# Describable Nuclei

# Negative Translations and

# Extension Stability

Proof Theory/Logic Online Seminar

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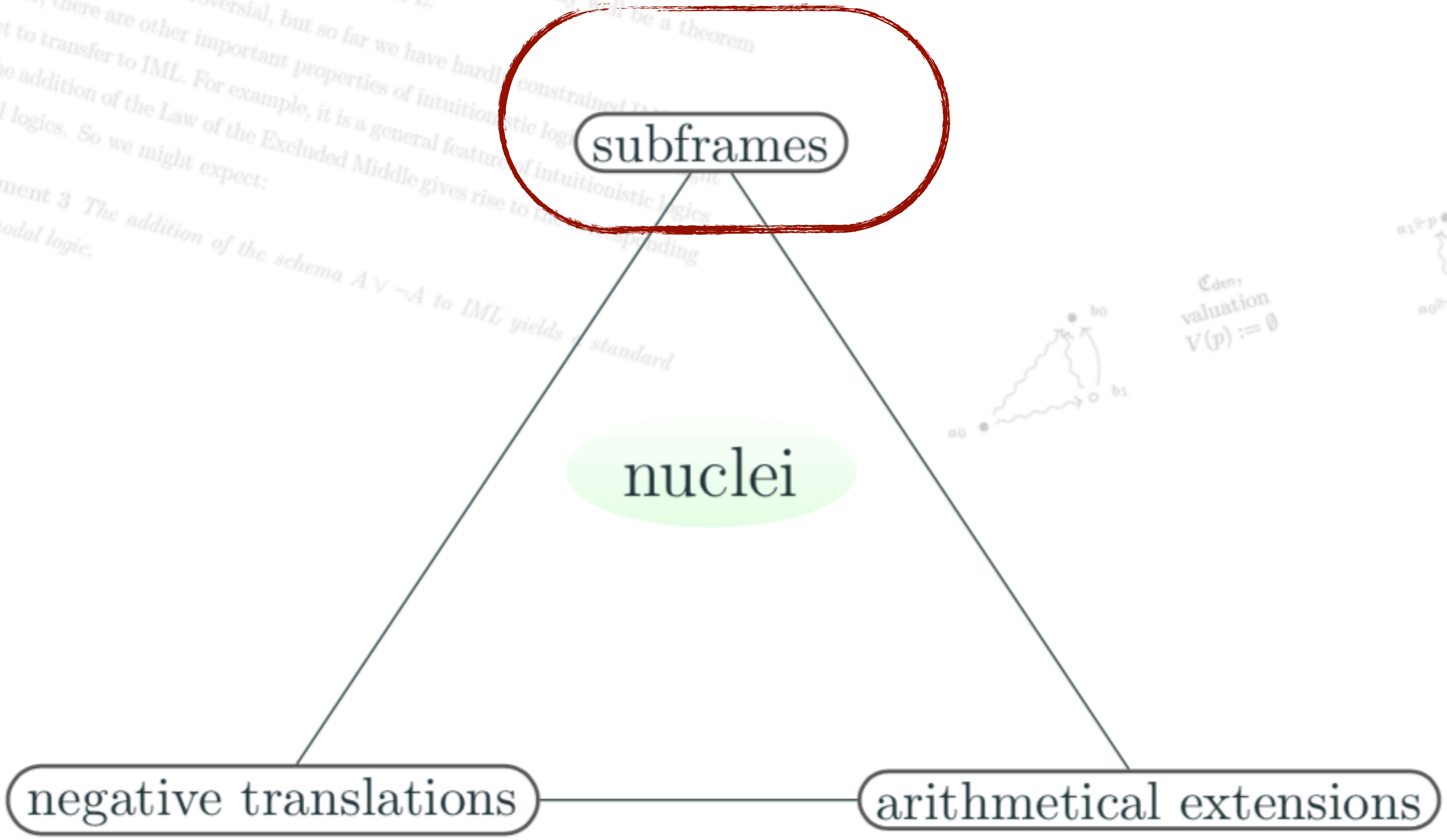
# stability of a logic under . . .

