

On operators on Hilbert C^* -modules

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Semi- \mathcal{A} -Fredholm operators on $H_{\mathcal{A}}$

In this presentation we let \mathcal{A} be a unital C^* -algebra, $H_{\mathcal{A}}$ be the standard module over \mathcal{A} (this is $H_{\mathcal{A}} = l_2(\mathcal{A})$) and $B^a(H_{\mathcal{A}})$ be the set of all \mathcal{A} -linear, bounded adjointable operators on $H_{\mathcal{A}}$. Inspired by definition of \mathcal{A} -Fredholm operator given in [MF], we give now the following definition.

Definition

Let $F \in B^a(H_{\mathcal{A}})$. We say that F is an upper semi- \mathcal{A} -Fredholm operator if there exists a decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

with respect to which F has the matrix

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism M_1, M_2, N_1, N_2 are closed submodules of $H_{\mathcal{A}}$ and N_1 is finitely generated. Similarly, we say that F is a lower semi- \mathcal{A} -Fredholm operator if all the above conditions hold except that in this case we assume that N_2 (and not N_1) is finitely generated.

Set

$$\mathcal{M}\Phi_+(H_{\mathcal{A}}) = \{F \in B^a(H_{\mathcal{A}}) \mid F \text{ is upper semi-}\mathcal{A}\text{-Fredholm} \},$$

$$\mathcal{M}\Phi_-(H_{\mathcal{A}}) = \{F \in B^a(H_{\mathcal{A}}) \mid F \text{ is lower semi-}\mathcal{A}\text{-Fredholm} \},$$

$\mathcal{M}\Phi(H_{\mathcal{A}}) = \{F \in B^a(H_{\mathcal{A}}) \mid F \text{ is } \mathcal{A}\text{-Fredholm operator on } H_{\mathcal{A}}\}$. Then obviously $\mathcal{M}\Phi(H_{\mathcal{A}}) \subseteq \mathcal{M}\Phi_+(H_{\mathcal{A}}) \cap \mathcal{M}\Phi_-(H_{\mathcal{A}})$. We are going to show later in this section that actually "=" holds.

Notice that if M, N are two arbitrary Hilbert modules C^* -modules, the definition above could be generalized to the classes $\mathcal{M}\Phi_+(M, N)$ and $\mathcal{M}\Phi_-(M, N)$.

We let now $K^*(H_{\mathcal{A}})$ denote the closed, two sided ideal of adjointable compact operators in $B^a(H_{\mathcal{A}})$, see [MT].

Theorem

Let $F \in B^a(H_{\mathcal{A}})$. The following statements are equivalent

- 1) $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$*
- 2) There exists $D \in B^a(H_{\mathcal{A}})$ such that $DF = I + K$ for some $K \in K^*(H_{\mathcal{A}})$*

Theorem

Let $D \in B^a(H_{\mathcal{A}})$. Then the following statements are equivalent:

- 1) $D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$*
- 2) There exist $F \in B^a(H_{\mathcal{A}})$, $K \in K^*(H_{\mathcal{A}})$ s.t. $DF = I + K$*

Corollary

$$\mathcal{M}\Phi(H_{\mathcal{A}}) = \mathcal{M}\Phi_+(H_{\mathcal{A}}) \cap \mathcal{M}\Phi_-(H_{\mathcal{A}})$$

Corollary

$\mathcal{M}\Phi_+(H_{\mathcal{A}})$ and $\mathcal{M}\Phi_-(H_{\mathcal{A}})$ are semigroups under multiplication.

Corollary

Let $F \in B^a(M, N)$. Then $F \in \mathcal{M}\Phi_+(M, N)$ if and only if $F^* \in \mathcal{M}\Phi_-(N, M)$. Moreover, if $F \in \mathcal{M}\Phi(H_{\mathcal{A}})$, then $F^* \in \mathcal{M}\Phi(H_{\mathcal{A}})$ and $\text{index} F = -\text{index} F^*$.

Lemma

Let M be a closed submodule of $H_{\mathcal{A}}$ s.t. $H_{\mathcal{A}} = M \tilde{\oplus} N$ for some finitely generated submodule N . Let $F \in B^a(H_{\mathcal{A}})$, J_M be the inclusion map from M into $H_{\mathcal{A}}$ and suppose that $FJ_M \in \mathcal{M}\Phi_+(M, H_{\mathcal{A}})$. Then $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$.

Lemma

Suppose that $D, F \in B^a(H_{\mathcal{A}})$ $DF \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$ and $\text{Im} F$ is closed. Then $DJ_{\text{Im} F} \in \mathcal{M}\Phi_+(\text{Im} F, H_{\mathcal{A}})$.

Lemma

Let $F \in \mathcal{M}\Phi(H_{\mathcal{A}})$ and suppose that there are two decompositions

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

$$H_{\mathcal{A}} = M'_1 \tilde{\oplus} N'_1 \xrightarrow{F} M'_2 \tilde{\oplus} N'_2 = H_{\mathcal{A}}$$

with respect to which F has matrices

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}, \begin{bmatrix} F'_1 & 0 \\ 0 & F'_4 \end{bmatrix},$$

respectively, where F_1, F'_1 are isomorphisms, N_1, N'_1, N_2 are closed, finitely generated and N'_2 is just closed. Then N'_2 is finitely generated also.

Lemma

Let $F \in \mathcal{M}\Phi(H_{\mathcal{A}})$ and let

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

be a decomposition with respect to which F has the matrix

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism, N_2 is finitely generated and N_1 is just closed. Then N_1 is finitely generated.

Lemma

Let $F \in \mathcal{M}\Phi_+(H_A)$ and suppose that $\text{Im} F$ is closed. If

$$H_A = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_A$$

$$H_A = M'_1 \tilde{\oplus} N'_1 \xrightarrow{F} M'_2 \tilde{\oplus} N'_2 = H_A$$

are two $\mathcal{M}\Phi_+$ decomposition for F then $F(N_1), F(N'_1)$ are closed finitely generated projective modules and

$$[N_1] - [F(N_1)] = [N'_1] - [F(N'_1)]$$

in $K(A)$.

Lemma

Let $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$. Then there is no sequence of unit vectors $\{x_n\}$ in $H_{\mathcal{A}}$ such that $\varphi(x_n) \rightarrow 0$ in \mathcal{A} for all $\varphi \in H'_{\mathcal{A}}$ and $\lim_{n \rightarrow \infty} \|Fx_n\| = 0$.

Generalized Schechter characterization of $\mathcal{M}\Phi_+$ operators on $H_{\mathcal{A}}$

Lemma

Let $F \in B^a(M, N)$. Then $F \in \mathcal{M}\Phi_+(M, N)$ if and only if there exists a closed, orthogonally complementable submodule $M' \subseteq M$ such that $F|_{M'}$ is bounded below and M'^{\perp} is finitely generated.

Lemma

Let $F \in B^a(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi_+(H_{\mathcal{A}})$. Then there exists a sequence $\{x_k\} \subseteq H_{\mathcal{A}}$ and an increasing sequence $\{n_k\} \subseteq \mathbb{N}$ s.t.

$$x_k \in L_{n_k} \setminus L_{n_{k-1}} \text{ for all } k \in \mathbb{N}, \|x_k\| \leq 1 \text{ for all } k \in \mathbb{N}$$

and

$$\|Fx_k\| \leq 2^{1-2k} \text{ for all } k \in \mathbb{N}.$$

Openness of the set of semi- \mathcal{A} -Fredholm operators on $H_{\mathcal{A}}$

Theorem

The sets $\mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$ and $\mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$ are open in $B^a(H_{\mathcal{A}})$, where $B^a(H_{\mathcal{A}})$ is equipped with the norm topology.

Corollary

If $F \in B^a(H_{\mathcal{A}})$ belongs to the boundary of $\mathcal{M}\Phi(H_{\mathcal{A}})$ in $B^a(H_{\mathcal{A}})$ then $F \notin \mathcal{M}\Phi_{\pm}(H_{\mathcal{A}})$.

Corollary

Let $f : [0, 1] \rightarrow B^a(H_{\mathcal{A}})$ be continuous and assume that $f([0, 1]) \subseteq \mathcal{M}\Phi_{\pm}(H_{\mathcal{A}})$. Then the following statements hold:

- 1) If $f(0) \in \mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$, then $f(1) \in \mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$*
- 2) If $f(0) \in \mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$, then $f(1) \in \mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$*
- 3) If $f(0) \in \mathcal{M}\Phi(H_{\mathcal{A}})$, then $f(1) \in \mathcal{M}\Phi(H_{\mathcal{A}})$ and $\text{index} f(0) = \text{index} f(1)$.*

$\mathcal{M}\Phi_+^-$ and $\mathcal{M}\Phi_-^+$ operators on H_A

Definition

Let $F \in \mathcal{M}\Phi(H_A)$. We say that $F \in \tilde{\mathcal{M}}\Phi_+^-(H_A)$ if there exists a decomposition

$$H_A = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_A$$

with respect to which F has the matrix

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism, N_1, N_2 are closed, finitely generated and $N_1 \preceq N_2$, that is N_1 is isomorphic to a closed submodule of N_2 . We define similarly the class $\tilde{\mathcal{M}}\Phi_-^+(H_A)$, the only difference in this case is that $N_2 \preceq N_1$. Then we set

$$\mathcal{M}\Phi_+^-(H_A) = (\tilde{\mathcal{M}}\Phi_+^-(H_A)) \cup (\mathcal{M}\Phi_+(H_A) \setminus \mathcal{M}\Phi(H_A))$$

and

$$\mathcal{M}\Phi_-^+(H_A) = (\tilde{\mathcal{M}}\Phi_-^+(H_A)) \cup (\mathcal{M}\Phi_-(H_A) \setminus \mathcal{M}\Phi(H_A))$$

Further, we define $\mathcal{M}\Phi_0(H_{\mathcal{A}})$ to be the set of all $F \in \mathcal{M}\Phi(H_{\mathcal{A}})$ for which there exists an $\mathcal{M}\Phi$ -decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

where $N_1 \cong N_2$.

Lemma

Suppose that $K(\mathcal{A})$ satisfies "the cancellation property". If $F \in \tilde{\mathcal{M}}\Phi_+^-(H_{\mathcal{A}})$, then for any decomposition

$$H_{\mathcal{A}} = M'_1 \tilde{\oplus} N'_1 \xrightarrow{F} M'_2 \tilde{\oplus} N'_2 = H_{\mathcal{A}}$$

with respect to which F has the matrix

$$\begin{bmatrix} F'_1 & 0 \\ 0 & F'_4 \end{bmatrix},$$

where F'_1 is an isomorphism, N'_1, N'_2 are finitely generated, we have $N'_1 \preceq N'_2$. Similarly $N'_1 \preceq N'_2$ if $F \in \tilde{\mathcal{M}}\Phi_+^+(H_{\mathcal{A}})$.

Lemma

$\tilde{\mathcal{M}}\Phi_+^-(H_{\mathcal{A}})$ and $\tilde{\mathcal{M}}\Phi_-^+(H_{\mathcal{A}})$ are semigroups under multiplication.

Lemma

$\mathcal{M}\Phi_+^-(H_{\mathcal{A}})$ and $\mathcal{M}\Phi_-^+(H_{\mathcal{A}})$ are semigroups under multiplication.

Lemma

$\tilde{\mathcal{M}}\Phi_+^-(H_{\mathcal{A}})$ and $\tilde{\mathcal{M}}\Phi_-^+(H_{\mathcal{A}})$ are open.

Definition

Let $F \in \mathcal{M}\Phi_+(H_A)$. We say that $F \in \mathcal{M}\Phi_+^{-'}(H_A)$ if there exists a decomposition

$$H_A = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_A$$

with respect to which

$$F = \begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism, N_1 is closed, finitely generated and $N_1 \preceq N_2$. Similarly, we define the class $\mathcal{M}\Phi_-^{+'}(H_A)$, only in this case $F \in \mathcal{M}\Phi_-(H_A)$, N_2 is finitely generated and $N_2 \preceq N_1$.

Proposition

$$\tilde{\mathcal{M}}\Phi_+^{-}(H_A) = \mathcal{M}\Phi_+^{-'}(H_A) \cap \mathcal{M}\Phi(H_A), \tilde{\mathcal{M}}\Phi_-^{+}(H_A) = \mathcal{M}\Phi_-^{+'}(H_A) \cap \mathcal{M}\Phi(H_A)$$

Lemma

The sets $\mathcal{M}\Phi_{-}^{+'}(H_{\mathcal{A}})$ and $\mathcal{M}\Phi_{+}^{-'}(H_{\mathcal{A}})$ are open. Moreover, if $F \in \mathcal{M}\Phi_{+}^{-'}(H_{\mathcal{A}})$ and $K \in K^{*}(H_{\mathcal{A}})$, then

$$(F + K) \in \mathcal{M}\Phi_{+}^{-'}(H_{\mathcal{A}}).$$

If $F \in \mathcal{M}\Phi_{-}^{+'}(H_{\mathcal{A}})$ and $K \in K^{*}(H_{\mathcal{A}})$, then

$$(F + K) \in \mathcal{M}\Phi_{-}^{+'}(H_{\mathcal{A}}).$$

Lemma

The sets $\mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi_{+}^{-'}(H_{\mathcal{A}})$, $\mathcal{M}\Phi_{-}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi_{-}^{+'}(H_{\mathcal{A}})$ and $\mathcal{M}\Phi(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi_0(H_{\mathcal{A}})$ are open.

