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The n-dimensional quadratic Heisenberg algebra as a "non–commutative" sl$(2, \mathbb{C})$

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Abstract

We prove that the commutation relations among the generators of the n–dimensional quadratic Heisenberg algebra ($n \in \mathbb{N}$), give rise to a kind of *non–commutative extension* of $\mathrm{sl}(2, \mathbb{C})$ (more precisely of its unique 1–dimensional central extension). This *non-commutativity* has a different nature from the one considered in quantum groups. In particular we prove that, for most values of n, this Lie algebra cannot be isomorphic to $\mathrm{sl}(N, \mathbb{C})$ for almost any value of N. We study under which conditions the vacuum distribution of an Hermitean element of this algebra can be reduced to a product measure by a change of coordinates in the test function space, as in the Gaussian case, and we find a necessary and sufficient condition for this. From this we deduce that vacuum distributions of generic Hermitean elements **cannot** be reduced to product measures. In particular this is true for the truncated Virasoro fields for which we give some examples where this reduction is possible.

Definition 1 The **complex** $2d+1$**–dimensional Heisenberg algebra**, denoted $\mathrm{heis}_{1,\mathbb{C}}(d)$, is the $*$–Lie algebra with generators

$$\mathbf{1} \text{ (central element) }, \; a_j^\dagger, a_k, \; j,k \in \{1,\ldots,n\}$$

commutation relations

$$[a_k, a_j^\dagger] = \delta_{j,k}\mathbf{1} \tag{1}$$

$$[a_k, \mathbf{1}] = [a_j^\dagger, \mathbf{1}] = [a_k, a_j] = [a_k^\dagger, a_j^\dagger] = 0$$

and involution given by

$$a_j^* = a_j^\dagger \quad ; \quad \left(a_j^\dagger\right)^* = a_j \quad ; \quad \mathbf{1}^* = \mathbf{1} \tag{2}$$

The complex n–dimensional quadratic Boson algebra

We are interested in the complex vector space, denoted

$$\text{heis}_{2;\mathbb{C}}(n)$$

of all **homogeneous quadratic expressions** in the generators of the Heisenberg algebra. These are called **homogeneous quadratic operators**
(simply **quadratic operators** in the following). Any such operator can be written as a multiple of the identity plus a sum of the following 3 types of quadratic operators:

$$a^\dagger A a^\dagger := \sum_{j,k=1}^{n} A_{j,k} a_j^\dagger a_k^\dagger$$

$$a^\dagger B a := \sum_{j,k=1}^{n} B_{j,k} a_j^\dagger a_k$$

$$aCa := \sum_{j,k=1}^{n} C_{j,k} a_j a_k$$

We denote

$$M_n(\mathbb{C}) := \{\text{of } n \times n \text{ complex matrices}\}$$

$$M_{n,sym}(\mathbb{C}) := \{n \times n \text{ complex symmetric matrices}\}$$

For $M \equiv (M_{j,k}) \in M_n(\mathbb{C})$, define

$$(M^T)_{j,k} := M_{k,j} \text{transpose of } \; M$$

$$(\overline{M})_{j,k} := \overline{M_{j,k}} \text{transpose of } \; M$$

$$(M^*)_{j,k} := (\overline{M})^T_{j,k} \text{adjoint of } \; M$$

Thus

$$M \in M_{n,sym}(\mathbb{C}) \;\Rightarrow\; (M^*)_{j,k} = \overline{(M)^T}_{j,k} = \overline{M_{k,j}}$$

Since creators (resp. annihilators) mutually commute, one has

$$a^\dagger A a^\dagger = \sum_{j,k=1}^{n} \frac{1}{2}\left(A_{j,k} + A_{k,j}\right) a_j^\dagger a_k^\dagger$$

$$aCa = \sum_{j,k=1}^{n} \frac{1}{2}\left(C_{j,k} + C_{k,j}\right) a_j a_k \qquad (3)$$

i.e. the expressions $a^\dagger A a^\dagger$ and aCa are parametrized by **symmetric matrices**, $A^T = A$ and $C^T = C$.