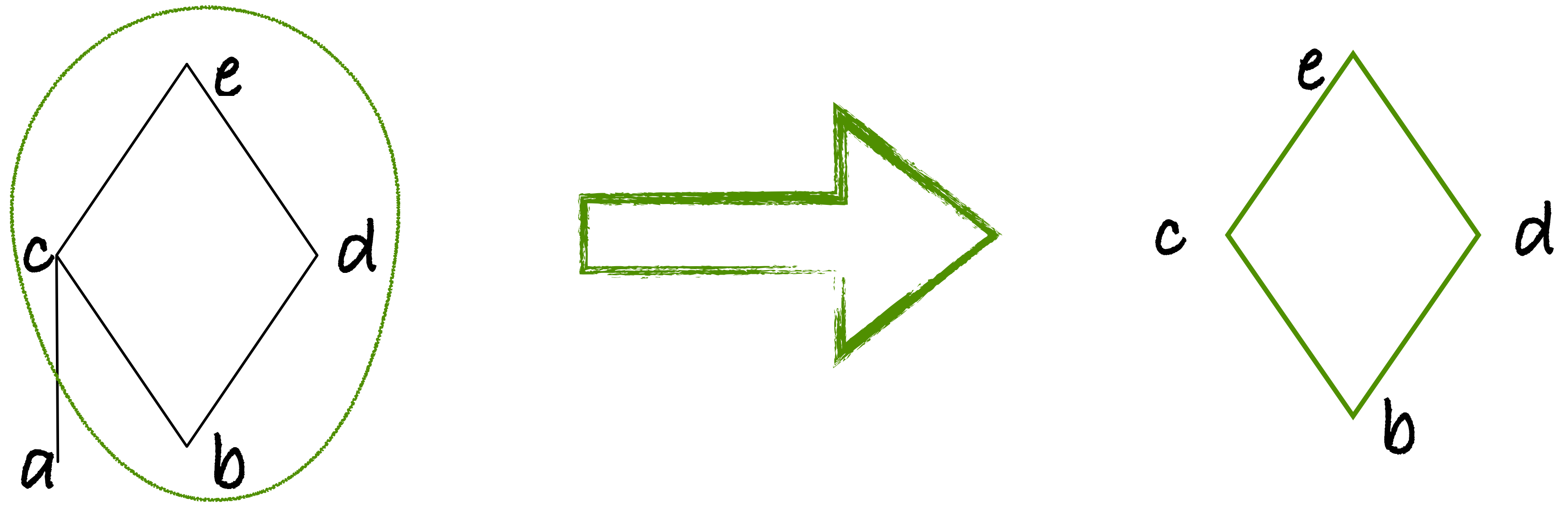


# Reminder of subframe logics I:

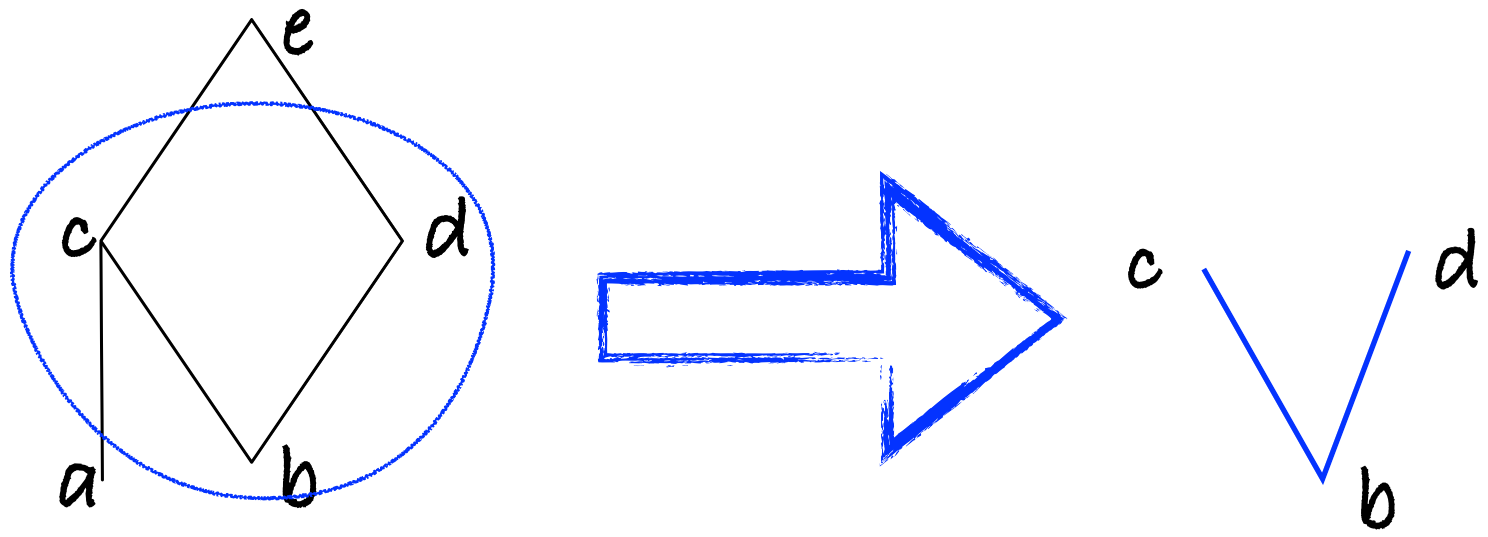
## how trivial nuclei seem over CPC

# Subframes & subcoalgebras classically

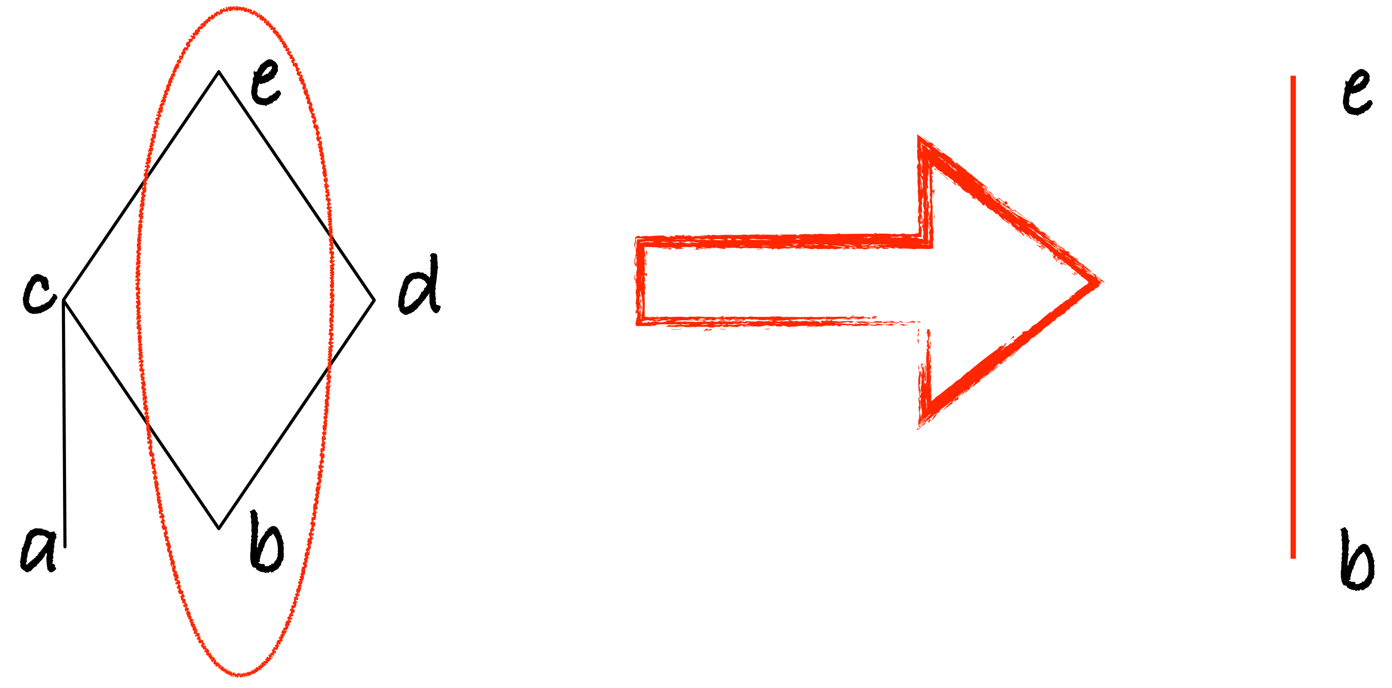
- Generated subframes = Kripke subcoalgebras preserve validity of all modal formulas




- Arbitrary subframes (submodels? substructures?) preserve much less




- Intermediate notions, like (transitive) cofinal subframes (in the set-theoretic sense of cofinality)



- Given a (modal Kripke) frame $\mathfrak{F} = \langle W, R \rangle$, and $U \subseteq W$, the subframe induced by $U$ is
  $\mathfrak{F}_U = \langle U, R\restriction_U \rangle$, where $R\restriction_U = R \cap (U \times U)$

- $\mathfrak{F}_U$ is a generated subframe (or a subcoalgebra) if
  $\forall uw\,.\, u \in U \,\&\, uR^* w \implies w \in U$ in modal notation: $U \subseteq \Box_R U$

- Under an additional assumption that $R$ is transitive, $\forall X\,.\, \Box_R X \subseteq \Box_R \Box_R X$
  $\mathfrak{F}_U$ is a co(n)final subframe if
  $\forall uw\,.\, u \in U \,\&\, uR^* w \implies \exists z\,.\, wR^* z \,\&\, z \in U$ $U \subseteq \Box_R (U \vee \Diamond_R U)$

Recall:

$$\Diamond_R X = \{w \in W \mid \exists v \in X\,.\, wRv\}$$

$$\Box_R X = \{w \in W \mid \forall v \in W\,.\, wRv \Rightarrow v \in X\}$$

$$= W - \Diamond_R (W - X)$$

# Subframe logics, Kripke-style

- Provisional definition, applying only to Kripke-complete logics:
  A logic is (Kripke-)subframe if determined by a class of frames closed under subframes

- The logic of transitive frames K4 given by $\Box\varphi \to \Box\Box\varphi$ is subframe

- Not the case with the opposite density axiom C4 $\Box\Box\varphi \to \Box\varphi$

- Basic model theory explains why:

  - transitivity definable by an universal sentence: $\forall xyz\,.\,(xRy \,\&\, yRz) \Rightarrow xRz$

  - not so with density: $\forall xz \underline{\exists y}\,.\, xRz \Rightarrow (xRy \,\&\, yRz)$

