Lower estimates of marginal density

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joint work with Hermann König

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Marginal density bounds

Theorem

Let $X = (X_1, \ldots, X_n) \in \mathbb{R}^n$ be a random vector with i.i.d. coordinates having bounded density $\left\| f_{X_j} \right\|_\infty \leq K$. Then for any $E \subset R^n$ with $\dim(E) = d$,

$$\|f_{P_E X}\|_\infty \leq (CK)^d.$$

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Can one derive a similar lower estimate?

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Can one derive a similar lower estimate?

More precisely: Assume that $f_{X_j}(y) \ge \kappa$ for $|y| \le a$. Is it true that

$$f_{P_E X}(v) \ge \phi(d)\kappa^d \quad \text{whenever } \|v\|_2 \le a?$$

Example: $X_j \sim N(0,1)$. Then $P_E X \sim N(0, I_d)$.

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Example: $X_j \sim N(0,1)$. Then $P_E X \sim N(0, I_d)$.
Counterexample: $X_j = \mathbf{1}_{[-\varepsilon,\varepsilon]} + \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$.
Then $\mathbb{E}X_j = 0$, $\mathbb{E}X_j^2 \approx 1$, but $f_{P_E X}(0) = O(1/\sqrt{n})$ if $E = \text{span}(1, 1, \dots, 1)$.

Probability vs geometry

Modified question

Can one derive a lower estimate for some densities?

Test case: uniform density: $X_j \sim Uni([-\frac{1}{2}, \frac{1}{2}])$. Is it true that

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Geometric formulation

Consider the cube $Q_n \subset \mathbb{R}^n$ of a unit volume. Then

$$f_{P_E X}(v) = \text{vol}_{n-d}(Q_n \cap (E^\perp + v))$$

Is it true that the volume of any section of the cube Q_n by a subset having distance at most $\frac{1}{2}$ from the origin is bounded below independently of the ambient dimension?

Sections of a cube

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$$\text{Is it true that} \quad \mathrm{vol}_{n-d}(Q_n \cap (E+v)) \geq \phi(d) \quad \text{whenever } \|v\|_2 \leq \frac{1}{2}$$

for $\dim(E) = n - d$?

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Central sections

- Minimal section: coordinate $\mathrm{vol}_{n-d}(Q_n \cap (E+0)) \geq 1$ (Vaaler).
- Maximal section: $\mathrm{vol}_{n-d}(Q_n \cap (E+0)) \leq 2^{d/2}$ (Ball).

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Non-central sections

- Maximal hyperplane section: $\mathrm{vol}_{n-d}(Q_n \cap (E+v))$ is maximal for $E = (1,1,,\ldots,1)^{\perp}$ whenever $\|v\|_2 \in (\sqrt{n-1}, \sqrt{n})$ (Moody, Stone, Zach, Zvavitch).
- Upper estimate for hyperplane sections (Koldobsky, König).

The position of the maximal section depends on the distance.

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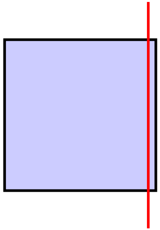
Distance $\frac{1}{2}$ is critical.

- Let $E = e_1^{\perp}$. If $v = \left(\frac{1}{2} - \varepsilon\right) e_1$, then $\mathrm{vol}_{n-1}(Q_n \cap (E+v)) = 1$.



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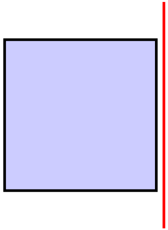

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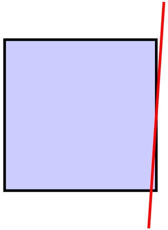
- Let $E = (e_1 + \varepsilon e_2)^{\perp}$. If $v = \frac{1}{2\sqrt{1+\varepsilon^2}}(e_1 + \varepsilon e_2)$, then $\text{vol}_{n-1}(Q_n \cap (E+v)) \approx \frac{1}{2}$.



Lower bound - general dimension

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Theorem (König-R')

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Remark

The bound $\phi(d)$ is not efficient: the proof yields $\phi(d) = O(\exp(Cd^c))$.
We will return to this later.

Proof ideas

1. We repeatedly pass from probabilistic to geometric version of the question until it becomes elementary.

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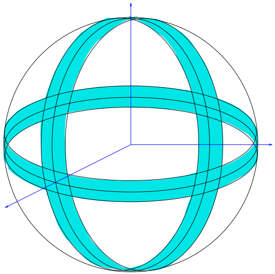
1. We repeatedly pass from probabilistic to geometric version of the question until it becomes elementary.
2. Divide and concur: treat compressible and incompressible vectors differently.



Step 1: vectors with a large ℓ_∞ norm.

$$\text{Our goal:} \qquad f_{PX}\left(\frac{1}{2}v\right) \geq \phi(d) \qquad (*)$$

1. Probability. Let $P = P_{E^\perp}$. Then $P = \sum_{j=1}^{n} (Pe_j)(Pe_j)^\top$.
   This allows to prove $(*)$ using characteristic functions
   if $v \parallel Pe_j$ and $\|Pe_j\|_2 \geq 1 - \varepsilon_1(d)$.

