Global properties of Spherically symmetric solutions in General Relativity with an Electromagnetic field and a Cosmological constant

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Abstract

We discuss following issues:

- Symmetry of the metric arises as the consequence of field equations ("spontaneous symmetry emergence"). We assume no symmetry of the metric from the very beginning.
- There are many qualitatively different global solutions. In particular, there are 11 spherically symmetric global solutions. To construct Carter-Penrose diagrams, we use the conformal block method[1].
- There is a new global spherically symmetric solution, which describes changing topology of spatial sections during the time evolution at the classical level.

[1] M. O. Katanaev, Global solutions in gravity. Lorentzian signature, *Proc. Steklov Inst. Math.*, 228 (2000) 158.

Metric of the warped product of two surfaces

- Space-time manifold $\mathbb{M}$ is assumed to be the warped product $\mathbb{M} = \mathbb{U} \times \mathbb{V}$ of Lorentzian $\mathbb{U}$ and Riemannian $\mathbb{V}$ surfaces.
- Local coordinates on $\mathbb{U}$, $\mathbb{V}$ and $\mathbb{M}$ are denoted as

$$(x^\alpha) \in \mathbb{U}, \ \alpha = 0, 1; \ (y^\mu) \in \mathbb{V}, \ \mu = 2, 3; \ (x^\alpha, y^\mu) =: \left(\widehat{x}^i\right) \in \mathbb{M}.$$

- The metric of the warped product has the form

$$\widehat{g}_{ij} = \begin{pmatrix} k(y) g_{\alpha\beta}(x) & 0 \\ 0 & m(x) h_{\mu\nu}(y) \end{pmatrix}, \qquad (1)$$

where $g_{\alpha\beta}$, $h_{\mu\nu}$, $m(x) \neq 0$ and $k(y) \neq 0$ are metrics and scalar fields on $\mathbb{U}$ and $\mathbb{V}$, respectively. Signature of metrics $g_{\alpha\beta}$, $h_{\mu\nu}$ are assumed to be $(+-)$ and $(++)$, respectively.

Field equations

- We consider the action

$$S = \int d^4x \sqrt{|\widehat{g}|} \left(\widehat{R} - 2\Lambda - \frac{1}{4}\widehat{F}^2 \right) \tag{2}$$

- Variation of action yields Einstein's and Maxwell's equations

$$\widehat{R}_{ij} - \frac{1}{2}\widehat{g}_{ij}\widehat{R} + \widehat{g}_{ij}\Lambda = -\frac{1}{2}\widehat{T}_{\mathrm{EM}ij}, \quad \partial_j\left(\sqrt{|\widehat{g}|}\widehat{F}^{ji}\right) = 0, \tag{3}$$

- Potential $\widehat{A}_i$ is assumed to be $\widehat{A}_i = (A_\alpha(x), A_\mu(y))$. Then electromagnetic field strength takes the form

$$\widehat{F}_{ij} = \begin{pmatrix} F_{\alpha\beta}(x) & 0 \\ 0 & F_{\mu\nu}(y) \end{pmatrix},$$

where

$$F_{\alpha\beta} := \partial_\alpha A_\beta - \partial_\beta A_\alpha, \quad F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Solution for electromagnetic field

- General solution to Maxwell's equations is

$$F^{\alpha\beta} = \frac{2Q}{|m|}\varepsilon^{\alpha\beta}, \quad F^{\mu\nu} = \frac{2P}{|k|}\varepsilon^{\mu\nu},$$

  where $\varepsilon^{\alpha\beta}$ and $\varepsilon^{\mu\nu}$ are the totally antisymmetric second rank tensors, Q and P are constants of integration.