Lemma 1 $\text{heis}_{2;\mathbb{C}}(n)$ is a $*$–Lie sub–algebra of the universal enveloping algebra of $\text{heis}_{\mathbb{C}}(n)$ with involution given by

$$(a^\dagger A a^\dagger)^* = aA^*a$$

$$(a^\dagger B a)^* = a^\dagger B^* a \ ; \ (aCa)^* = a^\dagger C^* a^\dagger \tag{4}$$

central element 1 and with the following commutation relations.
For $M, N \in M_{n,sym}(\mathbb{C})$:

$$[aMa, a^\dagger N a^\dagger] = 2 \ \text{Tr}(MN) + 4\, a^\dagger NMa \tag{5}$$

$$[a^\dagger M a^\dagger, a^\dagger N a^\dagger] = [aMa, aNa] = 0 \tag{6}$$

For $M \in M_{n,sym}(\mathbb{C})$, $N \in M_n(\mathbb{C})$:

$$[aMa, a^\dagger Na] = a\left(MN + (MN)^T\right)a \tag{7}$$

$$[a^\dagger M a^\dagger, a^\dagger Na] = -a^\dagger \left(NM + (NM)^T\right)a^\dagger \tag{8}$$

For $M, N \in M(\mathbb{C})$:

$$[a^\dagger Ma, a^\dagger Na] = a^\dagger [M, N] a \tag{9}$$

The commutation relations (7) and (2) suggest to introduce **the binary composition law**:

$$X \circ Y := XY + (XY)^T \tag{10}$$

With this notation one can write the commutation relations in the form

Corollary 1 For $M, N \in M_{n,sym}(\mathbb{C})$:

$$[aMa, a^\dagger N a^\dagger] = 2 \ \mathrm{Tr}(MN) + 4\, a^\dagger N M a \tag{11}$$

$$[a^\dagger M a^\dagger, a^\dagger N a^\dagger] = [aMa, aNa] = 0 \tag{12}$$

For $M \in M_{n,sym}(\mathbb{C})$, $N \in M_n(\mathbb{C})$:

$$[aMa, a^\dagger N a] = a(M \circ N)a \tag{13}$$

$$[a^\dagger M a^\dagger, a^\dagger N a] = -a^\dagger (N \circ M) a^\dagger \tag{14}$$

For $M, N \in M_n(\mathbb{C})$:

$$[a^\dagger M a, a^\dagger N a] = a^\dagger [M, N] a \tag{15}$$

One can consider heis$_{2;\mathbb{C}}(n)$ as a concrete realization of the following abstract $*$–Lie algebra.

Definition 2 The $*$–Lie algebra $sl(2, M_n(\mathbb{C}))$, with central element denoted E, is characterized by the following properties. There exists a vector space isomorphism

$$(\imath, B^+, \Lambda, B^-) : (c1_{M_n}, A, B, C) \in \tag{16}$$

$$\mathbb{C} \cdot 1_{M_n} \times M_{n,sym}(\mathbb{C}) \times M_n(\mathbb{C}) \times M_{n,sym}(\mathbb{C})$$

$$\to c\mathbf{1} + B^+(A) + \Lambda(B) + B^-(C) \in sl(2, M_n(\mathbb{C}))$$

satisfying

$$B^+(A)^* = B^-(A^*)$$

$$\Lambda(B)^* = \Lambda(B^*) \ ; \ B^-(C)^* = B^+(C^*) \tag{17}$$

and with the following commutation relations.

For $M, N \in M_n(\mathbb{C})$ and with $\circ$ defined by $X \circ Y := XY + (XY)^T$. For $M, N \in M_{n,sym}(\mathbb{C})$:

$$[B^-(M), B^+(N)] = 2\ \mathrm{Tr}(MN) + 4\Lambda(NM) \tag{18}$$

$$[B^+(M), B^+(N)] = [B^-(M), B^-(N)] = 0 \tag{19}$$

For $M \in M_{n,sym}(\mathbb{C})$, $N \in M_n(\mathbb{C})$:

$$[B^-(M), \Lambda(N)] = B^-(M \circ N) \tag{20}$$

$$[B^+(M), \Lambda(N)] = -B^+(N \circ M) \tag{21}$$

For $M, N \in M_n(\mathbb{C})$:

$$[\Lambda(M), \Lambda(N)] = \Lambda([M, N]) \tag{22}$$

The **real $*$–Lie sub–algebra of the skew–adjoint** elements of $sl(2, M_n(\mathbb{C}))$ will be denoted $sl_{\mathbb{R}}(2, M_n(\mathbb{C}))$.