Theorem

Let $F \in B^a(H_A)$. The following statements are equivalent

- 1) $F \in \mathcal{M}\Phi_+^{-'}(H_A)$
- 2) There exist $D \in B^a(H_A)$, $K \in K^*(H_A)$ such that D is bounded below and $F = D + K$

Proposition

- 1) $F \in \mathcal{M}\Phi_+^{-'}(H_A) \Leftrightarrow F^* \in \mathcal{M}\Phi_-^{+'}(H_A)$
- 2) $F \in \tilde{\mathcal{M}}\Phi_+^{-}(H_A) \Leftrightarrow F^* \in \tilde{\mathcal{M}}\Phi_-^{+}(H_A)$
- 3) $F \in \mathcal{M}\Phi_+^{-}(H_A) \Leftrightarrow F^* \in \mathcal{M}\Phi_-^{+}(H_A)$

Definition

We set $M^a(H_A) = \{F \in B^a(H_A) \mid F \text{ is bounded below}\}$ and $Q^a(H_A) = \{D \in B^a(H_A) \mid D \text{ is surjective}\}$.

Lemma

Let $B^a(H_A)$. Then $F \in M^a(H_A)$ if and only if $F^* \in Q^a(H_A)$.

Corollary

Let $D \in B^a(H_A)$. The following statements are equivalent:

- 1) $D \in \mathcal{M}\Phi_-^{+'}(H_A)$
- 2) There exist $Q \in Q^a(H_A)$, $K \in K^*(H_A)$ s.t. $D = Q + K$.

Theorem

Let $B^a(H_A)$. Then the following statements are equivalent:

- 1) $F \in \mathcal{M}\Phi_0(H_A)$
- 2) There exist an invertible $D \in B^a(H_A)$ and $K \in K^*(H_A)$ such that $F = D + K$.

On non-adjointable semi-Fredholm operators over a C^* -algebra

Non adjointable semi- \mathcal{A} -Fredholm operators on $H_{\mathcal{A}}$

Definition

Let $F \in B(H_{\mathcal{A}})$, where $B(H_{\mathcal{A}})$ is the set of all bounded, (not necessarily adjointable) \mathcal{A} -linear operators on $H_{\mathcal{A}}$. We say that F is an upper semi- \mathcal{A} -Fredholm operator if there exists a decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

with respect to which F has the matrix

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism M_1, M_2, N_1, N_2 are closed submodules of $H_{\mathcal{A}}$ and N_1 is finitely generated. Similarly, we say that F is a lower semi- \mathcal{A} -Fredholm operator if all the above conditions hold except that in this case we assume that N_2 (and not N_1) is finitely generated.

Set

$$\widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}}) = \{F \in B(H_{\mathcal{A}}) \mid F \text{ is upper semi-}\mathcal{A}\text{-Fredholm} \},$$

$$\widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}}) = \{F \in B(H_{\mathcal{A}}) \mid F \text{ is lower semi-}\mathcal{A}\text{-Fredholm} \},$$

$$\widehat{\mathcal{M}\Phi}(H_{\mathcal{A}}) = \{F \in B(H_{\mathcal{A}}) \mid F \text{ is } \mathcal{A}\text{-Fredholm operator on } H_{\mathcal{A}}\}.$$

Then, by definition we have

$$\mathcal{M}\Phi_+(H_{\mathcal{A}}) = \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}}) \cap B^a(H_{\mathcal{A}}),$$

$$\mathcal{M}\Phi_-(H_{\mathcal{A}}) = \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}}) \cap B^a(H_{\mathcal{A}})$$

and

$$\mathcal{M}\Phi(H_{\mathcal{A}}) = \widehat{\mathcal{M}\Phi}(H_{\mathcal{A}}) \cap B^a(H_{\mathcal{A}}).$$

Definition

[IM] An \mathcal{A} -operator $K : H_{\mathcal{A}} \rightarrow H_{\mathcal{A}}$ is called a finitely generated \mathcal{A} -operator if it can be represented as a composition of bounded \mathcal{A} -operators f_1 and f_2 :

$$K : H_{\mathcal{A}} \xrightarrow{f_1} M \xrightarrow{f_2} H_{\mathcal{A}},$$

where M is a finitely generated Hilbert C^* -module. The set $FG(\mathcal{A}) \subset B(H_{\mathcal{A}})$ of all finitely generated \mathcal{A} -operators forms a two sided ideal. By definition, an \mathcal{A} -operator K is called compact if it belongs to the closure

$$K(H_{\mathcal{A}}) = \overline{FG(\mathcal{A})} \subset B(H_{\mathcal{A}}),$$

which also forms two sided ideal.

Clearly, any operator $F \in \widehat{\mathcal{M}\Phi}_l(H_A)$ is also left invertible in $B(H_A)/K(H_A)$, whereas any operator $G \in \widehat{\mathcal{M}\Phi}_r(H_A)$ is right invertible in $B(H_A)/K(H_A)$. The converse also holds:

Proposition

If F is left invertible in $B(H_A)/K(H_A)$, then $F \in \widehat{\mathcal{M}\Phi}_l(H_A)$. If F is right invertible in $B(H_A)/K(H_A)$, then $F \in \widehat{\mathcal{M}\Phi}_r(H_A)$.

Corollary

The sets $\widehat{\mathcal{M}\Phi}_l(H_A)$ and $\widehat{\mathcal{M}\Phi}_r(H_A)$ are closed under multiplication.

Inspired by definition of external (Noether) decomposition given in [IM], we give the following definition.

Definition

We say that F has an upper external (Noether) decomposition if there exist two closed C^* -modules X_1, X_2 and two bounded \mathcal{A} -operators E_2, E_3 , where X_2 finitely generated, the operator F_0 given by the operator matrix $\begin{pmatrix} F & E_2 \\ E_3 & 0 \end{pmatrix}$ with respect to the decomposition $H_{\mathcal{A}} \oplus X_1 \xrightarrow{F_0} H_{\mathcal{A}} \oplus X_2$ is invertible and $Im E_2$ is complementable in $H_{\mathcal{A}}$. Similarly, we say that F has a lower external (Noether) decomposition if the above decomposition exists and F_0 is invertible, only in this case we assume that X_1 is finitely generated and that $\ker E_3$ is complementable in $H_{\mathcal{A}}$.

Proposition

A bounded \mathcal{A} -linear operator $F : H_{\mathcal{A}} \longrightarrow H_{\mathcal{A}}$ belongs to $\widehat{\mathcal{M}}\Phi_l(H_{\mathcal{A}})$ if and only if it admits an upper external (Noether) decomposition. Similarly, F belongs to $\widehat{\mathcal{M}}\Phi_r(H_{\mathcal{A}})$ if and only if F admits a lower external (Noether) decomposition.

Lemma

Let $F, G \in B(H_A)$ and suppose that $GF \in \widehat{\mathcal{M}\Phi}(H_A)$. Then there exist decompositions

$$H_A = M_1 \oplus N_1 \xrightarrow{F} H_A = M_3 \oplus N_3 \xrightarrow{G} H_A = M_2 \oplus N_2$$

with respect to which F, G have matrices $\begin{pmatrix} F_1 & 0 \\ 0 & F_4 \end{pmatrix}, \begin{pmatrix} G_1 & G_2 \\ 0 & G_4 \end{pmatrix}$, respectively, where F_1, G_1 are isomorphisms and N_1, N_2 are finitely generated.

Lemma

Let V be a finitely generated Hilbert submodule of H_A , $F \in B(H_A)$ and suppose that $P_{V^\perp} F \in \widehat{\mathcal{M}\Phi}(H_A, V^\perp)$, where P_{V^\perp} denotes the orthogonal projection onto V^\perp along V . Then $F \in \widehat{\mathcal{M}\Phi}_r(H_A)$.

Lemma

Let $G, F \in B(H_A)$, suppose that $\text{Im} G$ is closed. Assume in addition that $\ker G$ and $\text{Im} G$ are complementable in H_A . If $GF \in \widehat{\mathcal{M}\Phi}_r(H_A)$, then

$$\sqcap F \in \widehat{\mathcal{M}\Phi}_r(H_A, N),$$

where $\ker G \oplus N = H_A$ and \sqcap denotes the projection onto N along $\ker G$.

Lemma

Let $F \in \widehat{\mathcal{M}\Phi}(H_A)$ and suppose that

$$H_A = M'_1 \tilde{\oplus} N'_1 \xrightarrow{F} M'_2 \tilde{\oplus} N'_2 = H_A$$

is a decomposition with respect to which F has the matrix $\begin{bmatrix} F'_1 & 0 \\ 0 & F'_4 \end{bmatrix}$, where F'_1 is an isomorphism, N'_2 is finitely generated and N'_1 is just closed. Then N'_1 is finitely generated.

Lemma

Let $F \in B(H_A)$. Then F admits an upper external (Noether) decomposition with the property that $X_2 \preceq X_1$ if and only if $F \in \mathcal{M}\Phi_+^{-'}(H_A)$. Similarly, F admits a lower external (Noether) decomposition with the property that $X_1 \preceq X_2$ if and only if $F \in \mathcal{M}\Phi_+^{-'}(H_A)$.

Recall now the definition of the classes $\mathcal{M}\Phi_+^{-'}(H_A)$ and $\mathcal{M}\Phi_-^{+'}(H_A)$. We are going to keep this notion in the next results, but without assuming the adjointability of operators.

Lemma

Let $F \in \mathcal{M}\Phi_-^{+'}(H_A)$. Then $F + K \in \mathcal{M}\Phi_-^{+'}(H_A)$ for all $K \in K(H_A)$.

Lemma

Let $F \in B(H_A)$ and suppose that

$$H_A = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_A$$

is a decomposition w.r.t. which F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$, where F_1 is an isomorphism. Then $N_1 = F^{-1}(N_2)$.

Lemma

Let $F \in \mathcal{M}\Phi_+^{-'}(H_A)$ and $K \in K(H_A)$. Then $F + K \in \mathcal{M}\Phi_+^{-'}(H_A)$.

Semi-Fredholm operators over W^* -algebras

Proposition

Let $F \in \widehat{\mathcal{M}\Phi}_I(H_{\mathcal{A}})$ or $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$. Then there exists a decomposition.

$$H_{\mathcal{A}} = M_0 \tilde{\oplus} M'_1 \tilde{\oplus} \ker F \xrightarrow{F} N_0 \tilde{\oplus} N'_1 \tilde{\oplus} N''_1 = H_{\mathcal{A}}$$

w.r.t. which F has the matrix

$$\begin{bmatrix} F_0 & 0 & 0 \\ 0 & F_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where F_0 is an isomorphism, M'_1 and $\ker F$ are finitely generated.

Moreover $M'_1 \cong N'_1$ If $F \in \widehat{\mathcal{M}\Phi}_I(H_{\mathcal{A}})$ and $\text{Im} F$ is closed, then $\text{Im} F$ is complementable in $H_{\mathcal{A}}$.

In this case F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & 0 \end{bmatrix}$, w.r.t. the decomposition

$$H_{\mathcal{A}} = \ker F^0 \tilde{\oplus} \ker F \xrightarrow{F} \operatorname{Im} F \tilde{\oplus} \operatorname{Im} F^0 = H_{\mathcal{A}}$$

where F_1 is an isomorphism and $\ker F^0, \operatorname{Im} F^0$ denote the complements of $\ker F, \operatorname{Im} F$ respectively.

Proposition

If $D \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$ and $\operatorname{Im} D$ is closed and complementable in $H_{\mathcal{A}}$, then the decomposition given above exists for the operator D . In this case, instead of $\ker D$, we have that N_1'' is finitely generated and N_1'' is the complement of $\operatorname{Im} D$.

Lemma

If $F \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}}) \setminus \widehat{\mathcal{M}\Phi}(H_{\mathcal{A}})$, $\text{Im} F$ is closed and complementable, then the complement of $\text{Im} F$ is not finitely generated.

Theorem

Let $F \in B^a(H_{\mathcal{A}})$. Then $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$ if and only if $\ker(F - K)$ is finitely generated for all $K \in K^*(H_{\mathcal{A}})$.

Moreover, $F \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$ if and only if $\text{Im}(F - K)^\perp$ is finitely generated for all $K \in K^*(H_{\mathcal{A}})$.

Definition

Let $F \in B(H_{\mathcal{A}})$. We say that $F \in \widehat{\mathcal{M}\Phi}_+(H_{\mathcal{A}})$ if there exist a closed submodule M and a finitely generated submodule N s.t. $H_{\mathcal{A}} = M \dot{\oplus} N$ and $F|_M$ is bounded below.

Lemma

Let $F \in B(H_A)$. Then $F \in \widehat{\mathcal{M}\Phi}_+(H_A)$ iff $\ker(F - K)$ is finitely generated for all $K \in K^*(H_A)$.

Set $\widehat{\mathcal{M}\Phi}_-(H_A) = \{G \in B(H_A) \mid \text{there exists closed submodules } M, N, M' \text{ of } H_A \text{ s.t. } H_A = M \tilde{\oplus} N, N \text{ is finitely generated and } G|_{M'}, \text{ is an isomorphism onto } M\}$.

Proposition

Let $G \in \widehat{\mathcal{M}\Phi}_-(H_A)$. Then for every $K \in K(H_A)$ there exists an inner product equivalent to the initial one and such that the orthogonal complement of $\overline{\text{Im}(G + K)}$ w.r.t this new inner product is finitely generated.

Lemma

$$\begin{aligned}\mathcal{M}\Phi_+(H_A) &= \widehat{\mathcal{M}\Phi}_+(H_A) \cap B^a(H_A), \\ \mathcal{M}\Phi_-(H_A) &= \widehat{\mathcal{M}\Phi}_-(H_A) \cap B^a(H_A).\end{aligned}$$

Proposition

Let $F, G \in \widehat{\mathcal{M}\Phi}_l(H_A)$ with closed images and suppose that $\text{Im}GF$ is closed. Then $\text{Im}F, \text{Im}G$ and $\text{Im}GF$ are complementable in H_A . Moreover, if $\text{Im}F^0, \text{Im}G^0, \text{Im}GF^0$ denote the complements of $\text{Im}F, \text{Im}G, \text{Im}GF$, respectively, then

$$\text{Im}GF^0 \preceq \text{Im}F^0 \oplus \text{Im}G^0,$$

$$\ker GF \preceq \ker G \oplus \ker F.$$

If $F, G \in \widehat{\mathcal{M}\Phi}_r(H_A)$ and $\text{Im}F, \text{Im}G, \text{Im}GF$ are closed, then the statement above holds under additional assumption that $\text{Im}F, \text{Im}G, \text{Im}GF$ are complementable in H_A .