- Energy-momentum tensor has a block diagonal form

$$\widehat{T}_{ij} = \begin{pmatrix} \widehat{T}_{\alpha\beta} & 0 \\ 0 & \widehat{T}_{\mu\nu}, \end{pmatrix}$$

  where

$$\widehat{T}_{\alpha\beta} = \frac{2g_{\alpha\beta}}{km^2}(Q^2 + P^2), \quad \widehat{T}_{\mu\nu} = -\frac{2h_{\mu\nu}}{k^2 m}(Q^2 + P^2).$$

Einstein's equations

We put $P = 0$. The full system of Einstein's equations can be represented as

$$R^{(g)} + \frac{\nabla^2 m}{m} - \frac{(\nabla m)^2}{2m^2} + \frac{\nabla^2 k}{m} - 2k\Lambda + \frac{2Q^2}{m^2 k} = 0, \qquad (4)$$

$$R^{(h)} + \frac{\nabla^2 k}{k} - \frac{(\nabla k)^2}{2k^2} + \frac{\nabla^2 m}{k} - 2m\Lambda - \frac{2Q^2}{mk^2} = 0, \qquad (5)$$

$$\nabla_\alpha \nabla_\beta m - \frac{\nabla_\alpha m \nabla_\beta m}{2m} - \frac{1}{2} g_{\alpha\beta} \left(\nabla^2 m - \frac{(\nabla m)^2}{2m} \right) = 0, \qquad (6)$$

$$\nabla_\mu \nabla_\nu k - \frac{\nabla_\mu k \nabla_\nu k}{2k} - \frac{1}{2} h_{\mu\nu} \left(\nabla^2 k - \frac{(\nabla k)^2}{2k} \right) = 0, \qquad (7)$$

$$\nabla_\alpha m \nabla_\mu k = 0, \qquad (8)$$

where $R^{(g)}$ and $R^{(h)}$ are the scalar curvatures of $\mathbb{U}$ and $\mathbb{V}$, respectively, and

$$\nabla^2 m := g^{\alpha\beta} \nabla_\alpha \nabla_\beta m, \quad \nabla^2 k := h^{\mu\nu} \nabla_\mu \nabla_\nu k,$$

$$(\nabla m)^2 := g^{\alpha\beta} \nabla_\alpha m \nabla_\beta m, \quad (\nabla k)^2 := h^{\mu\nu} \nabla_\mu k \nabla_\nu k.$$

The restrictions on dilaton fields

- The Eq.(8) imposes strong restrictions on dilaton fields. At least one of the dilaton fields must be constant.
- There are only three cases

$$\mathbf{A} \quad : m = c_1 \neq 0, k = c_2 \neq 0; \quad \mathbf{B} : \nabla_\alpha m \neq 0, k = c \neq 0;$$ [2]
$$\mathbf{C} \quad : m = c \neq 0, \nabla_\mu k \neq 0,$$ [3]

  where c, c_1 and c_2 are constants.
- We will see that this leads to "spontaneous symmetry emergence".

[2] D. E. Afanasev and M. O. Katanaev. Global properties of warped solutions in general relativity with an electromagnetic field and a cosmological constant. *Phys. Rev. D*, 100 (2):024052, 2019.

[3] D. E. Afanasev and M. O. Katanaev. Global properties of warped solutions in general relativity with an electromagnetic field and a cosmological constant. II *Phys. Rev. D*, 101 (12):124025, 2020

Case **A**. Product of surfaces of constant curvature

- Einstein's equations have the simple form:

$$R^{(g)} = 2\left(\Lambda - Q^2\right) = -2K^{(g)}, \quad R^{(h)} = 2\left(\Lambda + Q^2\right) = -2K^{(h)}$$

- Space-time is the product of constant curvature surfaces:

$$\begin{aligned} \Lambda > -Q^2, \quad \Lambda \neq Q^2: &\quad \mathbb{M} = \quad \mathbb{L}^2 \times \mathbb{S}^2, \\ \Lambda = -Q^2: &\quad \mathbb{M} = \quad \mathbb{L}^2 \times \mathbb{R}^2, \\ \Lambda = Q^2: &\quad \mathbb{M} = \quad \mathbb{R}^{1,1} \times \mathbb{S}^2, \\ \Lambda < -Q^2: &\quad \mathbb{M} = \quad \mathbb{L}^2 \times \mathbb{H}^2, \end{aligned}$$

  where $\mathbb{L}^2$ is the one-sheeted hyperboloid, embedded in Minkowskian space $\mathbb{R}^{1,2}$, $\mathbb{R}^2$ is the Euclidean plane, $\mathbb{H}^2$ is the Lobachevsky plane, $\mathbb{S}^2$ is the two-dimensional sphere and $\mathbb{R}^{1,1}$ is the Minkowskian plane.