- the logic of confluent quasiorders is cofinal: S4 together with $\Diamond\Box\varphi \to \Box\Diamond\varphi$
  $\forall xyy'\,.\,(xRy \,\&\, xRy') \Rightarrow \exists z\,.\,(yRz \,\&\, y'Rz)$

# But why people cared at all?

- Nice marriage of model-theoretic methods with modal-theoretic ones (selection-of-points type of arguments)
- Covers logics not covered by typical Sahlqvist-style techniques (e.g., the combination of transitivity & Noetherianity)
- And subframe logics do have some nice properties
- E.g. all (weakly) transitive (cofinal) subframe logic have the fmp (finite model property): determined by a class of finite frames

- Some other results and observations taken from Wolter's 1993 PhD:
- A Kripke-subframe logic is complete wrt countable frames
- TFAE for a Kripke-subframe logic:
  - ✳ being determined by an universal class of frames
  - ✳ being determined by an elementary class of frames
  - ✳ being canonical (and a few other related properties)
- An universal class of frames is modal axiomatic iff closed under bounded morphic images and disjoint unions

# But …

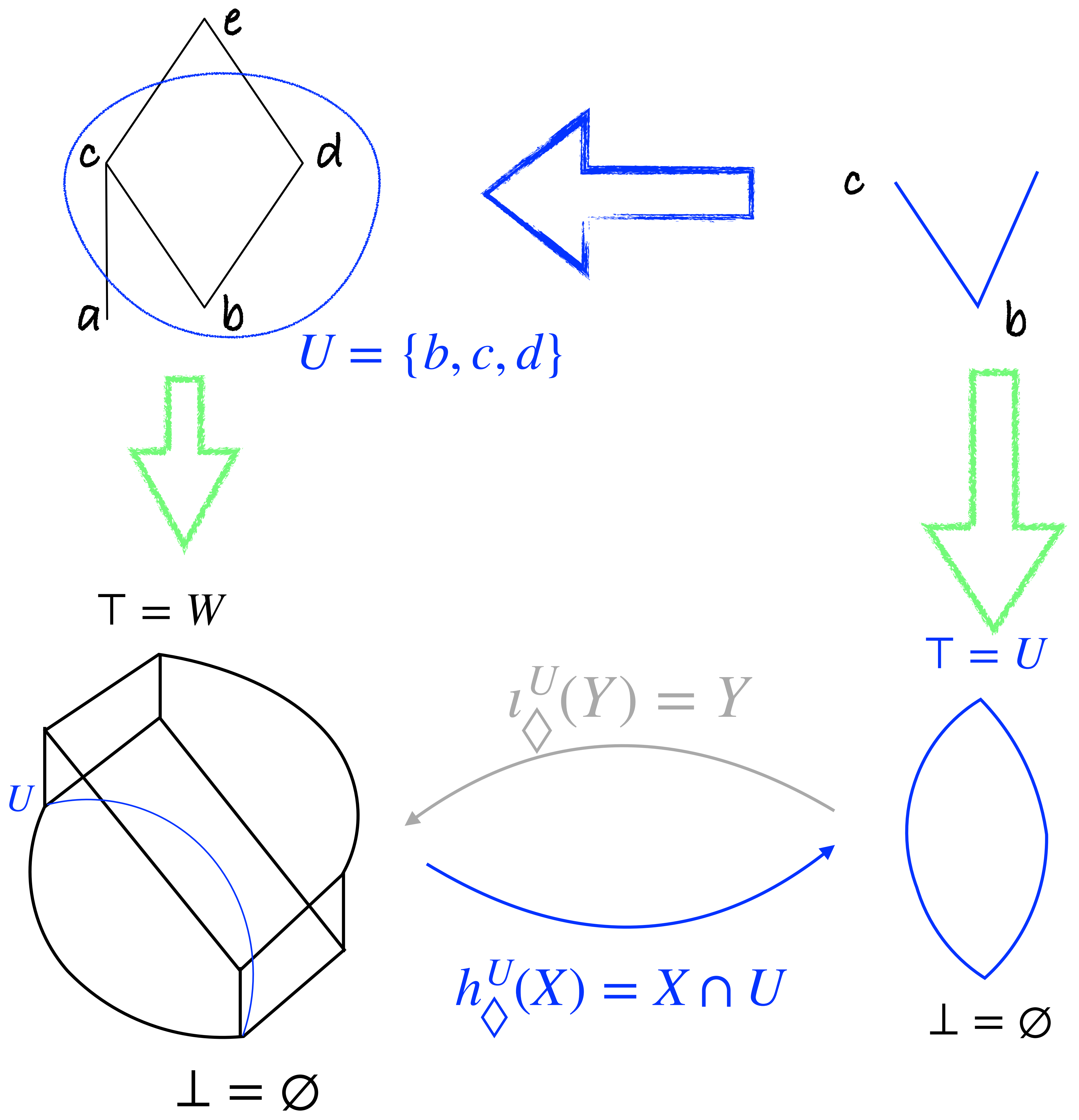
- Our initial restriction to Kripke-complete logics cripples such results
- E.g., the fmp of transitive subframe logics can be stated w/o such an explicit assumption
- And how to generalize even to logics over CPC with different semantics? (topological, neighbourhood, conditional, probabilistic etc. coalgebraic … or the interpretability logic of Peano Arithmetic with its Veltman semantics)
- Furthermore, how to move to other propositional bases?
- Algebra & duality to the rescue!