The Lie groups associated to $sl(2, M_n(\mathbb{C}))$ and $sl_{\mathbb{R}}(2, M_n(\mathbb{C}))$ are denoted respectively $SL(2, M_n(\mathbb{C}))$ and $SL_{\mathbb{R}}(2, M_n(\mathbb{C}))$.

The case $n = 1$

If $n = 1$, the above written commutation relations reduce to

$$[B^-, B^+] = M \quad ; \quad [M, B^\pm] = \pm 2B^\pm$$

$$(B^-)^* = B^+ \quad ; \quad M^* = M$$

which are the defining relations of the $*$–Lie algebra $sl(2, \mathbb{C})$.

The one–mode quadratic Weyl operators were introduced in the paper
L. Accardi, H. Ouerdiane, H. Rebei:
On the quadratic Heisenberg group,
Infin. Dimens. Anal.Quantum Probab. Relat. Top., Vol.13 (4), 551-587 (2010)

where it was shown that they satisfy the **quadratic analogue of the Weyl relations**.
In the same paper the **quadratic analogue of the Heisenberg group**, denoted QHeis(1), was constructed and the **underling manifold** $\mathfrak{D}_+ \subset \mathbb{R}^3$ explicitly determined

$$\mathfrak{D}_+ := \Big\{ Z = (z, \lambda) \in \mathbb{C} \times \mathbb{R} \ : \ w_Z \in \mathbb{R}_+ \cup i]0, \frac{\pi}{2}]$$

$$\text{with } \lambda > 0 \text{ if } w_Z = i\frac{\pi}{2} \Big\}$$

$$w_Z := \left(|z|^2 - \lambda^2 \right)^{\frac{1}{2}} \tag{23}$$

(see figure).
Also the **analytic form of the group law** was determined in the above paper and later simplified in
L. Accardi, A. Dhahri and H. Rebei:
C^***–Quadratic Quantization**,
Journal of statistical physics, 22.(1) (2018)

However the problem to identify QHesi(1) with some of the known classical groups was left open in those papers.
This problem was recently solved in the paper:

H. Rebeï, H. Rguigui A. Riahi and Z.A. Al-Hussain:
Identification of the $1-$mode quadratic Heisenberg grooup with the projective group PSU$(1,1)$ **and holomorphic representation**,
IDAQP (2020) N. 4
where it is shown that
QHesi(1) **is isomorphic to the projective group PSU$(1,1)$ with its usual law**.
This is, a posteriori, natural because $su(1,1)$ is the real form of the $*-$Lie algebra $sl(2,\mathbb{C})$ and the quadratic Weyl operators are exponentials of the skew–adjoint Lie sub–algebra of $sl(2,\mathbb{C})$.
Recall that SU$(1,1)$ be the sub-group of $SL2(\mathbb{C})$ given by

$$\mathrm{SU}(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in M_2(\mathbb{C}) \quad : |\alpha|^2 - |\beta|^2 = 1 \right\}$$

and that the projective group associated with SU$(1,1)$ is by definition the quotient

$$\mathrm{PSU}(1,1) = \mathrm{SU}(1,1)/\{\pm I_2\}$$

where I_2 is the identity matrix in $M_2(\mathbb{C})$.