Lemma

Let $F, D \in B^a(H_A)$ and suppose that $\text{Im}F, \text{Im}D$ and $\text{Im}DF$ are closed. Then

$$\text{Im}DF^\perp \preceq \text{Im}F^\perp \oplus \text{Im}D^\perp$$

$$\ker DF \preceq \ker D \oplus \ker F$$

Lemma

Let $F \in \mathcal{MP}(M)$ be such that $\text{Im} F$ is closed, where M is a Hilbert W^* -module. Then there exists an $\epsilon > 0$ such that for every $D \in B^a(M)$ with $\|D\| < \epsilon$, we have

$$\ker(F + D) \preceq \ker F, \quad \text{Im}(F + D)^\perp \preceq \text{Im} F^\perp.$$

Definition

Let M be a countably generated Hilbert W^* -module. For $F \in \mathcal{MP}(M)$, we say that F satisfies the condition $(*)$ if the following holds:

- 1) $\text{Im} F^n$ is closed for all n
- 2) $F\left(\bigcap_{n=1}^{\infty} \text{Im}(F^n)\right) = \bigcap_{n=1}^{\infty} \text{Im}(F^n)$

Theorem

Let $F \in \mathcal{MP}(\tilde{M})$ where \tilde{M} is countably generated Hilbert \mathcal{A} -module and suppose that F satisfies $(*)$. Then there exists an $\epsilon > 0$ such that, if $\alpha \in Z(\mathcal{A}) \cap G(\mathcal{A})$ and $\|\alpha\| < \epsilon$, then $[\ker(F - \alpha I)] + [N_1] = [\ker F]$ and $[\text{Im}(F - \alpha I)^\perp] + [N_1] = [\text{Im}(F)^\perp]$ for some fixed, finitely generated closed submodule N_1 .

Theorem

Let \tilde{M} be a Hilbert module over a C^* -algebra \mathcal{A} , $\alpha \in \mathbb{C}$ and $F \in B^a(H_{\mathcal{A}})$. Suppose that $\alpha \in \text{iso } \sigma(F)$ and assume either that $R(F - \alpha I)$ is closed or that $R(P_0)$ is self dual and that \mathcal{A} is a W^* -algebra, where P_0 denotes the spectral projection corresponding to α of the operator F . Then the following conditions are equivalent:

- a) $(F - \alpha I) \in \mathcal{M}\Phi_0(\tilde{M})$
- b) There exist closed submodules $M, N \subseteq \tilde{M}$ such that. $(F - \alpha I)$ has the matrix

$$\begin{bmatrix} (F - \alpha I)_1 & 0 \\ 0 & (F - \alpha I)_4 \end{bmatrix}$$

w.r.t. the decomposition $\tilde{M} = M \oplus N \xrightarrow{F - \alpha I} M \oplus N = \tilde{M}$, where $(F - \alpha I)_1$ is an isomorphism and N is finitely generated. Moreover, if $(F - \alpha I)$ is not invertible in $B(\tilde{M})$, then $N(F - \alpha I) \neq \{0\}$.

On generalized \mathcal{A} -Fredholm and \mathcal{A} -Weyl operators

Definition

Let $F \in B^a(H_{\mathcal{A}})$.

- 1) We say that $F \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$ if ImF is closed and in addition $\ker F$ and ImF^{\perp} are self-dual.
- 2) We say that $F \in \mathcal{M}\Phi_0^{gc}(H_{\mathcal{A}})$ if ImF is closed and $\ker F \cong ImF^{\perp}$ (here we do not require the self-duality of $\ker F$, ImF^{\perp}).

Proposition

Let $F, D \in \mathcal{M}\Phi_0^{gc}(H_{\mathcal{A}})$ and suppose that $ImDF$ is closed. Then $DF \in \mathcal{M}\Phi_0^{gc}(H_{\mathcal{A}})$.

Definition

Let M_1, \dots, M_n be Hilbert submodules of $H_{\mathcal{A}}$. We say that the sequence $0 \rightarrow M_1 \rightarrow M_2 \rightarrow \dots \rightarrow M_n \rightarrow 0$ is exact if for each $k \in \{2, \dots, n-1\}$ there exist closed submodules M'_k and M''_k such that the following holds:

- 1) $M_k = M'_k \tilde{\oplus} M''_k$ for all $k \in \{2, \dots, n-1\}$;
- 2) $M'_2 \cong M_1$ and $M''_{n-1} \cong M_n$;
- 3) $M''_k \cong M'_{k+1}$ for all $k \in \{2, \dots, n-2\}$.

Lemma

Let $F, D \in B^a(H_{\mathcal{A}})$ and suppose that $\text{Im}F, \text{Im}D, \text{Im}DF$ are closed. Then the sequence

$$0 \rightarrow \ker F \rightarrow \ker DF \rightarrow \ker D \rightarrow \text{Im}F^{\perp} \rightarrow \text{Im}DF^{\perp} \rightarrow \text{Im}D^{\perp} \rightarrow 0$$

is exact.

Lemma

Let $F, D \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$ and suppose that $\text{Im}DF$ is closed. Then $DF \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$.

Lemma

Let $F \in B^a(H_A)$. Then $F \in \mathcal{M}\Phi^{gc}(H_A)$ if and only if $F^* \in \mathcal{M}\Phi^{gc}(H_A)$.

Proposition

Let $F, D \in B^a(H_A)$, suppose that $\text{Im}F, \text{Im}D$ are closed and $DF \in \mathcal{M}\Phi^{gc}(H_A)$. Then the following statements hold:

- a) $D \in \mathcal{M}\Phi^{gc}(H_A) \Leftrightarrow F \in \mathcal{M}\Phi^{gc}(H_A)$;
- b) if $\ker D$ is self-dual, then $F, D \in \mathcal{M}\Phi^{gc}(H_A)$;
- c) if $\text{Im}F^\perp$ is self-dual, then $F, D \in \mathcal{M}\Phi^{gc}(H_A)$.

Lemma

Let $F \in B^a(H_A)$ and suppose that $\text{Im}F$ is closed. Moreover, assume that there exist operators $D, D' \in B^a(H_A)$ with closed images such that $D'F, FD \in \mathcal{M}\Phi^{gc}(H_A)$. Then $F \in \mathcal{M}\Phi^{gc}(H_A)$.

Definition

Let X, Y be Banach spaces and $T \in B(X, Y)$. Then T is called a regular operator if $T(X)$ is closed in Y and in addition $T^{-1}(0)$ and $T(X)$ are complementable in X and Y , respectively.

Definition

[DDj2] Let X, Y be Banach spaces and $T \in B(X, Y)$. Then we say that T is generalized Weyl, if $T(X)$ is closed in Y , and $T^{-1}(0)$ and $Y/T(X)$ are mutually isomorphic Banach spaces.

Proposition

Let X, Y, Z be Banach spaces and let $T \in B(X, Y), S \in B(Y, Z)$. Suppose that T, S, ST are regular, that is $T(X), S(Y), ST(X)$ are closed and T, S, ST admit generalized inverse. If T and S are generalized Weyl operators, then ST is a generalized Weyl operator.

Definition

Let X, Y be Banach spaces and $T \in B(X, Y)$ be a regular operator. Then T is said to be a generalized upper semi-Weyl operator if $\ker T \preceq Y \setminus R(T)$. Similarly T is said to be a generalized lower semi-Weyl operator if $Y \setminus R(T) \preceq \ker T$.

Lemma

Let $T \in B(X, Y)$ $S \in B(Y, Z)$ and suppose that S, T, ST are regular. If S and T are upper (or lower) generalized semi-Weyl operators, then ST is an upper (or respectively lower) generalized semi-Weyl operator.

Definition

For two Hilbert C^* -modules M and M' , We set $\tilde{\mathcal{M}}\Phi_0^{gc}(M, M')$ to be the class of all closed range operators $F \in B^a(M, M')$ such that there exist finitely generated Hilbert submodules N, \tilde{N} with the property that $N \oplus \ker F \cong \tilde{N} \oplus \operatorname{Im} F^\perp$.

Lemma

Let $T \in \tilde{\mathcal{M}}\Phi_0^{gc}(H_A)$ and $F \in B^a(H_A)$ s.t. $\text{Im}F$ is closed, finitely generated. Suppose that $\text{Im}(T + F)$, $T(\ker F)$, $P(\ker T)$, $P(\ker(T + F))$ are closed, where P denotes the orthogonal projection onto $\ker F^\perp$. Then $T + F \in \tilde{\mathcal{M}}\Phi_0^{gc}(H_A)$.

Corollary

Let $T \in \mathcal{M}\Phi_0^{gc}(H_A)$ and suppose that $\ker T \cong \text{Im}T^\perp \cong H_A$. If $F \in B^a(H_A)$ satisfies the assumptions of Lemma 64, then $\ker(T + F) \cong \text{Im}(T + F)^\perp \cong H_A$. In particular, $T + F \in \mathcal{M}\Phi_0^{gc}(H_A)$.

Lemma

Let $F \in B^a(M)$ where M is a Hilbert C^* -module and suppose that $\text{Im}F$ is closed. Then the following statements hold:

- a) $F \in \mathcal{M}\Phi_+(M)$, if and only if $\ker F$ is finitely generated;
- b) $F \in \mathcal{M}\Phi_-(M)$, if and only if $\text{Im}F^\perp$ is finitely generated.

Lemma

Let $T \in \mathcal{M}\Phi(H_A)$ and suppose that $\text{Im}T$ is closed. Then $T \in \tilde{\mathcal{M}}\Phi_0^{gc}(H_A)$.

On semi- \mathcal{A} - B -Fredholm operators

Definition

Let $F \in B^a(H_{\mathcal{A}})$. Then F is said to be an upper semi- \mathcal{A} - B -Fredholm operator if there exists some $n \in \mathbb{N}$ such that $\text{Im} F^m$ is closed for all $m \geq n$ and $F|_{\text{Im} F^n}$ is an upper semi- \mathcal{A} -Fredholm operator.

Similarly, F is said to be a lower semi- \mathcal{A} - B -Fredholm operator if the conditions above hold except that in this case we assume that $F|_{\text{Im} F^n}$ is a lower semi- \mathcal{A} -Fredholm operator and not an upper semi- \mathcal{A} -Fredholm operator.

Proposition

If F is an upper semi- \mathcal{A} - B -Fredholm operator (respectively, a lower semi- \mathcal{A} - B -Fredholm operator) and $n \in \mathbb{N}$ is such that $\text{Im} F^m$ is closed for all $m \geq n$ and $F|_{\text{Im} F^n}$ is an upper semi- \mathcal{A} -Fredholm operator (respectively, a lower semi- \mathcal{A} -Fredholm operator), then $F|_{\text{Im} F^m}$ is an upper semi- \mathcal{A} -Fredholm operator (respectively, a lower semi- \mathcal{A} -Fredholm operator) for all $m \geq n$. Moreover, if F is an \mathcal{A} - B -Fredholm operator and $n \in \mathbb{N}$ is such that $\text{Im} F^n \cong H_{\mathcal{A}}$, $\text{Im} F^m$ is closed for all $m \geq n$ and $F|_{\text{Im} F^n}$ is an \mathcal{A} -Fredholm operator, then $\text{Im} F^m \cong H_{\mathcal{A}}$, $F|_{\text{Im} F^m}$ is an \mathcal{A} -Fredholm operator and $\text{index } F|_{\text{Im} F^m} = \text{index } F|_{\text{Im} F^n}$ for all $m \geq n$.

Lemma

Let $F \in \mathcal{M}\Phi(H_{\mathcal{A}})$, let $P \in B(H_{\mathcal{A}})$ be a projection such that $N(P)$ is finitely generated. Then $PF|_{R(P)} \in \mathcal{M}\Phi(R(P))$ and $\text{index} PF|_{R(P)} = \text{index} F$.

Theorem

Let T be an \mathcal{A} - B -Fredholm operator on $H_{\mathcal{A}}$ and suppose that $m \in \mathbb{N}$ is such that $T|_{\text{Im} T^m}$ is an \mathcal{A} -Fredholm operator and $\text{Im} T^n$ is closed for all $n \geq m$. Let F be in the linear span of elementary operators and suppose that $\text{Im}(T + F)^n$ is closed for all $n \geq m$. Finally, assume that $\text{Im} T^m \cong H_{\mathcal{A}}$ and that $\text{Im}(\tilde{F})$, $T^m(\ker \tilde{F})$ are closed, where $\tilde{F} = (T + F)^m - T^m$. Then $T + F$ is an \mathcal{A} - B -Fredholm operator and $\text{index} T + F = \text{index} T$.

Proposition

Let $F \in B(H_{\mathcal{A}})$. If $n \in \mathbb{N}$ is s.t. $\text{Im} F^n$ closed, $\text{Im} F^n \cong H_{\mathcal{A}}$, $F|_{\text{Im} F^n}$ is upper semi- \mathcal{A} -Fredholm and $\text{Im} F^m$ is closed for all $m \geq n$, then $F|_{\text{Im} F^m}$ is upper semi- \mathcal{A} -Fredholm and $\text{Im} F^m \cong H_{\mathcal{A}}$ for all $m \geq n$. If $n \in \mathbb{N}$ is s.t. $\text{Im} F^n$ is closed, $\text{Im} F^n \cong H_{\mathcal{A}}$, $\text{Im} F^m$ is closed and complementable in $\text{Im} F^n$ for all $m \geq n$ and $F|_{\text{Im} F^n}$ is lower semi- \mathcal{A} -Fredholm, then $F|_{\text{Im} F^m}$ is lower semi- \mathcal{A} -Fredholm for all $m \geq n$ and $\text{Im} F^m \cong H_{\mathcal{A}}$ for all $m \geq n$.

