# Belyi pairs of the cell decomposition of oriented covering of the Deligne-Mumford compactification for $M^{\mathbb{R}}_{0,5}$

Natalia Amburg, Elena Kreines

ITEP, HSE          MSU, ITEP

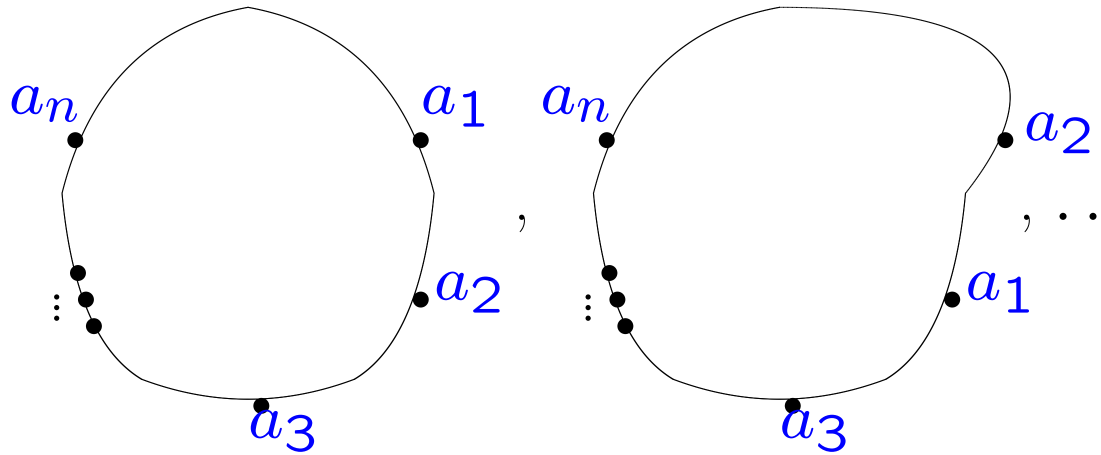
# The real moduli space of genus 0 curves

Real curve of $g = 0$ with $n$ marked points labeled $1, \ldots, n$ is a pair $\mathbb{RP}^1$ and $(x_1, x_2, \ldots, x_n) \in \mathbb{RP}^1$.

Definition. $(x_1, x_2, \ldots, x_n) \in \mathbb{RP}^1$ and $(y_1, y_2, \ldots, y_n) \in \mathbb{RP}^1$ are red iff $\exists f(x) = \frac{ax+b}{cx+d}$ such that $f(x_j) = y_j, j = 1, \ldots, n$.

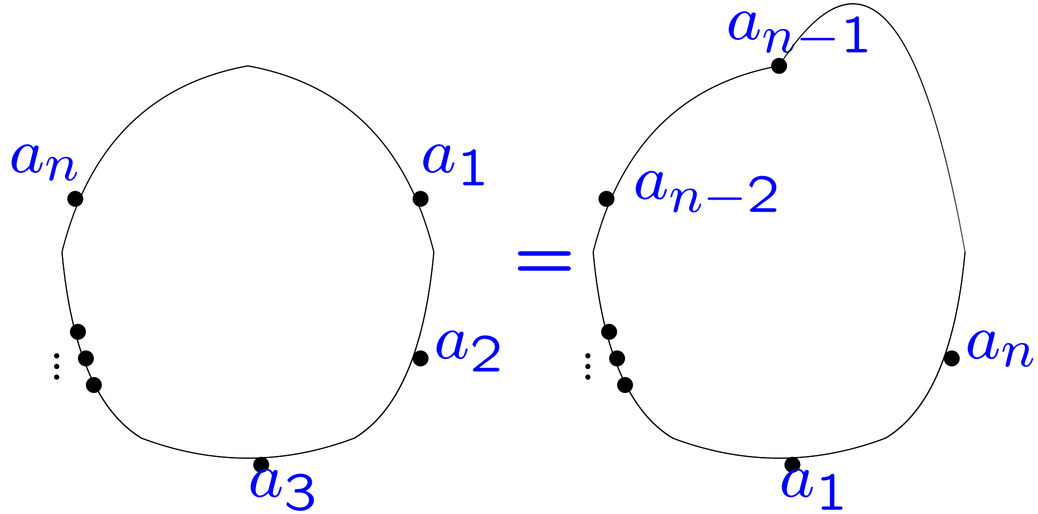
Definition. The set of equivalence classes of $(x_1, x_2, \ldots, x_n) \in \mathbb{RP}^1$ is called the open moduli space of curves of genus 0 with $n$ marked points labeled $1, \ldots, n$. We denote it by $\mathcal{M}^{\mathbb{R}}_{0,n}$.

$$\mathcal{M}^{\mathbb{R}}_{0,n} = \Big\{ \quad , \quad , \ \dots \Big\} / \sim$$




— real algebraic curves of genus $0$ with $n$ marked and numbered points.

$$\mathcal{M}^{\mathbb{R}}_{0,n} = \left\{ \quad \right.$$




$$\left. \quad \right\} / \sim$$

— real algebraic curves of genus $0$ with $n$ marked and numbered points.