They introduce the quadratic exponential vectors

$$\Phi(z) := e^{zB^+}\Phi = \sum_{n=0}^{+\infty} \frac{z^n}{n!}(B^+)^n\Phi \qquad (24)$$

$$z \in \mathbb{D} = \{z \in \mathbb{C} \,:\, |z| < 1\}$$

and prove that their scalar product is

$$\langle \Phi(w), \Phi(z)\rangle = (1 - \overline{w}z)^{-\mu} \quad \forall\, w, z \in \mathbb{D} \quad (25)$$

and show that the action of the re–scaled quadratic Weyl operators on the quadratic exponential vectors is given by:

$$W_r(Z)\Phi(t) = (P_Z - it\overline{M}_Z)^{-\mu}\Phi\Big(\frac{iM_Z + t\overline{P_Z}}{P_Z - it\overline{M}_Z}\Big) \qquad (26)$$

for all $t \in \mathbb{D}$, where

$$M_Z = \frac{\sinh(w_Z)}{zw_Z}$$

$$P_Z = \cosh(w_Z) - i\frac{\sinh(w_Z)}{\lambda w_Z}$$

and λ and w_Z are those in the definition of $\mathfrak{D}_+$.

Relations with sl$(N, \mathbb{R})$

Lemma 2 sl$(N, \mathbb{R})$ and $sl_{\mathbb{R}}(2, M_n(\mathbb{C}))$ are isomorphic as vector spaces if and only if N and n have the form

$$n = 2n_1 + 1 \ ; \ N = 2(2p_1 + 1) \ ; \ n_1, p_1 \in \mathbb{N}$$

where the pair $(n_1, p_1) \in \mathbb{N}^2$ is any solution of the quadratic diophantine equation

$$2(2p_1 + 1)^2 = (2n_1 + 1)^2 + 1 \tag{27}$$

Remark. Notice that, $p_1 = n_1 = 0$ is a solution of (27) and in fact in this case we know that the vector space isomorphism is a $*-$Lie algebra isomorphism.

Group elements and their 1–st and 2–d kind coordinates

The elements of $heis_{\mathbb{C}}(2; n)$ are parametrized by quadruples

$$(x, A, B, C) \in \mathbb{C} \times M_{n,sym}(\mathbb{C}) \times M_n(\mathbb{C}) \times M_{n,sym}(\mathbb{C})$$

and we consider the natural topology induced by this parametrization.
We have proved that, for (z, A, B, C) near the origin, the corresponding element of $heis_{\mathbb{C}}(2; n)$ can be exponentiated in the sense that the corresponding exponential series converges on a dense sub–space of the fock space.
In any finite–dimensional representation, there is no convergence problem.

Following the general theory of Lie groups, we say that the quadruple (x, A, B, C) defines the **the second kind coordinates of** the group element $G(x, A, B, C) \in Heis_{\mathbb{C}}(2; n)$ if

$$G(x, A, B, C)$$

$$= e^{x\mathbf{1}} e^{a^\dagger A a^\dagger} e^{a^\dagger B a} e^{aCa} = e^{x\mathbf{1}} e^{B^+(A)} e^{\Lambda(B)} e^{B^-(C)}$$

Similarly **the first kind coordinates of** $G(x, A, B, C) \in Heis_{\mathbb{C}}(2; n)$ are defined by

$$W(x, A, B, C) \tag{28}$$

$$= e^{x\mathbf{1} + a^\dagger A a^\dagger + a^\dagger B a + aCa} = e^{x\mathbf{1} + B^+(A) + \Lambda(B) + B^-(C)}$$

In both kinds of coordinates one can find a **sub–set** of the whole domain of the coordinates, i.e.
$\mathbb{C} \cdot 1_{M_n} \times M_{n,sym}(\mathbb{C}) \times M_n(\mathbb{C}) \times M_{n,sym}(\mathbb{C})$,
in which the correspondences

$$W(x, A, B, C) \mapsto (x, A, B, C)$$

$$G(x, A, B, C) \mapsto (x, A, B, C)$$

is one–to–one. This domain can be considered as a local **embedding** of the group manifold of $\text{Heis}_{2;\mathbb{C}}(n)$ into the vector space
$\mathbb{C} \cdot 1_{M_n} \times M_{n,sym}(\mathbb{C}) \times M_n(\mathbb{C}) \times M_{n,sym}(\mathbb{C})$.