- This is the example of "Spontaneous symmetry emergence".

Case B. Solutions with spatial symmetry

- We put $k = 1$. Einstein's equations take the form:

$$R^{(g)} + \frac{\nabla^2 m}{m} - \frac{(\nabla m)^2}{2m^2} - 2\Lambda + \frac{2Q^2}{m^2} = 0, \tag{9}$$

$$R^{(h)} + \nabla^2 m - 2m\Lambda - \frac{2Q^2}{m} = 0, \tag{10}$$

$$\nabla_\alpha \nabla_\beta m - \frac{\nabla_\alpha m \nabla_\beta m}{2m} - \frac{1}{2} g_{\alpha\beta} \left(\nabla^2 m - \frac{(\nabla m)^2}{2m} \right) = 0. \tag{11}$$

- Here $R^{(h)} = R^{(h)}(y)$, $m = m(x)$. Therefore, Eq.(10) leads to $R^{(h)} = \text{const} := 2K^{(h)}$. It means that $\mathbb{V}$ is the surface of constant curvature.
- There are three possibilities: planar solutions $K^{(h)} = 0$ ($\mathbb{V} = \mathbb{R}^2$), spherically symmetric solutions $K^{(h)} = 1$ ($\mathbb{V} = \mathbb{S}^2$), hyperbolic global solutions $K^{(h)} = -1$ ($\mathbb{V} = \mathbb{H}^2$).
- This is "spontaneous spatial symmetry emergence".

Case B. Solutions with spatial symmetry

General solution of Einstein's equations (9)–(11) can be represented as

$$ds^2 = |\Phi(q)| \left(d\tau^2 - d\sigma^2\right) - q^2 d\Omega^2, \tag{12}$$

where conformal factor is

$$\Phi(q) = K^{(h)} - \frac{2M}{q} + \frac{Q^2}{q^2} - \frac{\Lambda q^2}{3}.$$

Here the variable q depends on σ (static local solution) or τ (homogeneous local solution) through the differential equation

$$\left|\frac{dq}{d\zeta}\right| = \pm\Phi(q),$$

where the sign rule holds

$$\begin{aligned} \Phi > 0: &\quad \zeta = \sigma, \quad \text{sign } + \quad \text{(static local solution)},\\ \Phi < 0: &\quad \zeta = \tau, \quad \text{sign } - \quad \text{(homogeneous local solution)}. \end{aligned}$$

Case **B**. Spherically symmetric solutions $K^{(h)} = 1$

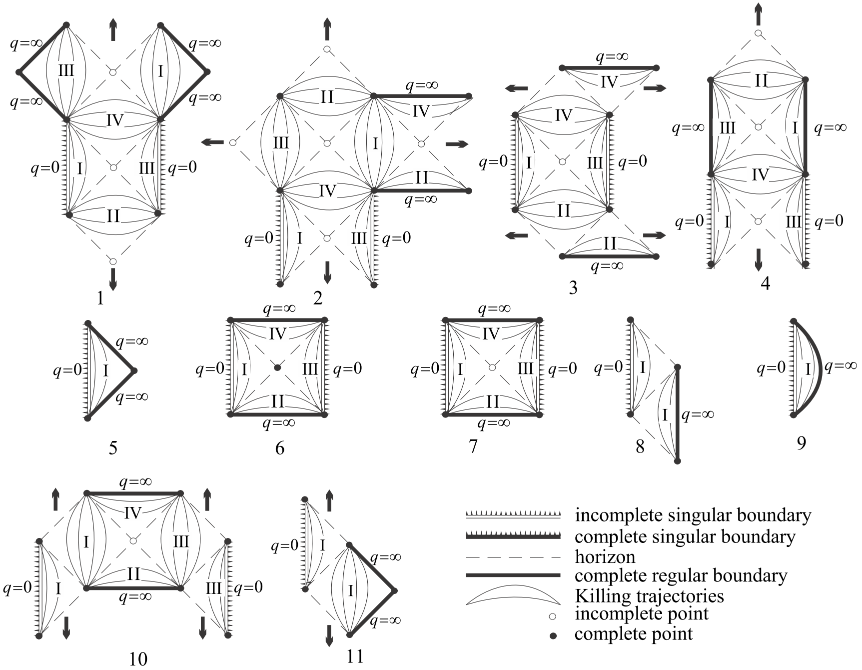
- Four-dimensional space-time manifold has the form $\mathbb{M} = \mathbb{U} \times \mathbb{S}^2$
- In this case metric (12) has the form