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2. Geometry. Assume that $\|Pe_j\|_2 \geq 1 - \varepsilon_2(d)$ and v is
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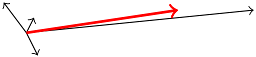

$$v' = v + tw \quad \text{where } w \in E^\perp,\ w \perp v$$

Then $(*)$ holds for v' if $|t| \leq \delta(d)$
(log concavity implies that $f_{PX}\left(\frac{1}{2}(v + tw)\right)$ cannot decay too fast).

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   (log concavity implies that $f_{PX}\left(\frac{1}{2}(v + tw)\right)$ cannot decay too fast).
3. Assume that $\|v\|_\infty \geq 1 - \varepsilon_3(d)$. Then $(*)$ holds
   (combination of 1 and 2).

Step 2: incompressible vectors.

Incompressible $=$ small coordinates carry non-negligible mass.

1. Probability. Let X be a random vector uniformly distributed in Q_n. For any $\varepsilon > 0$, there exist $\delta, \eta > 0$ such that if $J_\delta = \{j : |u_j| < \delta\}$ and $\sum_{j \in J_\delta} u_j^2 > \varepsilon^2$ then $\mathbb{P}(\langle X, u \rangle \geq 1) \geq \eta$.
   Proof: Berry-Esseen theorem.

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   Berry-Esseen theorem does not provide any density bounds.
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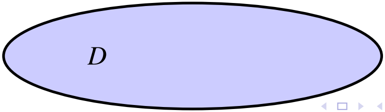

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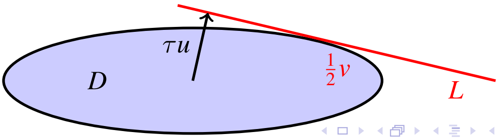
   Write $S = \tau u + L$, where $u \in E^\perp \cap S^{n-1}$ satisfies $u \perp L$, and $\tau \in [0, \frac{1}{2}]$.



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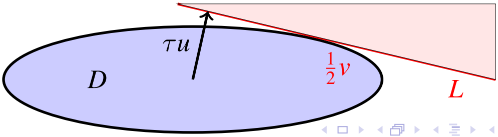
$$\text{Then} \qquad f_{PX}\left(\frac{1}{2}v\right) \geq c(d)\Big(\mathbb{P}(\langle X, u \rangle \geq \tau)\Big)^{1+d/2}.$$



Step 3: compressible vectors.

Compressible vectors: $\sum_{|u_j|<\delta} u_j^2 < \varepsilon(d)$.

$$\text{Need:} \qquad \mathbb{P}\left(\langle X, u\rangle \geq \tau\right) \geq \psi(d).$$

$$\langle X, u\rangle = \sum_{|u_j|<\delta} u_j X_j + \sum_{|u_j|\geq\delta} u_j X_j =: Y + Z.$$

- Incompressible vectors: drop Z and work with Y.
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We have to bound below

$$\mathbb{P}\left(\langle X, \frac{w}{\|w\|_2}\rangle > \frac{\tau}{\|w\|_2}\right), \quad \text{where } w = \sum_{|u_j|\geq\delta} u_j e_j.$$

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We reduced the original question to a similar one in dimension $\leq \delta^{-2} = \delta^{-2}(d)$ independent of n.

We can now allow a bound depending on the ambient dimension.

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Here $\tau < \frac{1}{2}$. However, $\frac{\tau}{\|w\|_2}$ can a priory be greater than $\frac{1}{2}$, and the section can miss the cube entirely. We have to analyze this situation.

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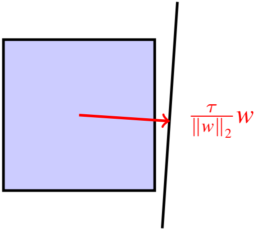
Recall that $\|w\|_2 \geq 1 - \varepsilon(d)$ since the vector is compressible.

Hence, $\frac{\tau}{\|w\|_2}$ can be only slightly greater than $\frac{1}{2}$.
In this case, if the section misses the cube, then

$$\left\| \frac{w}{\|w\|_2} \right\|_\infty > 1 - \varepsilon'(d)$$



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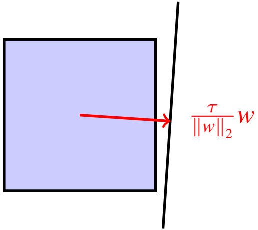

- The section does not miss the cube – elementary geometry + Vaaler's theorem.
- $\left\|\frac{w}{\|w\|_2}\right\|_\infty > 1 - \varepsilon'(d)$ – already excluded at the beginning. □

One-dimensional marginals a.k.a. hyperplane sections

A “reasonable ” bound

Theorem (König-R’)

Let $E \subset \mathbb{R}^n$ be a hyperplane. Then

$$f_{P_{E^\perp}X}(v) = \mathrm{vol}_{n-d}(Q_n \cap (E+v)) > \frac{1}{17} \quad \textit{whenever } \|v\|_2 \leq \frac{1}{2}$$

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Remark: this is at most 5.2 times smaller than the optimal bound.