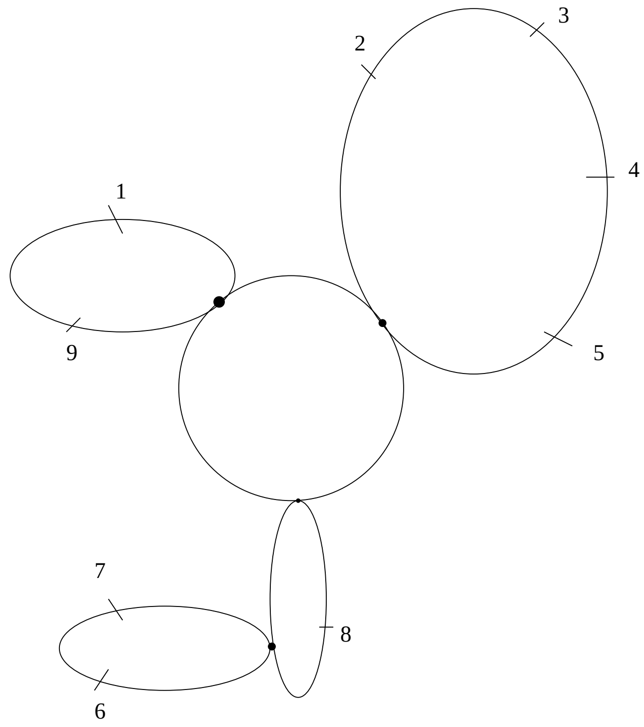
$\overline{\mathcal{M}_{0,n}^{\mathbb{R}}}$ is the Deligne-Mumford compactification of $\mathcal{M}_{0,n}^{\mathbb{R}}$

I.e., moduli of "cacti-like" structures: $3$-dimensional "trees" of flat circles with the points $\{1, 2, \ldots, n\}$ on them.

Definition. A stable curve of genus $0$ with $n$ marked points over $\mathbb{R}$ is $C = C_1 \cup \ldots \cup C_p$, $C_j = \mathbb{RP}^1$, with $n$ different points $z_1, z_2, \ldots, z_n \in C$, s.t.

- $\forall\ z_i\ \exists\,!$ line $C_j : z_i \in C_j$.
- $\forall$ pair $C_i, C_j$, $C_i \cap C_j$ is either $\emptyset$ or $\{X\}$, and it is transversal.
- The graph of $C$ ($C_1, C_2, \ldots, C_p$ are vertices; edges are intersections) is a tree.
- The number of special points (marked or intersection) in $C_j \geq 3\ \forall\ j$.

$p$ is the number of components.




A stable curve over $\mathbb{R}$ of genus $\texttt{0}$ with $\texttt{9}$ marked points

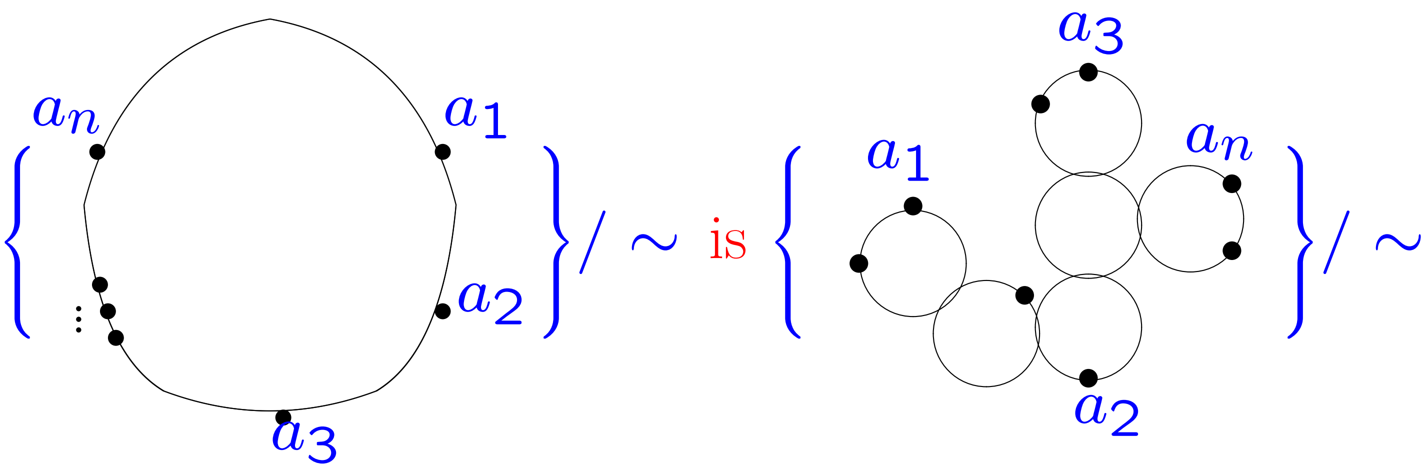
Definition.