On this domain the group multiplication law induces a group composition law through the identities

$$W(x_1, A_1, B_1, C_1)W(x_2, A_2, B_2, C_2)$$

$$=: W\left((x_1, A_1, B_1, C_1) \diamond_1 (x_2, A_2, B_2, C_2)\right)$$

$$G(x_1, A_1, B_1, C_1)G(x_2, A_2, B_2, C_2)$$

$$=: G\left((x_1, A_1, B_1, C_1) \diamond_2 (x_2, A_2, B_2, C_2)\right)$$

For the usual Heisenberg algebra, the composition law $\diamond_1$ is the composition law of the **Heisenberg group**.

Typically both composition laws $\diamond_1$ and $\diamond_2$ are strongly non–linear functions of the coordinates. In the 1–dimensional case and for the sub–group $\mathrm{Heis}_2(1)$ of $\mathrm{Heis}_{2;\mathbb{C}}(1)$, i.e. up to isomorphism $sl(2,\mathbb{R})$, both the domain and the explicit form of $\diamond_1$ were determined in the paper:
L. Accardi, H. Ouerdiane and H. Rebei,
On the Quadratic Heisenberg Group,
Infin. Dimens. Anal. Quantum Probab. Relat. Top. (IDA-QP) 13 (4) (2010) 551-587.

Our goal is to extend this result to the multi–dimensional case.

The splitting lemma

Formulas expressing second kind coordinates in terms of first kind ones are called **splitting or disentangling formulas**.
In the case of $Heis_{\mathbb{C}}(2;n)$, they are given by the following lemma due to
P. J. Feinsilver and G. Pap:
Calculation of Fourier transforms of a Brownian motion on the Heisenberg group using splitting formulas,
Journal of Functional Analysis
249 (2007) 1–30

Lemma 3 For $A, C \in Sym\,(M_n(\mathbb{C}))$, $B \in M_n(\mathbb{C})$, define

$$v := \begin{pmatrix} B & 2A \\ -2C & -B^T \end{pmatrix}$$

and P, Q, R, S by

$$\begin{pmatrix} P(t) & Q(t) \\ -R(t) & S(t) \end{pmatrix} := e^{tv}$$

Then, for $t \in \mathbb{R}$ sufficiently close to 0:

$$e^{t\left(a^\dagger A a^\dagger + a^\dagger B a + aCa\right)}$$

$$= e^{-\frac{t}{2}\ \mathrm{Tr}(B) + \frac{1}{2}\ \mathrm{Tr}(g_t(A,B,C))}$$

$$e^{\frac{1}{2} a^\dagger \widehat{f}_t(A,B,C) a^\dagger} \cdot e^{a^\dagger g_t(A,B,C) a} e^{\frac{1}{2} a \widehat{h}_t(A,B,C) a}$$

or, in the B-notation

$$e^{t\left(B^+(A) + \Lambda(B) + B^-(C)\right)}$$

$$= e^{-\frac{t}{2}\ \mathrm{Tr}(B) + \frac{1}{2}\ \mathrm{Tr}(g_t(A,B,C))} e^{\frac{1}{2} B^+(\widehat{f}_t(A,B,C))}$$

$$\cdot e^{\Lambda(g_t(A,B,C))} e^{\frac{1}{2} B^-(\widehat{h}_t(A,B,C))}$$

where

$$f_t(A,B,C) = Q(t) S(t)^{-1}$$

$$g_t(A,B,C) = -\log S(t)^T$$

$$h_t(A,B,C) = S(t)^{-1} R(t)$$

and

$$\widehat{f} = (f + f^T)/2 \quad , \quad \widehat{h} = (h + h^T)/2$$

denote the symmetric parts of f, h.

Remark. One proves separately the invertibility of $S(t)$ for small t.

Summarizing:
– A group element in coordinates of the first kind $W(x, A, B, C)$ (**quadratic Weyl opertor**) is determined by 1 scalar and 3 matrices (A, B, C).
– The triple (A, B, C) uniquely determines the 1–parameter family of matrices $Q(t), R(t), S(t)$.
– The matrices $Q(t), R(t), S(t)$ determine the matrices of the splitted form of $W(x, A, B, C)$ through the identities:

$$\widehat{f}_t(A,B,C) = (Q(t)S(t)^{-1})_{sym}$$

$$:= \frac{1}{2}((Q(t)S(t)^{-1}) + ((S(t)^T)^{-1}Q(t)^T))$$

$$g_t(A,B,C) = -\log S(t)^T$$

$$\widehat{h}_t(A,B,C) = S(t)^{-1}R(t))_{sym}$$

$$:= \frac{1}{2}((S(t)^{-1}R(t)) + (R(t)^T(S(t)^T)^{-1}))$$

and log is meant in the sense of its principal part.