# ¿ Dually ...

Modal frames

$W = \{a, b, c, d, e\}$

$\mathfrak{F} = \langle W, R \rangle$




$\mathfrak{F}_U = \langle U, R\restriction_S \rangle$

Dual modal algebras

$(\mathfrak{F}^+)_\Diamond = \langle \mathscr{P}(W), \Diamond_R \rangle$

$(\mathfrak{F}^+_U)_\Diamond = \langle \mathscr{P}(U), \Diamond^U_R \rangle$

$\Diamond^U_R Y = U \cap \Diamond_R(U \cap Y)$

$= h^U_\Diamond \Diamond_R(\iota^U_\Diamond Y)$

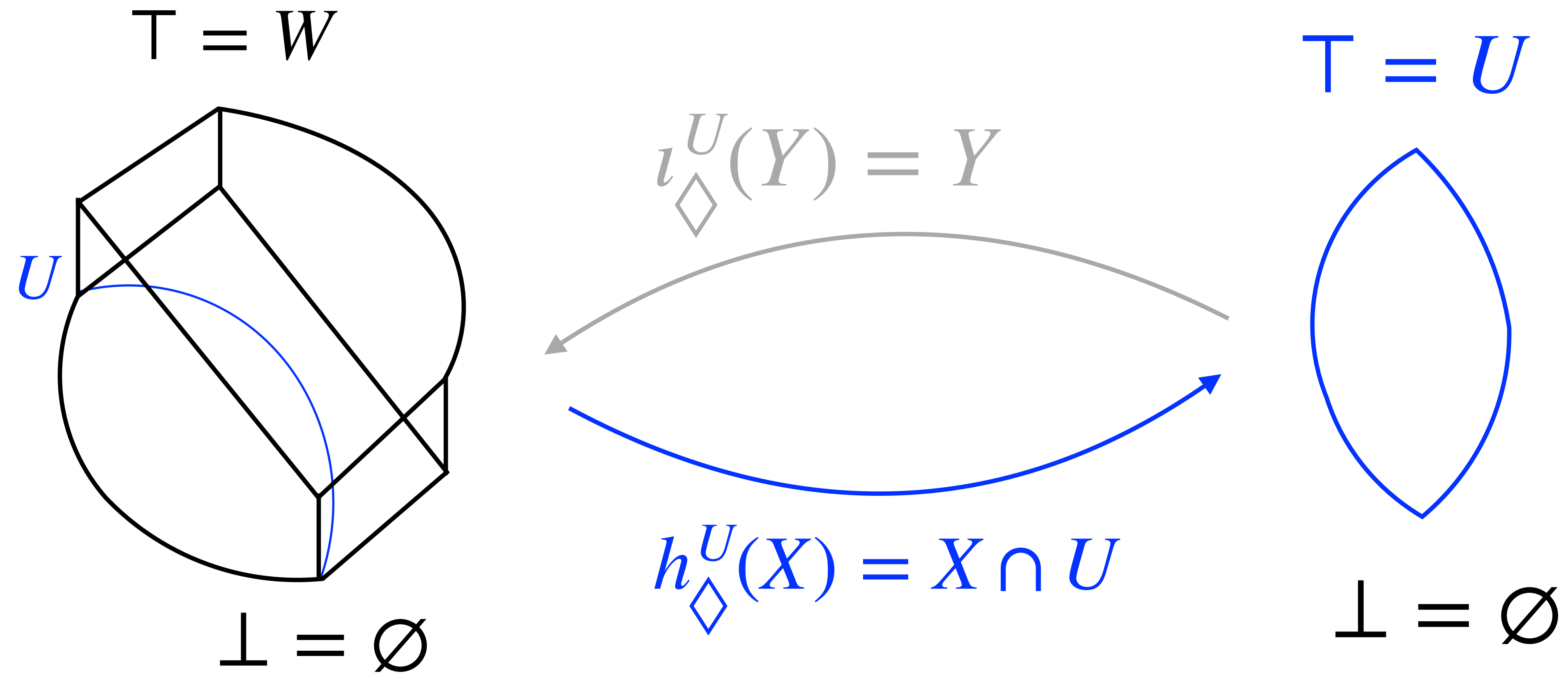
$\iota^U_\Diamond$ not a Boolean morphism and $h^U_\Diamond$ in general not a $\Diamond$-morphism: pick $Y = \{e\}$ to get $h^U_\Diamond(\Diamond_R Y) \neq \Diamond^U_R(h^U_\Diamond Y)$


