F_l^\dagger) = \delta_{kl}$$

Spectra of random generators

$$\mathcal{L}(\rho) = -i[H,\rho] + \sum_{k,l=1}^{N^2-1} K_{kl}\left(F_k\rho F_l^\dagger - \frac{1}{2}\{F_l^\dagger F_k,\rho\}\right)$$

Random H ; Random K_{ij}

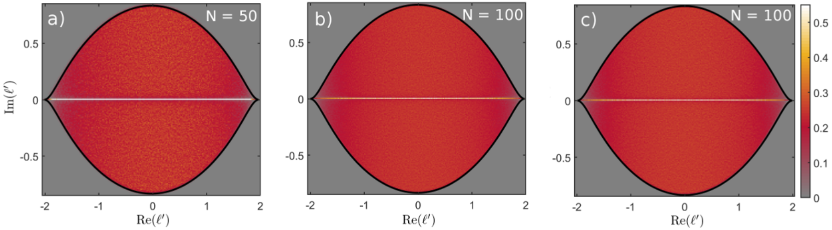
$H \in$ **GUE** ; $K \in$ **random Wishart**

$K = G^\dagger G$; $G \in$ **complex Ginibre**

Spectra of random generators: $H = 0$

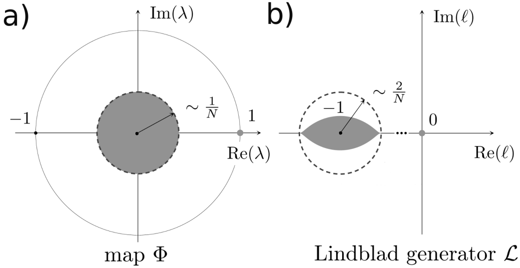
$$\mathcal{L}(\rho) = \sum_{k,l=1}^{N^2-1} K_{kl} \left(F_k \rho F_l^\dagger - \frac{1}{2}\{F_l^\dagger F_k, \rho\} \right)$$

(shifted by $+1$ and rescaled by N)



Spectra of a random map Φ and generator $\mathcal{L}$

a) spectrum of a **random map** Φ consist of the F-P leading eigenvalue, $\lambda_0 = 1$ and a **disk of eigenvalues** of radius $R \sim 1/N$



b) spectrum of a random **Lindblad generator** consists in $\ell_0 = 0$ and a **lemon of eigenvalues**

spectral gap $= 1 - \frac{2}{N}$

Denisov, Lapteva, Tarnowski, DC, Życzkowski, Phys.Rev.Lett. (2019)

Spectra of generators — Random Matrix Model

$$\mathcal{L}(\rho) = \sum_{k=1}^{N^2-1} \left(V_k \rho V_k^\dagger - \frac{1}{2}\{V_k^\dagger V_k, \rho\} \right)$$

vectorization: $\mathcal{L} \;\to\; \widehat{\mathcal{L}}$

$$\widehat{\mathcal{L}} = \widehat{\Phi} - \frac{1}{2}(B \otimes \mathbb{1} + \mathbb{1} \otimes \overline{B})$$

$$\Phi(\rho) = \sum_k V_k \rho V_k^\dagger \; ; \quad B = \sum_k V_k^\dagger V_k = \Phi^\dagger(\mathbb{1})$$

Spectra of random generators — Random Matrix Model

$$\widehat{\mathcal{L}} = \widehat{\Phi} - \frac{1}{2}(B \otimes \mathbb{1} + \mathbb{1} \otimes \overline{B})$$

If Φ is trace preserving, then $\Phi^\dagger$ is **unital**, hence $\widehat{\mathcal{L}} = \widehat{\Phi} - \mathbb{1} \otimes \mathbb{1}$. In the general case the dual map $\Phi^\dagger$ is **not unital**, so

$$\Phi^\dagger(\mathbb{1}) = \mathbb{1} + X$$

where a Hermitian matrix $X = X^\dagger \neq 0$.

$$\widehat{\mathcal{L}} = \widehat{\Phi} - \mathbb{1} \otimes \mathbb{1} - \frac{1}{2}(X \otimes \mathbb{1} + \mathbb{1} \otimes \overline{X})$$

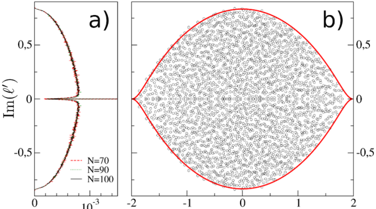
Spectra of random generators — Random Matrix Model

$$\widehat{\mathcal{L}} = \widehat{\Phi} - \mathbb{1} \otimes \mathbb{1} - \frac{1}{2}(X \otimes \mathbb{1} + \mathbb{1} \otimes \overline{X})$$

$$\widehat{\mathcal{L}} \;\to\; \widehat{\mathcal{L}}' = N(\widehat{\mathcal{L}} + \mathbb{1} \otimes \mathbb{1})$$