The group law in coordinates of the first kind

Theorem 1 In the notation of Section **??**, let $x_i \in \mathbb{C}$, $A_i, C_i \in M_{n,sym}(\mathbb{C})$ and $B_i \in M_n(\mathbb{C})$, $i = 1, 2$. Then

$$W(x_1, A_1, B_1, C_1)W(x_2, A_2, B_2, C_2) = W(x, A, B, C)$$

where x, A, B, C are determined by the system

$$x' = x - \frac{1}{2}\ \mathrm{Tr}(B) + \frac{1}{2}\ \mathrm{Tr}\left(g(A, B, C)\right)$$

$$A' = \frac{1}{2}\widehat{f}(A, B, C)\ ;\ B' = g(A, B, C)$$

$$C' = \frac{1}{2}\widehat{h}(A, B, C)$$

The explicit form of (x', A', B', C') is also determined, but it is complex and does not give much insight.

Denoting, for $i = 1, 2$,

$$\widehat{f}_i = \frac{1}{2}\widehat{f}(A_i, B_i, C_i) \; ; \; g_i = g(A_i, B_i, C_i)$$

$$\widehat{h}_i = \frac{1}{2}\widehat{h}(A_i, B_i, C_i)$$

one has

$$x' = x_1 + x_2 + \frac{1}{2} \ \mathsf{Tr}\,(-\,(g_1 + g_2) +$$

$$g\left(\widehat{f}_2, 4\widehat{f}_2\widehat{h}_1, 2(\widehat{h}_1\widehat{f}_2\widehat{h}_1 + (\widehat{h}_1\widehat{f}_2\widehat{h}_1)^T)\right)$$

$$+g\left(g_1X + (g_1X)^T, g_1, 0\right) + g\left(0, g_2, Zg_2 + (Zg_2)^T\right)\Big)$$

$$+\frac{1}{2} \ \mathsf{Tr}\,(-(B_1 + B_2) + g(A_1, B_1, C_1) + g(A_2, B_2, C_2))$$

$$A' = X + \widehat{f}_1 + \frac{1}{2}\widehat{f}\left(g_1X + (g_1X)^T, g_1, 0\right)$$

$$B' = Y + g\left(g_1X + (g_1X)^T, g_1, 0\right)$$

$$+g\left(0, g_2, Zg_2 + (Zg_2)^T\right)$$

$$+BCH\left(g\left(g_1X+(g_1X)^T,g_1,0\right),Y\right)$$

$$+BCH\left(E,g\left(0,g_2,Zg_2+(Zg_2)^T\right)\right)$$

$$C'=\widehat{h}_2+Z+\frac{1}{2}\widehat{h}\left(0,g_2,Zg_2+(Zg_2)^T\right)$$

and

$$E=Y+g\left(g_1X+(g_1X)^T,g_1,0\right)$$

$$+BCH\left(g\left(g_1X+(g_1X)^T,g_1,0\right),Y\right)$$

$$X=\frac{1}{2}\widehat{f}\left(\widehat{f}_2,4\widehat{f}_2\widehat{h}_1,2\left(\widehat{h}_1\widehat{f}_2\widehat{h}_1+\left(\widehat{h}_1\widehat{f}_2\widehat{h}_1\right)^T\right)\right)$$

$$Y=g\left(\widehat{f}_2,4\widehat{f}_2\widehat{h}_1,2\left(\widehat{h}_1\widehat{f}_2\widehat{h}_1+\left(\widehat{h}_1\widehat{f}_2\widehat{h}_1\right)^T\right)\right)$$

$$Z=\frac{1}{2}\widehat{h}\left(\widehat{f}_2,4\widehat{f}_2\widehat{h}_1,2\left(\widehat{h}_1\widehat{f}_2\widehat{h}_1+\left(\widehat{h}_1\widehat{f}_2\widehat{h}_1\right)^T\right)\right)$$