Lemma

Let $F \in \widehat{\mathcal{M}}\Phi(H_{\mathcal{A}})$, $K \in K(H_{\mathcal{A}})$. Then $\text{index} F = \text{index} F + K$.

Lemma

Let $F, D \in B^a(H_{\mathcal{A}})$ and suppose that $\text{Im} F, \text{Im} D$ are closed. If $\text{Im} F + \ker D$ is closed, then $\text{Im} F + \ker D$ is orthogonally complementable.

Corollary

Let $F, D \in B^a(H_{\mathcal{A}})$ and suppose that $\text{Im} F, \text{Im} D$ are closed. Then $\text{Im} DF$ is closed if and only if $\text{Im} F + \ker D$ is orthogonally complementable.

Definition

Given two closed submodules M, N of $H_{\mathcal{A}}$, we set

$$c_0(M, N) = \sup\{\|\langle x, y \rangle\| \mid x \in M, y \in N, \|x\|, \|y\| \leq 1\}.$$

We say then that the Dixmier angle between M and N is positive if $c_0(M, N) < 1$.

Lemma

Let M, N be two closed, submodules of $H_{\mathcal{A}}$, assume that M orthogonally complementable and suppose that $M \cap N = \{0\}$. Then $M + N$ is closed if the Dixmier angle between M and N is positive.

Corollary

Let $F, D \in B^a(H_A)$ and suppose that $\text{Im}F, \text{Im}D$ are closed. Set $M = \text{Im}F \cap (\ker D \cap \text{Im}F)^\perp$, $M' = \ker D \cap (\ker D \cap \text{Im}F)^\perp$. Assume that $\ker D \cap \text{Im}F$ is orthogonally complementable. Then $\text{Im}DF$ is closed if the Dixmier angle between M' and $\text{Im}F$, or equivalently the Dixmier angle between M and $\ker D$ is positive.

Lemma

Let M and N be two closed submodules of H_A . Suppose that M and N are orthogonally complementable in H_A and that $M \cap N = \{0\}$. Then $M + N$ is closed if and only if $P|_N$ is bounded below, where P denotes the orthogonal projection onto M^\perp .

Corollary

Let $F, D \in B^a(H_A)$ and suppose that $\text{Im}F, \text{Im}D$ are closed. Then $\text{Im}DF$ is closed if and only if $\ker D \cap \text{Im}F$ is orthogonally complementable and $P|_{\text{Im}F \cap (\ker D \cap \text{Im}F)^\perp}$ is bounded below, or equivalently $Q|_{\ker D \cap (\ker D \cap \text{Im}F)^\perp}$ is bounded below, where P and Q denote the orthogonal projections onto $\ker D^\perp$ and $\text{Im}F^\perp$, respectively.

Lemma

Let $F, G \in \widehat{\mathcal{M}\Phi}_I(H_A)$ and suppose that $\text{Im}G$ and $\text{Im}F$ are closed. Then $\text{Im}GF$ is closed if and only if $\text{Im}F + \ker G$ is closed and complementable. If $F, G \in \widehat{\mathcal{M}\Phi}_r(H_A)$ and $\text{Im}G, \text{Im}F$ are closed, then the statement above holds under additional assumption that $\text{Im}G, \text{Im}F$ are complementable. Moreover, if $F, G \in \widehat{\mathcal{M}\Phi}_I(H_A)$ and $\text{Im}F, \text{Im}G$ are closed and if the Dixmier angle between $\ker G$ and $\text{Im}F \cap (\ker G \cap \text{Im}F)^0$ is positive, or equivalently the Dixmier angle between $\text{Im}F$ and $\ker G \cap (\ker G \cap \text{Im}F)^0$ is positive, where $(\ker G \cap \text{Im}F)^0$ denotes the complement of $\ker G \cap \text{Im}F$, then $\text{Im}GF$ is closed.

Proposition

Let $F \in B^a(H_A)$. Then the following statements are equivalent:

- 1) $\text{Im} F$ is closed in H_A
- 2) $\text{Im} L_F$ is closed in $B^a(H_A)$
- 3) $\text{Im} R_F$ is closed in $B^a(H_A)$.

Lemma

Let $F \in M^a(H_A)$. If there exists a sequence $\{F_n\} \subseteq \mathcal{M}\Phi(H_A)$ of constant index such that $F_n \rightarrow F$, then $F \in \mathcal{M}\Phi(H_A)$ and $\text{index} F = \text{index} F_n$ for all n .

Lemma

Let $F \in B(H_A)$ and suppose that $\text{Im} F$ is closed. Then F is a regular operator with the property that $\text{Im} F^0, \ker F$ are finitely generated if and only if $F \in \widehat{\mathcal{M}\Phi}(H_A)$.

Proposition

Let $F \in B(H_{\mathcal{A}})$ be bounded below and suppose that there exists a sequence $\{F_n\} \subseteq \widehat{\mathcal{M}\Phi}(H_{\mathcal{A}})$ of constant index and such that $F_n \rightarrow F$. Suppose also that for each n there exists an $\widehat{\mathcal{M}\Phi}$ -decomposition

$$H_{\mathcal{A}} = M_1^{(n)} \oplus N_1^{(n)} \xrightarrow{F_n} M_2^{(n)} \oplus N_2^{(n)} = H_{\mathcal{A}}$$

such that the sequence of projections $\{\Pi_n\}$ is uniformly bounded, where Π_n denotes the projection onto $N_2^{(n)}$ along $M_2^{(n)}$ for each n . Then $F \in \widehat{\mathcal{M}\Phi}(H_{\mathcal{A}})$ and $\text{index} F_n = \text{index} F$ for all n .

Lemma

Let X, Y be Banach spaces and $F \in M(X, Y)$. Suppose that there exists a sequence $\{F_n\}$ of regular operators in $B(X, Y)$ such that $F_n \rightarrow F$. Moreover, assume that there exists a sequence of projections $\{\Pi_n\}$ in $B(Y)$ which is uniformly bounded in the norm and such that $\text{Im}(I - \Pi_n) = \text{Im} F_n$ for all n . Then, F is a regular operator, i.e. $\text{Im} F$ is complementable in Y .

Question: If \mathcal{A} is a C^* -algebra, then for $\alpha \in \mathcal{A}$ could we define in a suitable way the operator αI on $H_{\mathcal{A}}$ and the generalized spectra in \mathcal{A} of operators in $B^a(H_{\mathcal{A}})$ by setting for every $F \in B^a(H_{\mathcal{A}})$

$$\sigma^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \text{ is not invertible in } B^a(H_{\mathcal{A}})\}?$$

Answer: For $a \in \mathcal{A}$ we may let αI be the operator on $H_{\mathcal{A}}$ given by $\alpha I(x_1, x_2, \dots) = (\alpha x_1, \alpha x_2, \dots)$. It is straightforward to check that αI is an \mathcal{A} -linear operator on $H_{\mathcal{A}}$. Moreover, αI is bounded and $\|\alpha I\| = \|\alpha\|$. Finally, αI is adjointable and its adjoint is given by $(\alpha I)^* = \alpha^* I$.

We introduce then the following notion:

$$\sigma^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \text{ is not invertible in } B^a(H_{\mathcal{A}})\};$$

$$\sigma_p^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid \ker(F - \alpha I) \neq \{0\}\};$$

$$\sigma_{rl}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \text{ is bounded below, but not surjective on } H_{\mathcal{A}}\};$$

$$\sigma_{cl}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid \text{Im}(F - \alpha I) \text{ is not closed}\}. \text{ (where } F \in B^a(H_{\mathcal{A}})\text{)}.$$

Proposition

Let \mathcal{A} be a unital C^* -algebra, $\{e_k\}_{k \in \mathbb{N}}$ denote the standard orthonormal basis of $H_{\mathcal{A}}$ and S be the operator defined by $Se_k = e_{k+1}$, $k \in \mathbb{N}$, that is S is unilateral shift and $S^*e_{k+1} = e_k$ for all $k \in \mathbb{N}$. If $\mathcal{A} = L^\infty((0, 1))$ or if $\mathcal{A} = C([0, 1])$, then $\sigma^{\mathcal{A}}(S) = \{\alpha \in \mathcal{A} \mid \inf |\alpha| \leq 1\}$, (where in the case when $\mathcal{A} = L^\infty((0, 1))$, we set $\inf |\alpha| = \inf\{C > 0 \mid \mu(|\alpha|^{-1}[0, C]) > 0\} = \sup\{K > 0 \mid |\alpha| > K\}$ a.e. on $(0, 1)\}$). Moreover, $\sigma_p^{\mathcal{A}}(S) = \emptyset$ in both cases.

Corollary

Let \mathcal{A} be a commutative unital C^ -algebra. Then*

$$\sigma^{\mathcal{A}}(S) = \mathcal{A} \setminus G(\mathcal{A}) \cup \{\alpha \in G(\mathcal{A}) \mid (\alpha^{-1}, \alpha^{-2}, \dots, \alpha^{-k}, \dots) \notin H_{\mathcal{A}}\}.$$

Proposition

Let $\alpha \in \mathcal{A}$. We have

1. If $\alpha I - F$ is bounded below, and $F \in B^a(H_{\mathcal{A}})$ then $\alpha \in \sigma_{rl}^{\mathcal{A}}(F)$ if and only if $\alpha^* \in \sigma_p^{\mathcal{A}}(F^*)$.
2. If $F, D \in B^a(H_{\mathcal{A}})$ and $D = U^*FU$ for some unitary operator U , then $\sigma^{\mathcal{A}}(F) = \sigma^{\mathcal{A}}(D)$, $\sigma_p^{\mathcal{A}}(F) = \sigma_p^{\mathcal{A}}(D)$, $\sigma_{cl}^{\mathcal{A}}(F) = \sigma_{cl}^{\mathcal{A}}(D)$ and $\sigma_{rl}^{\mathcal{A}}(F) = \sigma_{rl}^{\mathcal{A}}(D)$.

Proposition

Let $U \in B^a(H_{\mathcal{A}})$ be unitary. Then $\sigma^{\mathcal{A}}(U) \subseteq \{\alpha \in \mathcal{A} \mid \|\alpha\| \geq 1\}$ and $\sigma^{\mathcal{A}}(U) \cap G(\mathcal{A}) \subseteq \{\alpha \in G(\mathcal{A}) \mid \|\alpha^{-1}\|, \|\alpha\| \geq 1\}$.

Consider again the orthonormal basis $\{e_k\}_{k \in \mathbb{N}}$ for $H_{\mathcal{A}}$. We may enumerate this basis by indexes in \mathbb{Z} . Then we get orthonormal basis $\{e_j\}_{j \in \mathbb{Z}}$ for $H_{\mathcal{A}}$ and we can consider bilateral shift operator V w.r.t. this basis i.e. $Ve_k = e_{k+1}$ all $k \in \mathbb{Z}$, which gives $V^*e_k = e_{k-1}$ for all $k \in \mathbb{Z}$.

Proposition

Let V be bilateral shift operator. Then the following holds

1) If $\mathcal{A} = C([0, 1])$, then $\sigma^{\mathcal{A}}(V) = \{f \in \mathcal{A} \mid |f|([0, 1]) \cap \{1\} \neq \emptyset\}$

2) If $\mathcal{A} = L^\infty([0, 1])$, then

$\sigma^{\mathcal{A}}(V) = \{f \in \mathcal{A} \mid \mu(|f|^{-1}((1 - \epsilon, 1 + \epsilon))) > 0 \ \forall \epsilon > 0\}$. In both cases

$\sigma_p^{\mathcal{A}}(V) = \emptyset$.

Lemma

If F is a self-adjoint operator on $H_{\mathcal{A}}$, then $\sigma_p^{\mathcal{A}}(F)$ is a self-adjoint subset of \mathcal{A} , that is $\alpha \in \sigma_p^{\mathcal{A}}(F)$ if and only if $\alpha^ \in \sigma_p^{\mathcal{A}}(F)$ in the case when \mathcal{A} is a commutative C^* -algebra.*

Lemma

Let \mathcal{A} be a commutative C^ -algebra. If F is a self-adjoint operator on $H_{\mathcal{A}}$ and $\alpha \in \mathcal{A} \setminus \sigma_p^{\mathcal{A}}(F)$, then $\overline{R(F - \alpha I)}^\perp = \{0\}$. Hence, if $\alpha \in \mathcal{A} \setminus \sigma_p^{\mathcal{A}}(F)$ and in addition $F - \alpha I$ is bounded below, then $\alpha \in \mathcal{A} \setminus \sigma^{\mathcal{A}}(F)$.*

Lemma

Let \mathcal{A} be a commutative unital C^* -algebra and F be a normal operator on $H_{\mathcal{A}}$, that is $FF^* = F^*F$. If $\alpha_1, \alpha_2 \in \sigma_p^{\mathcal{A}}(F)$ and $\alpha_1 - \alpha_2$ is invertible in \mathcal{A} , then $\ker(F - \alpha_1 I) \perp \ker(F - \alpha_2 I)$.

Lemma

Let \mathcal{A} be a commutative C^* -algebra and F be a normal operator on $H_{\mathcal{A}}$. Then $\sigma_{rl}^{\mathcal{A}}(F) = \emptyset$, hence $\sigma^{\mathcal{A}}(F) = \sigma_p^{\mathcal{A}}(F) \cup \sigma_{cl}^{\mathcal{A}}(F)$.

Lemma

Let $F \in B^a(H_{\mathcal{A}})$. Then the following statements are equivalent:

- a) $\alpha \in \mathcal{A} \setminus \sigma_a(F)$
- b) $\alpha \in \mathcal{A} \setminus \sigma_l(F)$
- c) $\alpha^* \in \mathcal{A} \setminus \sigma_r(F^*)$
- d) $\text{Im}(\alpha^* I - F^*) = H_{\mathcal{A}}$.