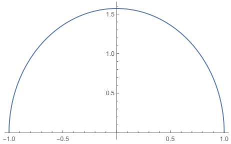
$$\widehat{\mathcal{L}}_{\mathrm{RMT}} = G_R - \mathbb{1} \otimes \mathbb{1} - \frac{1}{2}(C \otimes \mathbb{1} + \mathbb{1} \otimes C)$$

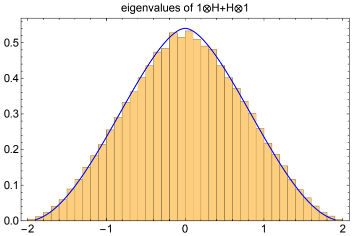
$G_R \in$ **real Ginibre** ; $\quad C \in$ **GOE**



$C \otimes \mathbb{1} + \mathbb{1} \otimes C$

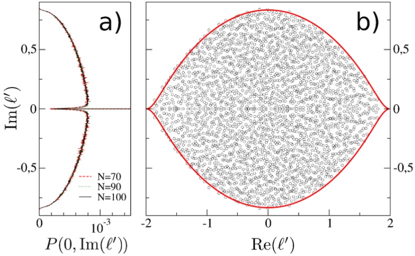
GOE $\rightarrow$ $\rho(x) = \frac{1}{2\pi}\sqrt{1-x^2}$; $|x| \leq 1$ (**Wigner semicircle**)





$G_R - (C \otimes \mathbb{1} + \mathbb{1} \otimes C)$

G_R from **real Ginibre** and $(C \otimes \mathbb{1} + \mathbb{1} \otimes C)$ with C from **GOE** are **asymptotically free**.
Free convolution of Girco disc with classical convolution of two Wigner semicircles gives



density $=?$; boundary $=?$

The boundary of the spectrum of $\mathcal{L}'_{RMT}$ is given by the solution of equation for complex z

$$\mathsf{Im}[z + G(z)] = 0, \quad \text{with}$$

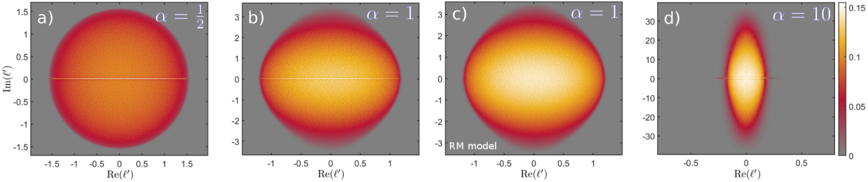
$$G(z) = 2z - \frac{2z}{3\pi}\left[(4+z^2)E\left(\frac{4}{z^2}\right) + (4-z^2)K\left(\frac{4}{z^2}\right)\right],$$

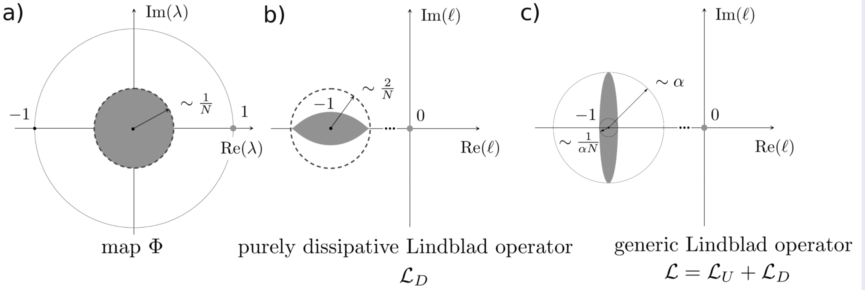
$E(k)$ and $K(k)$ are complete elliptic integrals of the first and second kind.

Adding a Hamiltonian part

$$G_R - \frac{1}{2}(W \otimes \mathbb{1} + \mathbb{1} \otimes \overline{W})$$

$$W = C + i\alpha H \;;\quad C \in \textbf{GOE} \;;\; H \in \textbf{GUE}$$




a)
Im(λ)
−1
∼ 1/N
1
Re(λ)
map Φ
b)
Im(ℓ)
∼ 2/N
−1
0
Re(ℓ)
purely dissipative Lindblad operator
$\mathcal{L}_D$
c)
Im(ℓ)
∼ α
−1
0
∼ 1/(αN)
Re(ℓ)
generic Lindblad operator
$\mathcal{L} = \mathcal{L}_U + \mathcal{L}_D$

Many body interaction (Wang, Piazza, Luitz, 2019)