NYPD

# … or maybe ?

Dual modal algebras

$(\mathfrak{F}^+)_\Diamond = \langle \mathscr{P}(W), \Diamond_R \rangle$




$(\mathfrak{F}^+_U)_\Diamond = \langle \mathscr{P}(U), \Diamond^U_R \rangle$

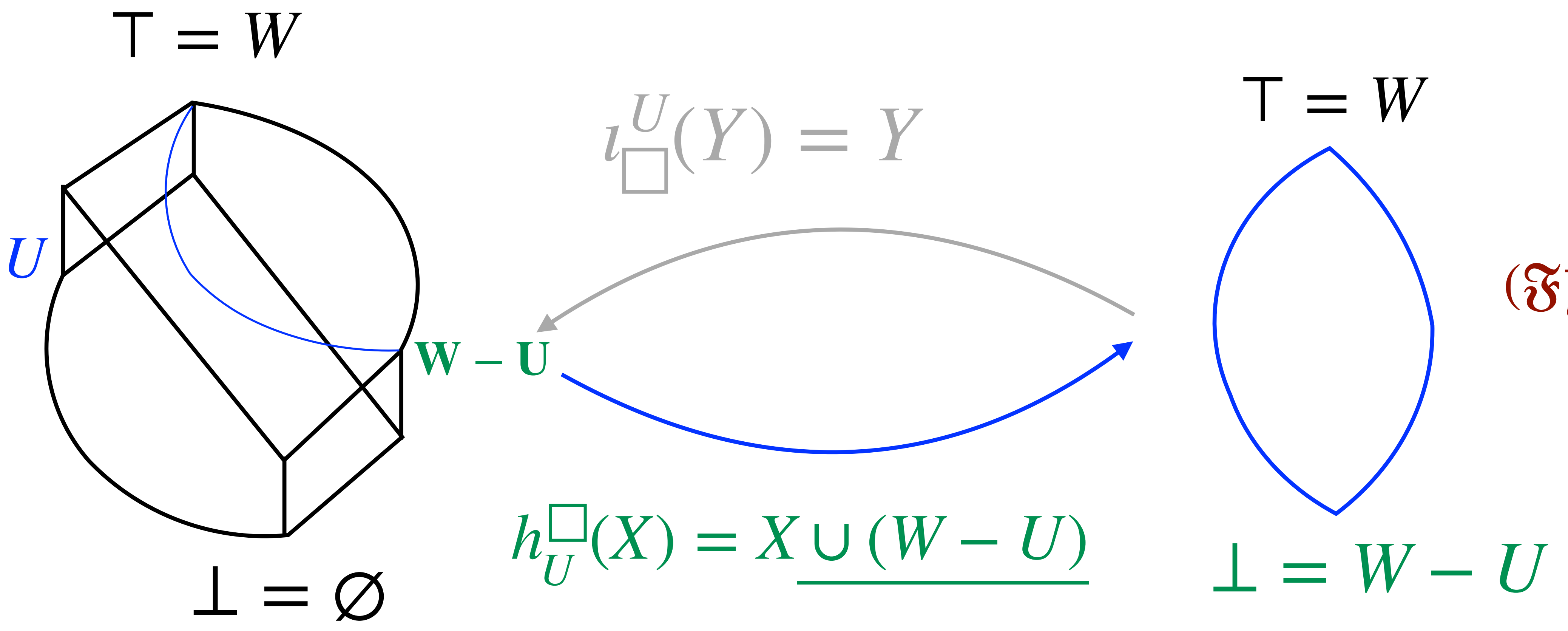
$\Diamond^U_R Y = U \cap \Diamond_R(U \cap Y)$

$= h^U_\Diamond \Diamond_R(\imath^U_\Diamond Y)$

term equivalent

Also dual modal algebras

$(\mathfrak{F}^+)_\Box = \langle \mathscr{P}(W), \Box_R \rangle$




$(\mathfrak{F}^+_U)_\Box = \langle \uparrow_{\mathscr{P}(W)} (W - U), \Box^U_R \rangle$

$\Box^U_R Y = W - \Diamond_R(U - Y)$

$= h^U_\Box \Box_R(\imath^U_\Box Y)$

$h^U_\Diamond$ and $h^U_\Box$ restrict to mutually inverse Boolean isomorphisms between $(\mathfrak{F}^+_U)_\Diamond$ and $(\mathfrak{F}^+_U)_\Box$

# Our first encounter with nuclei

- Given any Boolean (Heyting, distributive ...) algebra $\mathfrak{A}$ and $a \in A$,
  $\mathsf{J}_a : A \to A$ defined as $\mathsf{J}_a(x) = x \vee a$ is a nucleus
  which we can also call a strong monad on a poset category
  which we can also call a multiplicative closure operator
  which we can also call a lax modality