Characteristic functions

The adjoint action of the quadratic Lie group on the quadratic $*$–Lie algebra

Recall that

$$C \circ B = CB + (CB)^T = cB + B^c = 2CB$$

Lemma 4

$$e^{[aCa,\ \cdot]} aC'a = aC'a \tag{29}$$

$$e^{[aCa,\ \cdot]} a^+ Ba = a^+ Ba + a(C \circ B)a \tag{30}$$

$$e^{[aCa,\ \cdot]} a^+ Aa^+ \tag{31}$$

$$= a^+ Aa^+ + 4\, a^+ ACa + 4a(C \circ (AC))a$$

$$e^{[a^+ Aa^+,\ \cdot]} a^+ A' a^+ = a^+ A' a^+ \tag{32}$$

$$e^{[a^+ Aa^+,\ \cdot]} a^+ Ba = a^+ Ba - a(B \circ A)a \tag{33}$$

$$e^{[a^+ Aa^+,\ \cdot]} aCa \tag{34}$$

$$= aCa - 4\ \mathrm{Tr}(AC) - 4\, a^+ (AC)^T a + 4a^+ ((AC) \circ A) a^+$$

The binary operation $(A, B) \mapsto A \circ B$ is a kind of multiplication, but it is not associative. So when we iterate it, in particular if we want to define an **exponential function** for this multiplication, we have to keep in mind the order of the iteration.

Lemma 5 Introducing, for $n \in \mathbb{N}$ and $B, C \in M_n(\mathbb{C})$, the inductively defined notations

$$C \,\widehat{\circ}\, B^{\widehat{\circ}\,(n+1)} := (C \,\widehat{\circ}\, B^{\widehat{\circ}\,n}) \circ B \,,\; B^{\widehat{\circ}\,0} \,\widehat{\circ}\, B := B \tag{35}$$

$$C \,\widehat{\circ}\, e^{\widehat{\circ}\,(-B)} := \sum_{n \geq 0} \frac{1}{n!} (-1)^n C \,\widehat{\circ}\, B^{\widehat{\circ}\,n} \tag{36}$$

one has

$$e^{[a^+ Ba,\ \cdot]}(aCa) = a \left(C \,\widehat{\circ}\, e^{\widehat{\circ}\,(-B)} \right) a \tag{37}$$

Lemma 6 Introducing, for $n \in \mathbb{N}$ and $B, C, G, H \in M_n(\mathbb{C})$, the inductively defined notations

$$B^{\widehat{\circ}\,(n+1)}\,\widehat{\circ}\,C := B \circ (B^{\widehat{\circ}\,n}\,\widehat{\circ}\,C)\,,\; B^{\widehat{\circ}\,0}\,\widehat{\circ}\,C := C$$

$$e^{\widehat{\circ}\,(-B)}\,\widehat{\circ}\,C := \sum_{n\geq 0} \frac{1}{n!}(-1)^n B^{\widehat{\circ}\,n}\,\widehat{\circ}\,C$$

the following identities hold:

$$\left(G\,\widehat{\circ}\,e^{\widehat{\circ}\,(-H)}\right)^* = e^{\widehat{\circ}\,(-H^*)}\,\widehat{\circ}\,G^* \qquad (38)$$

$$e^{[a^+Ba,\ \cdot]}a^+Aa^+ \qquad (39)$$

$$= \left(e^{[-a^+B^*a,\ \cdot]}aA^*a\right)^* = a^+\left(e^{\circ(-B)}\,\widehat{\circ}\,A\right)a^+$$

$$e^{[a^+Ba,\ \cdot]}a^+B'a = a^+\left(e^{[B,\ \cdot]}B'\right)a \qquad (40)$$

$$= a^+\left(e^B B' e^{-B}\right)a\ .$$

The following Lemma gives an idea of why $C \,\widehat{\circ}\, e^{\widehat{\circ}(-B)}$ is a generalization of the usual exponential.

Lemma 7 If C and B commute and are both symmetric, then

$$C \,\widehat{\circ}\, e^{\widehat{\circ}(-B)} = Ce^{-2B} \tag{41}$$