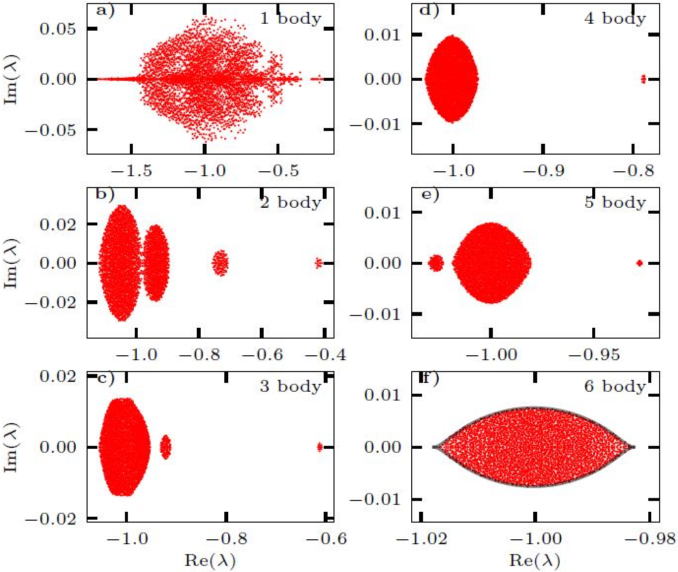
$$\mathcal{L}_D(\rho) = \sum_{i,j} K_{ij} \left(L_i \rho L_j^\dagger - \frac{1}{2}\{L_j^\dagger L_i, \rho\} \right)$$

ℓ qubits

$$L_i \propto \sigma_{x_1} \otimes \sigma_{x_2} \otimes \ldots \otimes \sigma_{x_\ell}$$

$$x_i \in \{0, 1, 2, 3\} \quad ; \quad (\sigma_0 = \mathbb{1})$$

k–body interaction $=$ at most k nonzero x_i


a)
1 body
b)
2 body
c)
3 body
d)
4 body
e)
5 body
f)
6 body
Im(λ)
Re(λ)
0.05
0.00
−0.05
−1.5
−1.0
−0.5
0.02
0.00
−0.02
−1.0
−0.8
−0.6
−0.4
0.02
0.00
−0.02
−1.0
−0.8
−0.6
0.01
0.00
−0.01
−1.0
−0.9
−0.8
0.01
0.00
−0.01
−1.00
−0.95
0.01
0.00
−0.01
−1.02
−1.00
−0.98

- **S. Denisov, T. Laptieva, W. Tarnowski, DC , and K. Życzkowski, Universal Spectra of Random Lindblad Operators, (2019)**
- K. Wang, F. Piazza, and D. J. Luitz, Hierarchy of relaxation timescales in local random Liouvillians, (2019)
- L. Sá, P. Ribeiro, and T. Prosen, Spectral and steady-state properties of random Liouvillians, (2019)
- T. Can, Random Lindblad dynamics, (2019)

Quantum $\to$ classical

Decoherence map

$$D \geq 0 \;\to\; \Phi_D(\rho) := \sum_{k=1}^{N} P_k \rho P_k$$

$$P_k = |k\rangle\langle k|$$

$$\mathcal{L} \;\to\; \mathcal{L}^{\lambda} = (1-\lambda)\mathcal{L} + \lambda \Phi_D \mathcal{L}$$

$\mathcal{L}^0 = \mathcal{L} \quad \mathcal{L}^1 =$ Classical Komlomogorov generator

$$\mathbf{K}_{ij} \geq 0 \; (i \neq j) \;, \quad \sum_{i=1}^{N} \mathbf{K}_{ij} = 0$$

Quantum $\to$ classical

$$\mathcal{L} \;\to\; \mathcal{L}^{\lambda} = (1-\lambda)\mathcal{L} + \lambda \Phi_D \mathcal{L}$$

$\lambda \sim 1/\sqrt{N}$ "phase transition

Needs further analysis

Concluding remarks

- Spectra of random **quantum operation** consist of the leading eigenvalue $\lambda_0 = 1$ and the bulk of eigenvalues covering the Girko disk of radius $r = 1/N$, described by **real Ginibre ensemble**
- Spectral properties determine long-time dynamics: existence of a spectral gap $1 - r$, implies exponential convergence to equilibrium.
- Spectra of random **Lindblad generators** consist of the leading eigenvalue $\ell_0 = 0$ and the bulk of eigenvalues covering the lemon of eigenvalues, so the spectral gap behaves as $1 - 2/N$
- Lemon-like boundary of the bulk was derived using free probability theory. The bulk can be described by convolution of **real Ginibre ensemble** with correction term $C_{\mathrm{GOE}} \otimes 1 + 1 \otimes C_{\mathrm{GOE}}$.
- open problem: density of eigenvalues