$$ds^2 = |\Phi(q)| \left(d\tau^2 - d\sigma^2\right) - q^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

where

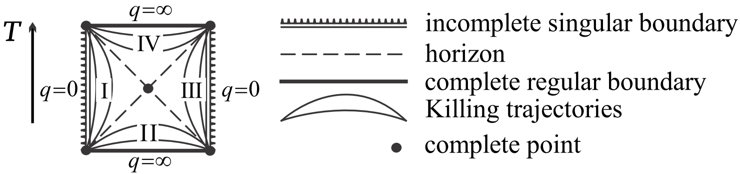
$$\Phi(q) = 1 - \frac{2M}{q} + \frac{Q^2}{q^2} - \frac{\Lambda q^2}{3} \tag{13}$$

- We can construct global solutions (maximally extended along geodesics) using the conformal block method. The number of singularities and zeroes of conformal factor (13) depends on relations between constants M, Q, and Λ.
- There are many qualitatively different global spherically symmetric solutions of Einstein's equations with electromagnetic field.



q=∞
q=0
I
II
III
IV
1
2
3
4
5
6
7
8
9
10
11
incomplete singular boundary
complete singular boundary
horizon
complete regular boundary
Killing trajectories
incomplete point
complete point

Changing topology of space in time



- The saddle point is complete and lying at space infinity.
- A section non-crossing saddle point is the interval of finite length. A section crossing saddle point is the union of two semi-infinite intervals.
- We can direct the time axis vertically up. Then spatial section is the union of two semi-infinite intervals for some value of T and it is finite interval for other values of T.
- This is the example of changing topology of space in time at the classical level.

Case C. Lorentz-invariant and planar solutions

- We put $m = 1$. Einstein's equations take the form:

$$R^{(h)} + \frac{\nabla^2 k}{k} - \frac{(\nabla k)^2}{2k^2} - 2\Lambda - \frac{2Q^2}{k^2} = 0, \quad (14)$$

$$R^{(g)} + \nabla^2 k - 2k\Lambda + \frac{2Q^2}{k} = 0, \quad (15)$$

$$\nabla_\mu \nabla_\nu k - \frac{\nabla_\mu k \nabla_\nu k}{2k} - \frac{1}{2} h_{\mu\nu} \left(\nabla^2 k - \frac{(\nabla k)^2}{2k} \right) = 0. \quad (16)$$

- Here $R^{(g)} = R^{(g)}(x)$, $k = k(y)$. Eq.(15) leads to $R^{(g)} = \mathrm{const} := 2K^{(g)}$. It means that Lorentzian surface $\mathbb{U}$ is of constant curvature.
- There are two cases: Lorentz-invariant solutions $K^{(g)} = \pm 1$ ($\mathbb{U} = \mathbb{L}^2$), solutions with Minkowskian plane $K^{(g)} = 0$ ($\mathbb{U} = \mathbb{R}^{1,1}$).

Conclusions

- We obtained many qualitatively different global solutions of Einstein's equations with an electromagnetic field and a cosmological constant. In particular, we obtained 11 global spherically symmetric solutions (case **B**).
- In all cases the symmetry arises as the consequence of field equations. This effect was called "spontaneous symmetry emergence".
- We obtain the new spherically symmetric solution. It describes changing topology of spatial sections during the time evolution at the classical level.

Thank you for your attention!