Next, for $F \in B^a(H_{\mathcal{A}})$, set $\sigma_a^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \text{ is not bounded below}\}$.

Proposition

For $F \in B^a(H_{\mathcal{A}})$, we have that $\sigma_a^{\mathcal{A}}(F)$ is a closed subset of \mathcal{A} in the norm topology and $\sigma^{\mathcal{A}}(F) = \sigma_a^{\mathcal{A}}(F) \cup \sigma_{rl}^{\mathcal{A}}(F)$.

Proposition

If $F \in B^a(H_{\mathcal{A}})$, then $\partial\sigma^{\mathcal{A}}(F) \subseteq \sigma_a^{\mathcal{A}}(F)$. Moreover, if M is a closed submodule of $H_{\mathcal{A}}$ and invariant with respect to F , and $F_0 = F|_M$, then we have $\partial\sigma^{\mathcal{A}}(F_0) \subseteq \sigma_a^{\mathcal{A}}(F)$, $\sigma^{\mathcal{A}}(F_0) \cap \sigma^{\mathcal{A}}(F) = \sigma_{rl}^{\mathcal{A}}(F_0)$.

Definition

Let $F \in B^a(H_{\mathcal{A}})$. We set

$$\sigma_{ew}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid (F - \alpha I) \notin \mathcal{M}\Phi_0(H_{\mathcal{A}})\},$$

$$\sigma_{e\alpha}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid (F - \alpha I) \notin \mathcal{M}\Phi_+(H_{\mathcal{A}})\},$$

$$\sigma_{e\beta}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid (F - \alpha I) \notin \mathcal{M}\Phi_-(H_{\mathcal{A}})\},$$

$$\sigma_{ek}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid (F - \alpha I) \notin \mathcal{M}\Phi_+(H_{\mathcal{A}}) \cup \mathcal{M}\Phi_-(H_{\mathcal{A}})\},$$

$$\sigma_{ef}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid (F - \alpha I) \notin \mathcal{M}\Phi(H_{\mathcal{A}})\}.$$

Definition

We set $ms_{\Phi}(F) = \inf\{\|\alpha\| \mid \alpha \in \mathcal{A}, F - \alpha I \notin \mathcal{M}\Phi(H_{\mathcal{A}})\},$

$$ms(F) = \inf\{\|\alpha\| \mid \alpha \in \mathcal{A}, F - \alpha I \notin (\mathcal{M}\Phi_+(H_{\mathcal{A}}) \cup \mathcal{M}\Phi_-(H_{\mathcal{A}}))\},$$

$$ms_+(F) = \inf\{\|\alpha\| \mid \alpha \in \mathcal{A}, F - \alpha I \notin \mathcal{M}\Phi_+(H_{\mathcal{A}})\},$$

$$ms_-(F) = \inf\{\|\alpha\| \mid \alpha \in \mathcal{A}, F - \alpha I \notin \mathcal{M}\Phi_-(H_{\mathcal{A}})\}.$$

It follows that $ms_{\Phi}(F) = \max\{\epsilon \geq 0 \mid \|\alpha\| < \epsilon \Rightarrow F - \alpha I \in \mathcal{M}\Phi(H_{\mathcal{A}})\},$

$$ms_+(F) = \max\{\epsilon \geq 0 \mid \|\alpha\| < \epsilon \Rightarrow F - \alpha I \in \mathcal{M}\Phi_+(H_{\mathcal{A}})\},$$

$$ms_-(F) = \max\{\epsilon \geq 0 \mid \|\alpha\| < \epsilon \Rightarrow F - \alpha I \in \mathcal{M}\Phi_-(H_{\mathcal{A}})\},$$

$$ms(F) = \max\{\epsilon \geq 0 \mid \|\alpha\| < \epsilon \Rightarrow F - \alpha I \in (\mathcal{M}\Phi_+(H_{\mathcal{A}}) \cup \mathcal{M}\Phi_-(H_{\mathcal{A}}))\},$$

it follows that $ms_{\Phi}(F) > 0 \Leftrightarrow F \in \mathcal{M}\Phi(H_{\mathcal{A}}),$

$$ms_+(F) > 0 \Leftrightarrow F \in \mathcal{M}\Phi_+(H_{\mathcal{A}}), ms_-(F) > 0 \Leftrightarrow F \in \mathcal{M}\Phi_-(H_{\mathcal{A}}),$$

$$ms(F) > 0 \Leftrightarrow F \in (\mathcal{M}\Phi_+(H_{\mathcal{A}}) \cup \mathcal{M}\Phi_-(H_{\mathcal{A}})), \text{ it follows that}$$

$$ms_+(F) = ms_-(F^*), ms_{\Phi}(F) = ms_{\Phi}(F^*), ms(F) = ms(F^*).$$

Lemma

Let $F \in B(H_{\mathcal{A}})$. If $ms_+(F) > 0$ and $ms_-(F) > 0$, then $ms_+(F) = ms_-(F)$.

Lemma

Let $F \in B(H_{\mathcal{A}})$. Then

- 1) $ms_{\Phi}(F) = \min\{ms_+(F), ms_-(F)\}$
- 2) $ms(F) = \max\{ms_+(F), ms_-(F)\}$.

Lemma

Let $F \in B(H_{\mathcal{A}})$, where \mathcal{A} be a W^* -algebra and suppose that $K(\mathcal{A})$ satisfies the cancellation property. Then

$$\sigma^{\mathcal{A}}(F) = \sigma_{ew}^{\mathcal{A}}(F) \cup \sigma_p^{\mathcal{A}}(F) \cup \sigma_{cl}^{\mathcal{A}}(F).$$

Lemma

Let now \mathcal{A} be an arbitrary C^* -algebra. For $F \in B^a(H_{\mathcal{A}})$ set

$\sigma_{\text{ewgc}}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid (F - \alpha I) \notin \mathcal{M}\Phi_0^{gc}(H_{\mathcal{A}})\}$. Then

$$\sigma^{\mathcal{A}}(F) = \sigma_{\text{ewgc}}^{\mathcal{A}}(F) \cup \sigma_p^{\mathcal{A}}(F).$$

Lemma

Let $F \in B^a(H_{\mathcal{A}})$ and suppose $K(\mathcal{A})$ satisfies the cancellation property.

Then $\sigma^{\mathcal{A}}(F) = \sigma_{\text{ew}}^{\mathcal{A}}(F) \cup \sigma_p^{\mathcal{A}}(F) \cup \sigma_{cl}^{\mathcal{A}}(F)$.

Proposition

If $F \in B^a(H_{\mathcal{A}})$ then the components of $\mathcal{A} \setminus (\sigma_{e\alpha}^{\mathcal{A}}(F) \cap \sigma_{e\beta}^{\mathcal{A}}(F))$ are either completely contained in $\mathcal{M}\Phi_+(F) \setminus \mathcal{M}\Phi(F)$ or in $\mathcal{M}\Phi_+(F) \setminus \mathcal{M}\Phi(F)$ or index $(F - \alpha I)$ is constant on them.

Lemma

Let $F \in B^a(H_{\mathcal{A}})$. If $\alpha \in \partial\sigma^{\mathcal{A}}(F) \setminus (\sigma_{e\alpha}^{\mathcal{A}}(F) \cap \sigma_{e\beta}^{\mathcal{A}}(F))$, then $\alpha \in \mathcal{M}\Phi_0(F)$.

Let now $\tilde{\mathcal{M}}\Phi_0(H_{\mathcal{A}})$ be the set of all $F \in B^a(H_{\mathcal{A}})$ such that there exists a decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

w.r.t. which F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$, where F_1 is an isomorphism, N_1, N_2 are finitely generated and

$$N \tilde{\oplus} N_1 = N \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

for some closed submodule $N \subseteq H_{\mathcal{A}}$.

Notice that this implies that $F \in \mathcal{M}\Phi(H_{\mathcal{A}})$ and $N_1 \cong N_2$, so that index $F = [N_1] - [N_2] = 0$. Hence $\tilde{\mathcal{M}}\Phi_0(H_{\mathcal{A}}) \subseteq \mathcal{M}\Phi_0(H_{\mathcal{A}})$.

Let $\mathcal{P}(H_{\mathcal{A}}) = \{P \in B(H_{\mathcal{A}}) \mid P \text{ is a projection and } N(P) \text{ is finitely generated}\}$

and let

$$\sigma_{eW}^{\mathcal{A}}(F) = \{\alpha \in Z(\mathcal{A}) \mid (F - \alpha I) \notin \tilde{\mathcal{M}}\Phi_0(H_{\mathcal{A}})\}$$

for $F \in B^a(H_{\mathcal{A}})$.

Theorem

Let $F \in B^a(H_{\mathcal{A}})$. Then

$$\sigma_{eW}^{\mathcal{A}}(F) = \cap \{ \sigma^{\mathcal{A}}(PF|_{R(P)}) \mid P \in P(H_{\mathcal{A}}) \}$$

where

$$\sigma^{\mathcal{A}}(PF|_{R(P)}) = \{ \alpha \in Z(\mathcal{A}) \mid (PF - \alpha I)|_{R(P)} \text{ is not invertible in } B(R(P)) \}.$$

Lemma

$\tilde{M}\Phi_0(H_{\mathcal{A}})$ is open in $B^a(H_{\mathcal{A}})$.

We let now $\widehat{\mathcal{M}\Phi}_+^-(H_{\mathcal{A}})$ be the space of all $F \in B^a(H_{\mathcal{A}})$ such that there exists a decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

w.r.t. which F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$, where F_1 is an isomorphism,

N_1 is finitely generated and such that there exist closed submodules N'_2, N where $N'_2 \subseteq N_2, N'_2 \cong N_1$,

$H_{\mathcal{A}} = N \tilde{\oplus} N_1 = N \tilde{\oplus} N'_2$ and the projection onto N along N'_2 is adjointable.

Then we set

$$\sigma_{e\tilde{a}}^{\mathcal{A}}(F) := \{\alpha \in Z(\mathcal{A}) \mid (F - \alpha I) \notin \widehat{\mathcal{M}\Phi}_+^-(H_{\mathcal{A}})\}.$$

Theorem

Let $F \in B^a(H_{\mathcal{A}})$. Then $\sigma_{e\tilde{a}}^{\mathcal{A}}(F) = \cap \{\sigma_a^{\mathcal{A}}(PF|_{R(P)}) \mid P \in P^a(H_{\mathcal{A}})\}$ where $\sigma_a^{\mathcal{A}}(PF|_{R(P)})$ is the set of all $\alpha \in Z(\mathcal{A})$ s.t. $(PF - \alpha I)|_{R(P)}$ is not bounded below on $R(P)$ and $P^a(H_{\mathcal{A}}) = P(H_{\mathcal{A}}) \cap B^a(H_{\mathcal{A}})$.

Definition

We set $\widehat{\mathcal{M}\Phi}_-^+(H_{\mathcal{A}})$ to be the set of all $D \in B^a(H_{\mathcal{A}})$ such that there exists a decomposition

$$H_{\mathcal{A}} = M'_1 \tilde{\oplus} N'_1 \xrightarrow{D} M'_2 \tilde{\oplus} N'_2 = H_{\mathcal{A}}$$

w.r.t. which D has the matrix $\begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix}$, where D_1 is an isomorphism,

N'_2 is finitely generated and such that $H_{\mathcal{A}} = M'_1 \tilde{\oplus} N \tilde{\oplus} N'_2$ for some closed submodule N , where the projection onto $M'_1 \tilde{\oplus} N$ along N'_2 is adjointable.

Then we set

$$\sigma_{ed}^{\mathcal{A}}(D) = \{\alpha \in Z(\mathcal{A}) \mid (D - \alpha I) \notin \widehat{\mathcal{M}\Phi}_-^+(H_{\mathcal{A}})\}$$

and for $P \in P^a(H_{\mathcal{A}})$ we set

$$\sigma_d^{\mathcal{A}}(PD|_{R(P)}) = \{\alpha \in Z(\mathcal{A}) \mid (PD - \alpha I)|_{R(P)} \text{ is not onto } R(P)\}.$$

Theorem

Let $D \in B^a(H_{\mathcal{A}})$. Then

$$\sigma_{ed}^{\mathcal{A}}(D) = \bigcap \{\sigma_d^{\mathcal{A}}(PD|_{R(P)}) \mid P \in P^a(H_{\mathcal{A}})\}$$

Definition

We let $\widehat{\widehat{\mathcal{M}\Phi_+}}^-(H_{\mathcal{A}})$ be the set of all $F \in B(H_{\mathcal{A}})$ such that there exists an $\mathcal{M}\Phi_+$ -decomposition for F

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

and closed submodules N, N'_2 with the property that N_1 is isomorphic to N'_2 , $N'_2 \subseteq N_2$ and

$$H_{\mathcal{A}} = N \tilde{\oplus} N_1 = N \tilde{\oplus} N'_2.$$

Theorem

For $F \in B(H_{\mathcal{A}})$ we have

$$\sigma_{e\tilde{a}0}^{\mathcal{A}}(F) = \cap \{ \sigma_{a0}^{\mathcal{A}}(PF|_{R(P)}) \mid P \in P(H_{\mathcal{A}}) \},$$

where $\sigma_{a0}^{\mathcal{A}}(PF|_{R(P)}) = \{ \alpha \in Z(\mathcal{A}) \mid (PF - \alpha I)|_{R(P)} \text{ is not bounded below on } R(P) \text{ or } R(PF - \alpha P) \text{ is not complementable in } R(P) \}$.

Definition

We set $\widehat{\widehat{\mathcal{M}\Phi_-}}^+(H_{\mathcal{A}})$ to be the set of all $G \in B(H_{\mathcal{A}})$ such that there exists an $\mathcal{M}\Phi_-$ -decomposition for G

$$H_{\mathcal{A}} = M'_1 \tilde{\oplus} N'_1 \xrightarrow{G} M'_2 \tilde{\oplus} N'_2 = H_{\mathcal{A}},$$

and a closed submodule N with the property that $H_{\mathcal{A}} = M'_1 \tilde{\oplus} N \tilde{\oplus} N'_2$.