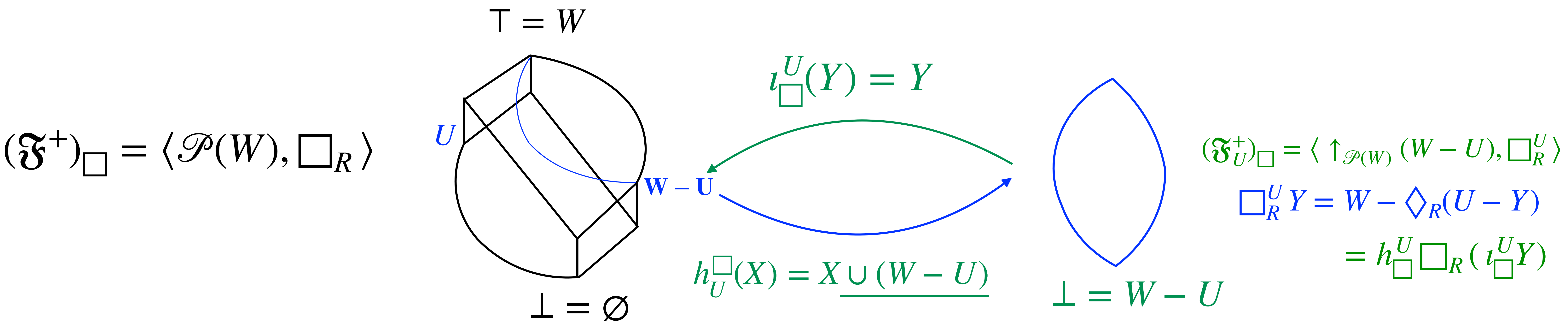
- Axioms: $x \leq j(x)$, $j(x) = j(j(x))$ and $j(x \wedge y) = j(x) \wedge j(y)$

- Boolean algebras are a Kindergarten setting for nuclei:
  any nucleus on a Boolean algebra $\mathfrak{A}$ is of the form $\mathsf{J}_a$ for some $a \in A$
  In Fourman-Scott terminology, any Boolean nucleic quotient is closed
  Note that we could use also the open quotient $\mathsf{J}^a : A \to A$ defined as $\mathsf{J}^a(x) = a \to x$

- For any $\mathfrak{A}$ and any nucleus $j : A \to A$, we can define
  $A_j = \{a \in A \mid j(a) = a\}$ (the collection of fixpoints of $j$)

- Any $n$-ary operation $\heartsuit : A^n \to A$ can be turned into $\heartsuit_j : A_j^n \to A_j$ by
  $\heartsuit_j(c_1, \ldots, c_n) = j(\heartsuit(c_1, \ldots, c_n))$
  (or, strictly speaking, $\heartsuit_j(c_1, \ldots, c_n) = j(\heartsuit(\iota_j(c_1), \ldots, \iota_j(c_n)))$ if the identity embedding $\iota_j : A_j \to A$ made visible)

- We can call $\mathfrak{A}_j$ the nucleic quotient of $\mathfrak{A}$ via $j$



# Subframe logics, for real

- We think of unary modal logic, with $\Box$ as the basic modal primitive

- Abstract algebraic logic (AAL) perspective:
  a logic $\Lambda$ as a set of theorems $\iff$ an equational theory Var($\Lambda$)

- Def: $\Lambda$ is a subframe logic if Var($\Lambda$) is closed under nucleic quotients
  That is, for any $\mathfrak{A} \in$ Var($\Lambda$) and any nucleus $j : A \to A$, $\mathfrak{A}_j \in$ Var($\Lambda$)
  (this definition follows G. Bezhanishvili & S. Ghilardi rather than Wolter)

- Theorem: Kripke-subframe logics are subframe in this sense. (Wolter, I guess)
  For transitive normal modal logics, the converse holds as well. (essentially Fine)
  (G. Bezhanishvili & S. Ghilardi & M. Jibladze: still holds for weak transitivity,
  F. Wolter: … but not for 2-transitivity)

- This definition can be re-used in a non-Boolean setting …

Reminder of subframe logics II:

over IPC, nuclei interesting even w/o modalities

- Syntactically, the intutionistic propositional calculus (IPC) can be seen as the $\Box$-fragment of S4: the modal logic of quasi-orders (via the Gödel-McKinsey-Tarski translation)

- An easy Kripke semantics in terms of upsets of partial orders (upsets do not distinguish quasi-orders and partial orders)

- Under this interpretation, e.g., the cofinal condition of confluence defined by $\neg\varphi \vee \neg\neg\varphi$ (the weak law of excluded middle)

- However, again, the most general semantics is algebraic
- Heyting algebras: bounded lattices where $\wedge$ has right adjoint $\rightarrow$ (hence distributive)
- G. Bezhanishvili & Ghilardi 2007: nuclei on Heyting algebras capture descriptive/Priestley/Esakia subframe constructions

- Recall the construction of $\mathfrak{A}_j$, i.e., the nucleic quotient of $\mathfrak{A}$ via $j$:
  For any $\mathfrak{A}$ and any nucleus $j : A \to A$, we can define
  $A_j = \{a \in A \mid j(a) = a\}$ (the collection of fixpoints of $j$)

- Any $n$-ary operation $\heartsuit : A^n \to A$ is turned into $\heartsuit_j : A_j^n \to A_j$ by
  $\heartsuit_j(c_1, \ldots, c_n) = j(\heartsuit(c_1, \ldots, c_n))$
  (or, strictly speaking, $\heartsuit_j(c_1, \ldots, c_n) = j(\heartsuit(\iota_j(c_1), \ldots, \iota_j(c_n)))$ if the identity embedding $\iota_j : A_j \to A$ made visible

- The only difference now is that we explicitly see the "extensional" connectives ( $\wedge$ , $\vee$ , $\to$ , $\top$ , $\bot$ ) of $\mathfrak{A}_j$ as obtained in the same way, but ...