Fourier analysis at work

Theorem (Polya)

Let $a \in S^{n-1}$ and let $E = a^{\perp} \subset \mathbb{R}^n$ be a hyperplane. Then

$$\mathrm{vol}_{n-d}(Q_n \cap (E + \frac{1}{2}a)) = \frac{2}{\pi} \int_0^{\infty} \prod_{j=1}^{n} \frac{\sin(a_j s)}{a_j s} \cos s \, ds$$

Remark: this is an oscillating integral. It is highly unstable.

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Let $U_1, \dots, U_n$ be a sequence of i.i.d. random vectors uniformly distributed on the sphere $S^2 \subset \mathbb{R}^3$.
Then for any $a \in S^{n-1}$,

$$\mathrm{vol}_{n-d}(Q_n \cap (E + \frac{1}{2}a)) = \int_{\left\| \sum_{j=1}^{n} a_j U_j \right\|_2 \geq 1} \frac{dm(U)}{\left\| \sum_{j=1}^{n} a_j U_j \right\|_2}$$

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If $e \in S^2$, then $\langle U, e \rangle \sim Uni([-1, 1])$. Hence,

$$\int_{S^2} \exp(it < e, U >)\, dm(U) = \frac{\sin(t)}{t}$$

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Applying this twice, we get

$$\prod_{j=1}^n \frac{\sin(a_j s)}{a_j s} = \int_{(S^2)^n} \frac{\sin\left(\left\| \sum_{j=1}^n a_j U_j \right\|_2 s\right)}{\left\| \sum_{j=1}^n a_j U_j \right\|_2 s}\, dm(U)$$

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$$\int_0^\infty \prod_{j=1}^n \frac{\sin(a_j s)}{a_j s} \cos s\, ds = \int_0^\infty \int_{(S^2)^n} \frac{\sin\left(\left\| \sum_{j=1}^n a_j U_j \right\|_2 s \right)}{\left\| \sum_{j=1}^n a_j U_j \right\|_2 s}\, dm(U) \cos s\, ds$$

From the integral to probability

$$\mathrm{vol}_{n-d}(Q_n \cap (E + \frac{1}{2}a)) = \int_{\left\|\sum_{j=1}^{n} a_j U_j\right\|_2 \geq 1} \frac{d\mathbb{P}}{\left\|\sum_{j=1}^{n} a_j U_j\right\|_2}$$

$$= \int_0^1 \mathbb{P}\left(1 \leq \left\|\sum_{j=1}^{n} a_j U_j\right\|_2 < \frac{1}{s}\right) ds$$

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Probability of positivity

We reduced the original problem to finding a lower bound for $\mathbb{P}(Y > 0)$, where

$$Y = \sum_{1 \le i < j \le n} a_i a_j \langle U_i, U_j \rangle \quad \text{with } a \in S^{n-1},\ U_1, \dots, U_n \text{ i.i.d. } Uni(S^2)$$

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Let $\|\cdot\|_L$, $\|\cdot\|_M$ be dual Orlicz norms. Then

$$\mathbb{E}Y_+ = \mathbb{E}(Y_+ \cdot \mathbf{1}_{(0,\infty)}) \le \|Y_+\|_L \cdot \left\|\mathbf{1}_{(0,\infty)}(Y)\right\|_M$$

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The faster L grows, the better the estimate is. Choose L of the exponential type.
We need to bound

$$\mathbb{E}Y_+ = \frac{1}{2}\mathbb{E}|Y| \text{ below} \qquad \text{and} \qquad \mathbb{E}\exp(\lambda Y_+) \text{ above}$$

Exponential moment

We need to bound

$$\mathbb{E}\exp(\lambda Y_+)$$

Lemma

Let Y be a real-valued random variable such that $\mathbb{E}Y = 0$. Then for any $\lambda > 0$,

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Remark. If $\mathbb{E}\exp(-\lambda Y) > 2$, then one can obtain a better bound

$$\mathbb{E}\exp(\lambda Y_+) \leq \mathbb{E}\exp(\lambda Y) - \big(\mathbb{E}\exp(-\lambda Y)\big)^{-1} + 1$$

It remains to bound the Laplace transform of Y.

Laplace transform

We need to bound

$$\mathbb{E}\exp(\lambda Y) \quad \text{for } Y = \sum_{1 \le i < j \le n} a_i a_j \langle U_i, U_j \rangle$$

Here $a \in S^{n-1}$, $U_1, \dots, U_n$ are i.i.d. $\mathrm{Uni}(S^2)$ random variables.

$U_1, \dots, U_n$ are subgaussian random vectors. Y is a quadratic form of their coordinates

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and the spectral structure of the quadratic form □