Theorem

For $G \in B(H_{\mathcal{A}})$ we have

$$\sigma_{ed0}^{\mathcal{A}}(G) = \cap \{ \sigma_{d0}^{\mathcal{A}}(PG|_{R(P)}) \mid P \in P(H_{\mathcal{A}}) \},$$

where $\sigma_{d0}^{\mathcal{A}}(PG|_{R(P)}) = \{ \alpha \in Z(\mathcal{A}) \mid R(P) \text{ does not split into the decomposition } R(P) = \tilde{N} \tilde{\oplus} \tilde{N} \text{ with the property that } PG|_{\tilde{N}} \text{ is an isomorphism onto } R(P) \}.$

The boundary of several kinds of Fredholm spectra in \mathcal{A}

Theorem

Let $F \in B^a(H_{\mathcal{A}})$. Then the following inclusions hold:

$$\partial\sigma_{ew}^{\mathcal{A}}(F) \subseteq \partial\sigma_{ef}^{\mathcal{A}}(F) \subseteq \frac{\partial\sigma_{e\beta}^{\mathcal{A}}(F)}{\partial\sigma_{e\alpha}^{\mathcal{A}}(F)} \subseteq \partial\sigma_{ek}^{\mathcal{A}}(F).$$

Theorem

Let $F \in B^a(H_{\mathcal{A}})$. Then

$$\partial\sigma_{ew}^{\mathcal{A}}(F) \subseteq \partial\sigma_{e\tilde{a}}^{\mathcal{A}}(F) \subseteq \partial\sigma_{ea}^{\mathcal{A}}(F)$$

Moreover, $\partial\sigma_{ea}^{\mathcal{A}}(F) \subseteq \partial\sigma_{e\alpha}^{\mathcal{A}}(F)$ if $K(\mathcal{A})$ satisfies the cancellation property.

Perturbations of the generalized spectra in \mathcal{A}

Lemma

$MI(H_{\mathcal{A}})$ is a closed two sided ideal in $B^a(H_{\mathcal{A}})$ and

$$\begin{aligned} MI(H_{\mathcal{A}}) &= \{D \in B^a(H_{\mathcal{A}}) \mid I + DF \in \mathcal{M}\Phi(H_{\mathcal{A}}) \ \forall F \in B^a(H_{\mathcal{A}})\} = \\ &= \{D \in B^a(H_{\mathcal{A}}) \mid I + DF \in \mathcal{M}\Phi(H_{\mathcal{A}}) \ \forall F \in \mathcal{M}\Phi(H_{\mathcal{A}})\} = \\ &= \{D \in B^a(H_{\mathcal{A}}) \mid I + FD \in \mathcal{M}\Phi(H_{\mathcal{A}}) \ \forall F \in B^a(H_{\mathcal{A}})\} = \\ &= \{D \in B^a(H_{\mathcal{A}}) \mid I + FD \in \mathcal{M}\Phi(H_{\mathcal{A}}) \ \forall F \in \mathcal{M}\Phi(H_{\mathcal{A}})\}. \end{aligned}$$

Lemma

- a) If $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$ and $D \in P(\mathcal{M}\Phi(H_{\mathcal{A}}))$, then $F + D \in \mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$.
- b) If $F \in \mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$ and $D \in P(\mathcal{M}\Phi(H_{\mathcal{A}}))$, then $F + D \in \mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$.
- c) If $\mathcal{M}\Phi(H_{\mathcal{A}})$ and $D \in P(\mathcal{M}\Phi(H_{\mathcal{A}}))$, then $D + F \in \mathcal{M}\Phi(H_{\mathcal{A}})$ and $\text{index } D + F = \text{index } F$.

Lemma

We have $P(\mathcal{M}\Phi_0(H_{\mathcal{A}})) = P(\mathcal{M}\Phi(H_{\mathcal{A}}))$.

Proposition

Let $F \in B^a(H_{\mathcal{A}})$. Then

$$\sigma_{ew}^{\mathcal{A}}(F) = \bigcap_{D \in K^*(H_{\mathcal{A}})} \sigma^{\mathcal{A}}(F + D) = \bigcap_{D \in \mathcal{M}I(H_{\mathcal{A}})} \sigma^{\mathcal{A}}(F + D).$$

Theorem

The operator $D \in B^a(H_{\mathcal{A}})$ satisfies the condition $\sigma_{ek}^{\mathcal{A}}(F + D) = \sigma_{ek}^{\mathcal{A}}(F)$ for every $F \in B^a(H_{\mathcal{A}})$ if and only if $D \in P(\mathcal{M}\Phi_+(H_{\mathcal{A}})) \cap P(\mathcal{M}\Phi_-(H_{\mathcal{A}})) = P(\mathcal{M}\Phi(H_{\mathcal{A}}))$.

Lemma

The operator $D \in B^a(H_A)$ satisfies the condition $\sigma_{e\alpha}^A(F + D) = \sigma_{e\alpha}^A(F)$ for every $F \in B^a(H_A)$ if and only if $D \in P(\mathcal{M}\Phi(H_A))$.

Lemma

The operator $D \in B^a(H_A)$ satisfies the condition $\sigma_{e\beta}^A(F + D) = \sigma_{e\beta}^A(F)$ for every $F \in B^a(H_A)$ if and only if $D \in P(\mathcal{M}\Phi(H_A))$.

Lemma

The operator $D \in B^a(H_A)$ satisfies the condition $\sigma_{ef}^A(F + D) = \sigma_{ef}^A(F)$ for every $F \in B^a(H_A)$ if and only if $D \in P(\mathcal{M}\Phi(H_A))$.

Lemma

The operator $D \in B^a(H_A)$ satisfies the condition $\sigma_{ew}^A(F + D) = \sigma_{ew}^A(F)$ for every $F \in B^a(H_A)$ if and only if $D \in P(\mathcal{M}\Phi(H_A))$.

Definition

For $F \in B^a(H_{\mathcal{A}})$ we set $\sigma_{e\alpha'}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \notin \mathcal{M}\Phi_+^{-'}(H_{\mathcal{A}})\}$ and $\sigma_{e\beta'}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \notin \mathcal{M}\Phi_-^{+'}(H_{\mathcal{A}})\}$.

Lemma

Let $F \in B^a(H_{\mathcal{A}})$. Then

$$\sigma_{e\alpha'}^{\mathcal{A}}(F) = \bigcap_{D \in K^*(H_{\mathcal{A}})} \sigma_a^{\mathcal{A}}(F + D) = \bigcap_{D \in P(\mathcal{M}\Phi_+^{-'}(H_{\mathcal{A}}))} \sigma_a^{\mathcal{A}}(F + D),$$

$$\sigma_{e\beta'}^{\mathcal{A}}(F) = \bigcap_{D \in K^*(H_{\mathcal{A}})} \sigma_d^{\mathcal{A}}(F + D) = \bigcap_{D \in P(\mathcal{M}\Phi_-^{+'}(H_{\mathcal{A}}))} \sigma_d^{\mathcal{A}}(F + D),$$

Lemma

Let $F \in B^a(H_{\mathcal{A}})$. Then

- 1) We have $\sigma_{e\alpha'}^{\mathcal{A}}(F + D) = \sigma_{e\alpha'}^{\mathcal{A}}(D)$ for every $D \in B^a(H_{\mathcal{A}})$ if and only if $F \in P(\mathcal{M}\Phi_+^{-'}(H_{\mathcal{A}}))$.
- 2) We have $\sigma_{e\beta'}^{\mathcal{A}}(D) = \sigma_{e\beta'}^{\mathcal{A}}(F + D)$ for every $D \in B^a(H_{\mathcal{A}})$ if and only if $F \in P(\mathcal{M}\Phi_-^{+'}(H_{\mathcal{A}}))$.

We will consider the operator $\mathbf{M}_C^A(F, D) : H_A \oplus H_A \rightarrow H_A \oplus H_A$ given as 2×2 operator matrix

$$\begin{bmatrix} F & C \\ 0 & D \end{bmatrix},$$

where $C \in B^a(H_A)$.

To simplify notation we will only write \mathbf{M}_C^A instead of $\mathbf{M}_C^A(F, D)$ when $F, D \in B^a(H_A)$ are given.

Proposition

For given $F, C, D \in B^a(H_A)$, one has

$$\sigma_e^A(\mathbf{M}_C^A) \subset (\sigma_e^A(F) \cup \sigma_e^A(D)).$$

Theorem

Let $F, D \in B^a(H_A)$. If $\mathbf{M}_C^A \in \mathcal{M}\Phi(H_A \oplus H_A)$ for some $C \in B^a(H_A)$, then $F \in \mathcal{M}\Phi_+(H_A)$, $D \in \mathcal{M}\Phi_-(H_A)$ and for all decompositions

$$H_A = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_A,$$

$$H_A = M'_1 \tilde{\oplus} N'_1 \xrightarrow{D} M'_2 \tilde{\oplus} N'_2 = H_A$$

w.r.t. which F, D have matrices $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$, $\begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix}$, respectively, where F_1, D_1 are isomorphisms, and N_1, N'_2 are finitely generated, there exist closed submodules $\tilde{N}'_1, \tilde{N}'_1, \tilde{N}_2, \tilde{N}_2$ such that $N_2 \cong \tilde{N}_2$, $N'_1 \cong \tilde{N}'_1$, \tilde{N}_2 and \tilde{N}'_1 are finitely generated and

$$\tilde{N}_2 \tilde{\oplus} \tilde{N}_2 \cong \tilde{N}'_1 \tilde{\oplus} \tilde{N}'_1.$$

Proposition

Suppose that there exists some $C \in B^a(H_A)$ such that the inclusion $\sigma_e^{\mathcal{A}}(\mathbf{M}_C^{\mathcal{A}}) \subset \sigma_e^{\mathcal{A}}(F) \cup \sigma_e^{\mathcal{A}}(D)$ is proper. Then for any

$$\alpha \in [\sigma_e^{\mathcal{A}}(F) \cup \sigma_e^{\mathcal{A}}(D)] \setminus \sigma_e^{\mathcal{A}}(\mathbf{M}_C^{\mathcal{A}})$$

we have

$$\alpha \in \sigma_e^{\mathcal{A}}(F) \cap \sigma_e^{\mathcal{A}}(D).$$

Next, we define the following classes of operators on H_A :

$$\mathcal{MS}_+(H_A) = \{F \in B^a(H_A) \mid (F - \alpha 1) \in \mathcal{M}\Phi_-^+(H_A)\}$$

$$\text{whenever } \alpha \in \mathcal{A} \text{ and } (F - \alpha 1) \in \mathcal{M}\Phi_{\pm}(H_A)\},$$

$$\mathcal{MS}_-(H_A) = \{F \in B^a(H_A) \mid (F - \alpha 1) \in \mathcal{M}\Phi_+^-(H_A)\}$$

$$\text{whenever } \alpha \in \mathcal{A} \text{ and } (F - \alpha 1) \in \mathcal{M}\Phi_{\pm}(H_A)\}.$$

Proposition

If $F \in \mathcal{MS}_+(H_A)$ or $D \in \mathcal{MS}_-(H_A)$, then for all $C \in B^a(H_A)$, we have

$$\sigma_e^{\mathcal{A}}(\mathbf{M}_C^{\mathcal{A}}) = \sigma_e^{\mathcal{A}}(F) \cup \sigma_e^{\mathcal{A}}(D)$$

Theorem

Let $F \in \mathcal{M}\Phi_+(H_A)$, $D \in \mathcal{M}\Phi_-(H_A)$ and suppose that there exist decompositions

$$H_A = M_1 \tilde{\oplus} N_1 \xrightarrow{F} N_2^\perp \oplus N_2 = H_A$$

$$H_A = N_1'^\perp \oplus N_1' \xrightarrow{D} M_2' \tilde{\oplus} N_2' = H_A$$

w.r.t. which F, D have matrices

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}, \begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix},$$

respectively, where F_1, D_1 are isomorphisms, N_1, N_2' are finitely generated and assume also that one of the following statements hold:

a) There exists some $J \in B^a(N_2, N_1')$ such that $N_2 \cong \text{Im} J$ and $\text{Im} J^\perp$ is finitely generated.

b) There exists some $J' \in B^a(N_1', N_2)$ such that $N_1' \cong \text{Im} J'$, $(\text{Im} J')^\perp$ is finitely generated.

Then $\mathbf{M}_C^A \in \mathcal{M}\Phi(H_A \oplus H_A)$ for some $C \in B^a(H_A)$.

Theorem

Suppose $\mathbf{M}_C^A \in \mathcal{M}\Phi_-(H_A \oplus H_A)$ for some $C \in B^a(H_A)$. Then $D \in \mathcal{M}\Phi_-(H_A)$ and in addition the following statement holds:
Either $F \in \mathcal{M}\Phi_-(H_A)$ or there exists decompositions

$$H_A \oplus H_A = M_1 \tilde{\oplus} N_1 \xrightarrow{F'} M_2 \tilde{\oplus} N_2 = H_A \oplus H_A,$$

$$H_A \oplus H_A = M'_1 \tilde{\oplus} N'_1 \xrightarrow{D'} M'_2 \tilde{\oplus} N'_2 = H_A \oplus H_A,$$

w.r.t. which F', D' have the matrices $\begin{bmatrix} F'_1 & 0 \\ 0 & F'_4 \end{bmatrix}, \begin{bmatrix} D'_1 & 0 \\ 0 & D'_4 \end{bmatrix}$, where F'_1, D'_1 are isomorphisms, N'_2 is finitely generated, N_1, N_2, N'_1 are closed, but not finitely generated, and $M_2 \cong M'_1, N_2 \cong N'_1$.