- As $j(\top) = \top$, $\quad j(j(a) \wedge j(b)) = j(a) \wedge j(b)$ and $j(j(a) \to j(b)) = j(a) \to j(b)$,
  $\mathfrak{A}_j$ is an implicative subsemilattice of $\mathfrak{A}$: we only need to prefix $j$ in front of $\vee$ and $\bot$

- Furthermore, $\mathfrak{A}_j$ obtained this way is a Heyting algebra in its own right!
  But not necessarily satisfying the same equational axioms as the original $\mathfrak{A}$:
  the subframe ones are precisely the safe ones

- Also, as for preservation of $\bot$:
  nuclei satisfying $j(\bot) = \bot$ are called dense

- G. Bezhanishvili & S. Ghilardi show that the (pre-existing) notion of (superintuitionistic) cofinal subframe logics corresponds to preservation by dense nuclei

- Furthermore, this is in turn equivalent to a seemingly stronger property: preservation by locally dense nuclei: those satisfying $j(\neg j(\bot)) = \top$ (correspond to strict lax modalities of Aczel 2001)

# Pleasant results in the pure Heyting signature

- Theorem (Fine, Zakharyaschev):
  - A (locally dense) nuclear superintuitionistic logic/variety has the finite frame/algebra property (in the modal setting, true only in the presence of wK4!)
  - A logic/variety is nuclear iff it is axiomatized by $( \wedge , \rightarrow )$-formulas/identities
  - A logic/variety is (locally) dense nuclear iff it is axiomatized by $( \wedge , \rightarrow , \bot )$-formulas/identities
- Theorem (quite a few good people):
  **TFAE** for a superintuitionistic logic $\Lambda$:
  - $\mathrm{Var}(\Lambda)$ is nuclear
  - $\Lambda$ is axiomatized by NNIL formulas (No Nesting of Implication to the Left)
    "NNIL" is pronounced as "NIL", where the first "N" is pronounced with some slight hesitation - Visser et al. 1995
  - $\Lambda$ is axiomatized by formulas preserved by submodels of Kripke models

# But we also begin to see first problems

- Nucleic quotient of a perfect BAO ($\mathscr{CAV}$-BAO or simply a Kripke algebra) is again the dual of a Kripke frame
- This does not hold anymore in the Heyting setting!
- More issues to follow …

# What happens when more connectives present?

- Intuitionistic modal logics: with box only ...? With diamond(s) too?
- Preservativity in Heyting Arithmetic and its extension?
  (generalized Veltman semantics)
- More broadly: constructive strict implication/Lewis arrow?
  (includes, e.g., the logic of type inhabitation of Haskell arrows?)
- Still more broadly: extensions of Weiss's ICK?
  (Basic Intutionistic Conditional Logic, JPL 2019:
  Chellas-Weiss semantics or generalized Routley-Meyer semantics)
- The logic of bunched implications BI?
  (variants of partial monoid semantics, also topological ones)

# Problems even in the pure Heyting signature

- The lattice of nuclei on a Heyting algebra is quite complex
- Let us look at several standard examples of nuclei, taken from
  - “Sheaves and Logic”, Fourman and Scott 1977
  - “Modal operators on Heyting algebras”, Macnab 1981

- $\mathsf{J}_a\varphi = a \vee \varphi$ (Macnab writes $\mathsf{u}_a$): the closed quotient, dense (identity) for a = ⊥.

- $\mathsf{J}^a\varphi = a \to \varphi$ (Macnab writes $\mathsf{v}_a$): the open quotient, dense (identity) for a = ⊤.

- $\mathsf{B}_a\varphi = (\varphi \to a) \to a$ (Macnab writes $\mathsf{w}_a$): the boolean quotient,
  dense for a = ⊥; even then identity not a special case.
  Denote the dense case as $\mathsf{B}_\bot\varphi = \neg\neg\varphi$ ($\mathsf{w}_\bot$): the double-negation quotient.

- $(\mathsf{J}_a \wedge \mathsf{J}^b)\varphi = (a \vee \varphi) \wedge (b \to \varphi)$: the forcing quotient,
  dense (identity) for a = ⊥.

- $(\mathsf{B}_a \wedge \mathsf{J}^a)\varphi = (\varphi \to a) \to \varphi$: a mixed quotient; identity a special case.