Theorem

Let $F, D \in B^a(H_A)$ and suppose that $D \in \mathcal{M}\Phi_-(H_A)$ and either $F \in \mathcal{M}\Phi_-(H_A)$ or that there exist decompositions

$$H_A = M_1 \tilde{\oplus} N_1 \xrightarrow{F} N_2^\perp \tilde{\oplus} N_2 = H_A,$$

$$H_A = N_1'^\perp \tilde{\oplus} N_1' \xrightarrow{D} M_2' \tilde{\oplus} N_2' = H_A,$$

w.r.t. which F, D have the matrices $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}, \begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix},$

respectively, where F_1, D_1 are isomorphisms $N_2',$ is finitely generated and that there exists some

$\iota \in B^a(N_2, N_1')$ such that ι is an isomorphism onto its image in $N_1'.$

Then $\mathbf{M}_C^A \in \mathcal{M}\Phi_-(H_A \oplus H_A)$ for some $C \in B^a(H_A).$

Theorem

Let $\mathbf{M}_C^A \in \mathcal{M}\Phi_+(H_A \oplus H_A)$. Then $F' \in \mathcal{M}\Phi_+(H_A \oplus H_A)$ and either $D \in \mathcal{M}\Phi_+(H_A)$ or there exist decompositions

$$H_A \oplus H_A = M_1 \tilde{\oplus} N_1 \xrightarrow{F'} M_2 \tilde{\oplus} N_2 = H_A \oplus H_A,$$

$$H_A \oplus H_A = M'_1 \tilde{\oplus} N'_1 \xrightarrow{D'} M'_2 \tilde{\oplus} N'_2 = H_A \oplus H_A,$$

w.r.t. which F', D' have matrices $\begin{bmatrix} F'_1 & 0 \\ 0 & F'_4 \end{bmatrix}, \begin{bmatrix} D'_1 & 0 \\ 0 & D'_4 \end{bmatrix},$

respectively, where F'_1, D'_1 are isomorphisms, $M_2 \cong M'_1$ and $N_2 \cong N'_1$, N_1 is finitely generated and N_2, N'_1 are closed, but not finitely generated.

Theorem

Let $F \in \mathcal{M}\Phi_+(H_A)$ and suppose that either $D \in \mathcal{M}\Phi_+(H_A)$ or that there exist decompositions

$$H_A = M_1 \tilde{\oplus} N_1 \xrightarrow{F} N_2^\perp \tilde{\oplus} N_2 = H_A,$$

$$H_A = N_1'^\perp \tilde{\oplus} N_1' \xrightarrow{D} M_2' \tilde{\oplus} N_2' = H_A$$

w.r.t. which F, D have matrices $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}, \begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix}$, respectively, where F_1, D_1 are isomorphisms, N_1' is finitely generated and in addition there exists some $\iota \in B^a(N_1', N_2)$ such that ι is an isomorphism onto its image. Then

$$\mathbf{M}_C^A \in \mathcal{M}\Phi_+(H_A \oplus H_A),$$

for some $C \in B^a(H_A)$.

Definition

Let \mathcal{X} be a Banach space. A sequence $(T_n)_{n \in \mathbb{N}_0}$ of operators in $B(\mathcal{X})$ is called *topologically transitive* if for each non-empty open subsets U, V of \mathcal{X} , $T_n(U) \cap V \neq \emptyset$ for some $n \in \mathbb{N}$. If $T_n(U) \cap V \neq \emptyset$ holds from some n onwards, then $(T_n)_{n \in \mathbb{N}_0}$ is called *topologically mixing*.

Definition

Let \mathcal{X} be a Banach space. A sequence $(T_n)_{n \in \mathbb{N}_0}$ of operators in $B(\mathcal{X})$ is called *hypercyclic* if there is an element $x \in \mathcal{X}$ (called *hypercyclic vector*) such that the orbit $\mathcal{O}_x := \{T_n x : n \in \mathbb{N}_0\}$ is dense in \mathcal{X} . The set of all hypercyclic vectors of a sequence $(T_n)_{n \in \mathbb{N}_0}$ is denoted by $HC((T_n)_{n \in \mathbb{N}_0})$. If $HC((T_n)_{n \in \mathbb{N}_0})$ is dense in \mathcal{X} , the sequence $(T_n)_{n \in \mathbb{N}_0}$ is called *densely hypercyclic*. An operator $T \in B(\mathcal{X})$ is called *hypercyclic* if the sequence $(T^n)_{n \in \mathbb{N}_0}$ is hypercyclic.

Definition

Let \mathcal{X} be a Banach space, and $(T_n)_{n \in \mathbb{N}_0}$ be a sequence of operators in $B(\mathcal{X})$. A vector $x \in \mathcal{X}$ is called a *periodic element* of $(T_n)_{n \in \mathbb{N}_0}$ if there exists a constant $N \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $T_{kN}x = x$. The set of all periodic elements of $(T_n)_{n \in \mathbb{N}_0}$ is denoted by $\mathcal{P}((T_n)_{n \in \mathbb{N}_0})$. The sequence $(T_n)_{n \in \mathbb{N}_0}$ is called *chaotic* if $(T_n)_{n \in \mathbb{N}_0}$ is topologically transitive and $\mathcal{P}((T_n)_{n \in \mathbb{N}_0})$ is dense in \mathcal{X} . An operator $T \in B(\mathcal{X})$ is called *chaotic* if the sequence $\{T^n\}_{n \in \mathbb{N}_0}$ is chaotic.

Linear dynamics of Translation Operators

Definition

Let $U, W \in B(\mathcal{H})$. We define the operator $T_{U,W} : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ by

$$T_{U,W}(F) := WFU \tag{1}$$

for all $F \in B(\mathcal{H})$.

Theorem

Let \mathcal{H} be a separable Hilbert space. Let $W \in B(\mathcal{H})$ be invertible and $U \in B(\mathcal{H})$ be unitary such that for each $k \in \mathbb{N}$ there exists an $N_k \in \mathbb{N}$ with

$$U^n(L_k) \perp L_k \quad \text{for all } n \geq N_k. \quad (2)$$

Then, the following statements are equivalent.

- (i) $T_{U,W}$ is hypercyclic on $B_0(\mathcal{H})$, where $B_0(\mathcal{H})$ is equipped with the operator norm $\|\cdot\|$.
- (ii) For each $m \in \mathbb{N}$ there exist a strictly increasing sequence $\{n_k\}$ in \mathbb{N} and the sequences $\{D_k\}$ and $\{G_k\}$ of operators in $B_0(\mathcal{H})$ such that

$$\lim_{k \rightarrow \infty} \|D_k - P_m\| = \lim_{k \rightarrow \infty} \|G_k - P_m\| = 0, \quad (3)$$

and

$$\lim_{k \rightarrow \infty} \|W^{n_k} G_k\| = \lim_{k \rightarrow \infty} \|W^{-n_k} D_k\| = 0, \quad (4)$$

where P_m denotes the orthogonal projection onto L_m .

Definition

Let \mathcal{X} be a Banach space, $a \in \mathcal{X}$, and $T \in B(\mathcal{X})$. We say that T is *a-transitive* if for each two non-empty open subsets \mathcal{O}_1 and \mathcal{O}_2 of \mathcal{X} with $a \in \mathcal{O}_1$, there are $m, n \in \mathbb{N}$ such that

$$T^n(\mathcal{O}_1) \cap \mathcal{O}_2 \neq \emptyset, \quad T^m(\mathcal{O}_2) \cap \mathcal{O}_1 \neq \emptyset.$$

Theorem

Let $U, W \in B(\mathcal{H})$ such that W is invertible and U is unitary. Then, the following statements are equivalent.

- (i) $T_{U,W}$ and $S_{U,W}$ are 0-transitive on $B_0(\mathcal{H})$.
- (ii) For every finite dimensional subspace K of \mathcal{H} there are strictly increasing sequences $\{n_j\}$ and $\{m_j\}$ in \mathbb{N} and sequences of operators $\{G_j\}$ and $\{D_j\}$ in $B_0(\mathcal{H})$ such that

$$\lim_{j \rightarrow \infty} \|G_j - P_K\| = \lim_{j \rightarrow \infty} \|D_j - P_K\| = 0, \quad (5)$$

and

$$\lim_{j \rightarrow \infty} \|W^{-m_j} G_j\| = \lim_{j \rightarrow \infty} \|W^{n_j} D_j\| = 0. \quad (6)$$

Theorem

Let $U, W \in B(\mathcal{H})$ such that W be invertible and U be unitary. If $T_{U,W}$ is hypercyclic on $B_0(\mathcal{H})$, then $m(W) < 1 < \|W\|$.

Theorem

Let $U, W \in B(\mathcal{H})$ such that W be invertible and U be unitary. Suppose that there is a finite dimensional subspace K of \mathcal{H} such that for a constant $N > 0$, $U^n(K) \perp K$ for all $n \geq N$. Then, we have (i) \Rightarrow (ii):

- (i) P_K belongs to the closure of $\mathcal{P}(\{S_{U,W}^n\}_{n \in \mathbb{N}_0})$ in $B_0(\mathcal{H})$.
- (ii) There exists an increasing sequence (n_k) in \mathbb{N} such that $m(W^{-n_k}) \rightarrow 0$ as $k \rightarrow \infty$.

Theorem

Let \mathcal{H} be a separable Hilbert space and $U, W \in B(\mathcal{H})$ such that W be invertible and U be unitary. Then, we have (ii) \Rightarrow (i):

- (i) the operators $T_{U,W}$ and $S_{U,W}$ are chaotic on $B_0(\mathcal{H})$.
- (ii) For each $m \in \mathbb{N}$ there is a strictly increasing sequence $\{n_k\} \subseteq \mathbb{N}$ such that

$$\lim_{k \rightarrow \infty} \sum_{l=1}^{\infty} \|W^{ln_k} P_m\| = \lim_{k \rightarrow \infty} \sum_{l=1}^{\infty} \|W^{-ln_k} P_m\| = 0,$$

where the corresponding series are convergent for each k .

Cosine Operator Functions

Theorem

Suppose that $U, W \in B(\mathcal{H})$ such that W is invertible and U is unitary. Then, we have (ii) \Rightarrow (i):

- (i) The sequence $(C_{U,W}^{(n)})_{n \in \mathbb{N}_0}$ is topologically transitive on $B_0(\mathcal{H})$.
- (ii) For each $m \in \mathbb{N}$, there are sequences (E_k) and (R_k) of subspaces of L_m and an strictly increasing sequence (n_k) of positive integers such that $L_m = E_k \oplus R_k$ and

$$\lim_{k \rightarrow \infty} \|W^{n_k} P_m\| = \lim_{k \rightarrow \infty} \|W^{-n_k} P_m\| = 0, \quad (7)$$

$$\lim_{k \rightarrow \infty} \|W^{2n_k} P_{E_k}\| = \lim_{k \rightarrow \infty} \|W^{-2n_k} P_{R_k}\| = 0. \quad (8)$$

Theorem

Suppose that $U, W \in B(\mathcal{H})$ such that W is invertible and U is unitary. Let there exist a closed subspace K of \mathcal{H} such that $U^n(K) \perp K$ for all $n \geq N$. Then, (i) \Rightarrow (ii).

- (i) $\mathcal{P}(C_{U,W}^{(n)})$ is dense in $B_0(\mathcal{H})$, and for each $F \in B_0(\mathcal{H})$,
 $\lim_{n \rightarrow \infty} S_{U,W}^n(F) = 0$ in $B_0(\mathcal{H})$.
- (ii) $m(W) < 1$.

Theorem

Suppose that $U, W \in B(\mathcal{H})$ such that W is invertible and U is unitary. Assume that there exists a closed subspace K of \mathcal{H} such that $U^n(K) \perp K$ for all $n \geq N$. We have (i) \Rightarrow (ii).

- (i) $\mathcal{P}(C_{U,W}^{(n)})$ is dense in $B_0(\mathcal{H})$, and $\lim_{n \rightarrow \infty} T_{U,W}^n F = F$ for all $F \in B_0(\mathcal{H})$.
- (ii) $m(W^{-1}) < 1$.

Theorem

Let \mathcal{H} be a separable Hilbert space. We have (ii) \Rightarrow (i):

- (i) The sequence $\{C_{U,W}^{(n)}\}$ is chaotic on $B_0(\mathcal{H})$.
- (ii) For each $m \in \mathbb{N}$, there exists a strictly increasing sequence $\{n_k\} \subseteq \mathbb{N}$ such that

$$\lim_{k \rightarrow \infty} \sum_{l=1}^{\infty} \|W^{ln_k} P_m\| = \lim_{k \rightarrow \infty} \sum_{l=1}^{\infty} \|W^{-ln_k} P_m\| = 0, \quad (9)$$

where the corresponding series are convergent for each k .

Remark

Our sufficient conditions for topological transitivity in the norm topology of $B_0(\mathcal{H})$ in Theorem 129 and Theorem 135 are also sufficient conditions for topological transitivity in the strong topology of $B(\mathcal{H})$. Indeed, since $\{e_n\}$ is an orthonormal basis for \mathcal{H} , it is easily seen that the set $\{P_n F : F \in B(\mathcal{H}), n \in \mathbb{N}\}$ is dense in $B(\mathcal{H})$ in the strong operator topology. Moreover, in this case the conditions (3)-(4) in Theorem 129 can even be relaxed by considering the strong limits instead of the limit in norm and by dropping the requirement that the sequences $\{D_k\}$ and $\{G_k\}$ should belong to $B_0(\mathcal{H})$. Hence, also in the case of strong operator topology on $B(\mathcal{H})$, the operator W in Example 147 satisfies the sufficient conditions for topological transitivity of $T_{U,W}$ and $\{C_{U,W}^{(n)}\}_n$.

Remark

Except from the implication (i) \Rightarrow (ii) in Theorem 129, all our results about sufficient conditions for topological transitivity, easily generalize to the case where $B_0(\mathcal{H})$ is replaced by an arbitrary non-unital C^* -algebra \mathcal{A} , and the set of all finite rank orthogonal projections on \mathcal{H} is replaced by the canonical approximate unit in \mathcal{A} . Indeed, if \mathcal{A} is a non-unital C^* -algebra, then it can be isometrically embedded into a unital C^* -algebra \mathcal{A}_1 such that \mathcal{A} becomes an ideal in \mathcal{A}_1 . If u and w are invertible elements in \mathcal{A}_1 and u is unitary (i.e. $uu^* = u^*u = 1_{\mathcal{A}_1}$), then we can define the operator $T_{u,w}$ on \mathcal{A} by $T_{u,w}(a) := wau$ for all $a \in \mathcal{A}$. Therefore, all our results regarding the sufficient conditions for $T_{u,w}$ to be topologically transitive or chaotic can be generalized in this setting.

Moreover, if \mathcal{A} is a unital C^* -algebra and $H_{\mathcal{A}}$ denotes the standard Hilbert module over \mathcal{A} , then all our results so far can be transferred directly to the case where $B_0(\mathcal{H})$ and $B(\mathcal{H})$ are replaced by $K(H_{\mathcal{A}})$ and $B(H_{\mathcal{A}})$, respectively. Here, $K(H_{\mathcal{A}})$ and $B(H_{\mathcal{A}})$ stand for the set of all compact and all bounded \mathcal{A} -linear operators on $H_{\mathcal{A}}$, respectively.

Theorem

Let $w \in \mathcal{A}_1$ be invertible and u be a unitary element of \mathcal{A}_1 . Suppose that there exist an element $a \in \mathcal{A}^+$ and an $N \in \mathbb{N}$ such that $au^n a = 0$ for all $n \geq N$. Then, (i) \Rightarrow (ii).

- (i) $\mathcal{P}((C_{u,w}^{(n)})_n)$ is dense in \mathcal{A} .
- (ii) $m(\varphi(w)) < 1 < \|\varphi(w)\|$, where (φ, \mathcal{H}) is the universal representation of \mathcal{A}_1 .

Dynamics of the Adjoint Operator

Theorem

Suppose that for every $m \in \mathbb{N}$ there exist sequences (E_k) and (R_k) of subspaces of L_m and an increasing sequence $(n_k) \subseteq \mathbb{N}$ such that for each k , $L_m = E_k \oplus R_k$ and

$$\lim_{k \rightarrow \infty} \|W^{n_k} P_m\| = \lim_{k \rightarrow \infty} \|W^{-n_k} P_m\| = 0, \quad (10)$$

$$\lim_{k \rightarrow \infty} \|W^{2n_k} P_{E_k}\| = \lim_{k \rightarrow \infty} \|W^{-2n_k} P_{R_k}\| = 0. \quad (11)$$

Then, $\{C_{U,W}^{*(n)}\}$ is topologically transitive on $B_1(\mathcal{H})$.

Theorem

Suppose that $U, W \in B(\mathcal{H})$ such that W is invertible and U is unitary. Assume that there exists a finite dimensional subspace K of \mathcal{H} such that $U^n(K) \perp K$ for all $n \geq N$. Then, (i) \Rightarrow (ii).

(i) $\mathcal{P}(C_{U,W}^{(n)*})$ is dense in $B_1(\mathcal{H})$, and for each $F \in B_1(\mathcal{H})$,
 $\lim_{n \rightarrow \infty} S_{U,W}^{*n}(F) = 0$ in $B(\mathcal{H})$.

(ii) $m(W) < 1$.

Theorem

Let $U, W \in B(\mathcal{H})$ be invertible such that U is unitary. Suppose that there exists a finite dimensional subspace K of \mathcal{H} and $N \in \mathbb{N}$ such that $U^n(K) \perp K$ for all $n \geq N$. Then, (i) \Rightarrow (ii):

- (i) $\mathcal{P}\{(C_{U,W}^*)^n\}$ is dense in $B(\mathcal{H})'$ and $\lim_{n \rightarrow \infty} (S_{U,W}^*)^n \varphi = 0$ for all $\varphi \in B(\mathcal{H})'$.
- (ii) $m(W) < 1$.

Theorem

We have (ii) \Rightarrow (i):

- (i) $(C_{U,W}^{(n)*})$ is topologically transitive in $B(\mathcal{H})'$.
- (ii) For every $m \in \mathbb{N}$ there exist sequences (E_k) and (R_k) of subspaces of L_m and an increasing sequence $(n_k) \subseteq \mathbb{N}$ such that for each k , $L_m = E_k \oplus R_k$ and

$$\lim_{k \rightarrow \infty} \|P_m W^{n_k}\| = \lim_{k \rightarrow \infty} \|P_m W^{-n_k}\| = 0, \quad (12)$$

$$\lim_{k \rightarrow \infty} \|P_{E_k} W^{2n_k}\| = \lim_{k \rightarrow \infty} \|P_{R_k} W^{-2n_k}\| = 0. \quad (13)$$

Theorem

We have (i) \Rightarrow (ii):

- (i) $P(T_{U,W}^{*^n})$ is dense in $B(\mathcal{H})'$.
- (ii) $m(W) < 1$.

Theorem

We have (i) \Rightarrow (ii):

- (i) $P(S_{U,W}^{*^n})$ is dense in $B(\mathcal{H})'$.
- (ii) $m(W^{-1}) = \|W\|^{-1} < 1$, that is $\|W\| > 1$.

Theorem

Let $B(\mathcal{H})$ be equipped with the strong topology, and $B(\mathcal{H})'$ be equipped with the w^* -topology, where $B(\mathcal{H})'$ is the dual of $B(\mathcal{H})$. Then we have

(ii) \Rightarrow (i):

- (i) $\{T_{U,W}^{*^n}\}$ and $\{S_{U,W}^{*^n}\}$ are topologically transitive on $B(\mathcal{H})'$.
- (ii) for every $n \in \mathbb{N}$ there exist an increasing sequence $\{n_k\} \subseteq \mathbb{N}$ and sequences of operators $\{G_k\}$ and $\{D_k\}$ in $B(\mathcal{H})$ such that same as theorem 3.2 in the draft with

$$\lim_{k \rightarrow \infty} \|G_k W^{n_k}\| = \lim_{k \rightarrow \infty} \|D_k W^{-n_k}\| = 0,$$

and

$$s\text{-}\lim_{k \rightarrow \infty} G_k = s\text{-}\lim_{k \rightarrow \infty} D_k = P_n,$$

where $s\text{-}\lim$ denotes the limit in the strong operator topology.

Example

Let $\{e_j\}_{j \in \mathbb{N}}$ be an orthonormal basis for a Hilbert space \mathcal{H} . Define $W \in B(\mathcal{H})$ by

$$W(e_j) := \begin{cases} \frac{1}{2} e_{j+2}, & \text{if } j \text{ is odd,} \\ 2 e_{j-2}, & \text{if } j \text{ is even and } j > 2, \\ e_1, & \text{if } j = 2. \end{cases}$$

Then, W is invertible and $\|W\| = 2$. For each fixed $k \in \mathbb{N}$ it is easily checked that $\|W^{2k-1+m}P_{2k}\| = \frac{1}{2^m}$ for all $m \in \mathbb{N}$. Consequently, $\|W^{2k-1+m}P_{2k-1}\| \leq \frac{1}{2^m}$. Further, it is also easily verified that for each $k, m \in \mathbb{N}$ we have $\|W^{-2k-m}P_{2k+1}\| = \frac{1}{2^{m-1}}$, and this gives that $\|W^{-2k-m}P_{2k}\| \leq \frac{1}{2^{m-1}}$. As above, P_n denotes the orthogonal projection onto $\text{span}\{e_1, \dots, e_n\}$.

It follows that

$$\|P_{2k}(W^*)^{2k-1+m}\| = \frac{1}{2^m}, \quad \|P_{2k+1}(W^*)^{-2k-m}\| = \frac{1}{2^{m-1}},$$

for all $k, m \in \mathbb{N}$.

Then W and W^* satisfy the sufficient condition in various results above on topological transitivity. If we instead of H consider $H_{\mathcal{A}}$ and let $\{e_j\}_{j \in \mathbb{N}}$ denote the standard basis, then the same arguments applies in this case also.

Example

Let $F(e_k) = e_{2k}$ for all k .

Then $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$

Example

Let $D(e_{2k-1}) = 0, D(e_{2k}) = e_k$.

Then $D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$

Example

In general, let $\iota : \mathbb{N} \rightarrow \iota(\mathbb{N})$ be a bijection such that $\iota(\mathbb{N}) \subseteq \mathbb{N}$ and $\mathbb{N} \setminus \iota(\mathbb{N})$ infinite. Moreover we may define ι in a such way s.t.

$\iota(1) < \iota(2) < \iota(3) < \dots$. Then, if we define an \mathcal{A} -linear operator F as $F(e_k) = e_{\iota(k)}$ for all k , we get that $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$. Moreover, if we define an \mathcal{A} -linear operator D as

$$D(e_k) = \begin{cases} e_{\iota^{-1}(k)}, & \text{for } k \in \iota(\mathbb{N}), \\ 0, & \text{else} \end{cases}$$

then $D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$.

Those examples are also valid in the case when $\mathcal{A} = \mathbb{C}$, that is when $H_{\mathcal{A}} = H$ is a Hilbert space. We will now introduce examples where we use the structure of \mathcal{A} itself in the case when $\mathcal{A} \neq \mathbb{C}$:

Example

Let $\mathcal{A} = (L^\infty([0, 1]), \mu)$, where μ is a Borel probability measure. Set

$$F(f_1, f_2, f_3, \dots) = (\chi_{[0, \frac{1}{2}]} f_1, \chi_{[\frac{1}{2}, 1]} f_1, \chi_{[0, \frac{1}{2}]} f_2, \chi_{[\frac{1}{2}, 1]} f_2, \dots) .$$

Then F is bounded \mathcal{A} -linear operator, $\ker F = \{0\}$,

$$\operatorname{Im} F = \operatorname{Span}_{\mathcal{A}} \{ \chi_{[0, \frac{1}{2}]} e_1, \chi_{[\frac{1}{2}, 1]} e_2, \chi_{[0, \frac{1}{2}]} e_3, \chi_{[\frac{1}{2}, 1]} e_4, \dots \},$$

and clearly $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$.

Example

Let again $\mathcal{A} = (L^\infty([0, 1]), \mu)$. Set

$$D(g_1, g_2, g_3, \dots) = (\chi_{[0, \frac{1}{2}]} g_1 + \chi_{[\frac{1}{2}, 1]} g_2, \chi_{[0, \frac{1}{2}]} g_3 + \chi_{[\frac{1}{2}, 1]} g_4, \dots) .$$

Then $\ker D = \operatorname{Im} F$, D is an \mathcal{A} -linear, bounded operator and $\operatorname{Im} D = H_{\mathcal{A}}$. Thus $D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$. Indeed, $D = F^*$.

Example

Let $\mathcal{A} = B(H)$, where H is a Hilbert space and let P be an orthogonal projection on H . Set

$$F(T_1, T_2, \dots) = (PT_1, (I - P)T_1, PT_2, (I - P)T_2, \dots),$$

$$D(S_1, S_2, \dots) = (PS_1 + (I - P)S_2, PS_3 + (I - P)S_4, \dots),$$

then by similar arguments $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$, $D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$.

Example

In general, suppose that $\{p_j^i\}_{j,i \in \mathbb{N}}$ is a family of projections in \mathcal{A} s.t.

$p_{j_1}^i p_{j_2}^i = 0$ for all i , whenever $j_1 \neq j_2$ and $\sum_{j=1}^k p_j^i = 1$ for some $k \in \mathbb{N}$.

Set

$$F'(\alpha_1, \dots, \alpha_n, \dots) = (p_1^1 \alpha_1, p_2^1 \alpha_1, \dots, p_k^1 \alpha_1, p_2^2 \alpha_2, p_2^2 \alpha_2, \dots, p_k^2 \alpha_2, \dots),$$

$$D'(\beta_1, \dots, \beta_n, \dots) = \left(\sum_{i=1}^k p_i^1 \beta_i, \sum_{i=1}^k p_i^2 \beta_{i+k}, \dots \right).$$

Then $F' \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$, $D' \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$.

Recalling now that a composition of two $\mathcal{M}\Phi_+$ operators is again an $\mathcal{M}\Phi_+$ operator and that the same is true for $\mathcal{M}\Phi_-$ operators, we may take suitable compositions of operators from these examples in order to construct more $\mathcal{M}\Phi_{\pm}$ operators.

Even more $\mathcal{M}\Phi_{\pm}$ operators can be obtained by composing these operators with isomorphisms of $H_{\mathcal{A}}$. We will present here also some isomorphisms of $H_{\mathcal{A}}$.

Example

Let $j : \mathbb{N} \rightarrow \mathbb{N}$ be a bijection. Then the operator U given by $U(e_k) = e_{j(k)}$ for all k is an isomorphism of $H_{\mathcal{A}}$. This is a classical well known example of an isomorphism.

Example







Let $(\alpha_1, \dots, \alpha_n, \dots) \in \mathcal{A}^{\mathbb{N}}$ be a sequence of invertible elements in \mathcal{A} s.t. $\|\alpha_k\| \leq M$ for all $k \in \mathbb{N}$ and some $M > 0$. If the operator V is given by $V(e_k) = e_k \cdot \alpha_k$ for all k , then V is an isomorphism of $H_{\mathcal{A}}$. Moreover, if $(\alpha_1, \dots, \alpha_n, \dots)$ is the sequence from above, we may let \tilde{V} be the operator on $H_{\mathcal{A}}$ given by $\tilde{V}(x_1, \dots, x_n) = (\alpha_1 x_1, \dots, \alpha_n x_n, \dots)$. Then \tilde{V} is also an isomorphism of $H_{\mathcal{A}}$.

Thank you for attention!

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