Let $C = (C_1, C_2, \ldots, C_p, z_1, z_2, \ldots, z_n)$ and

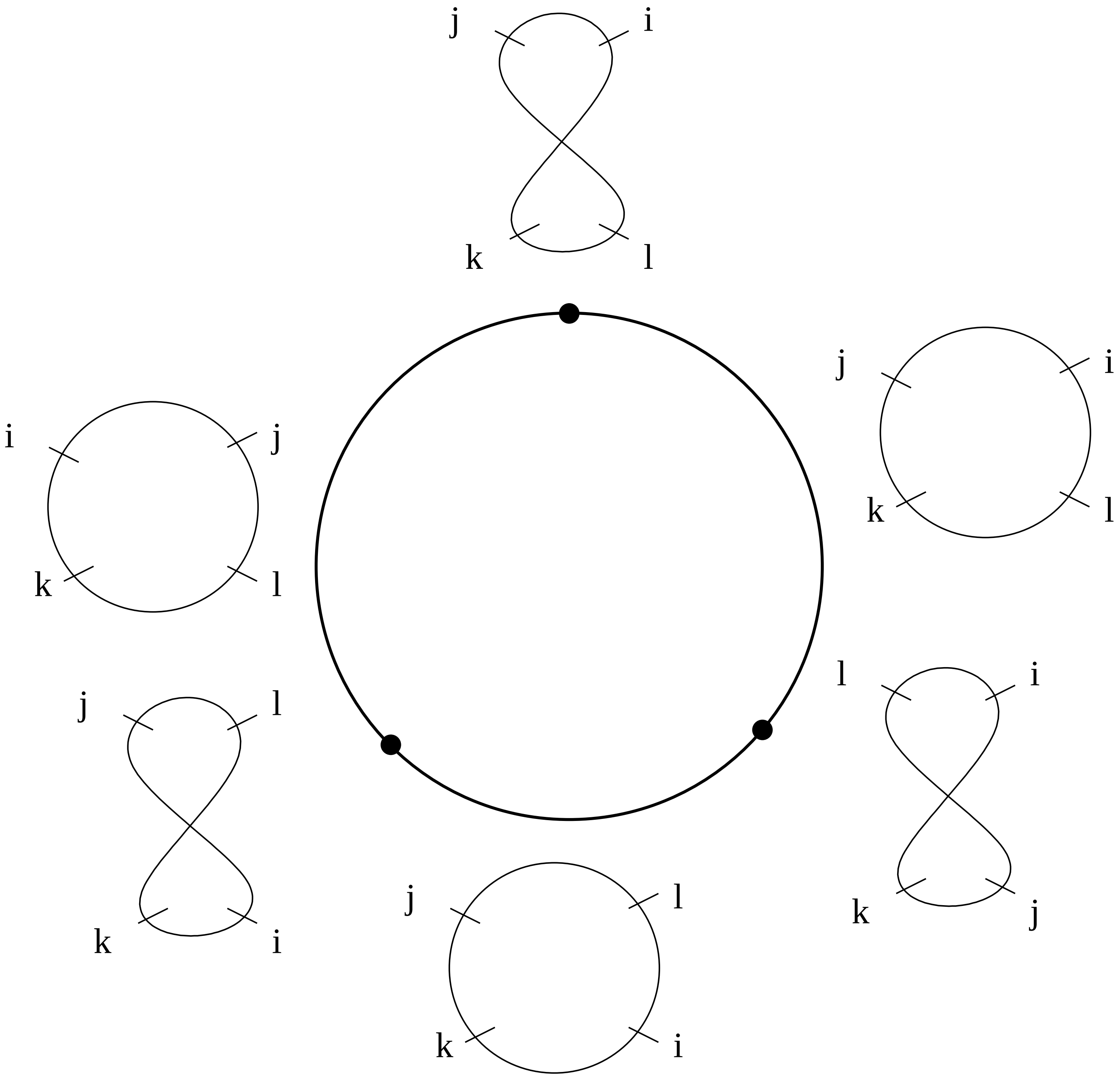
$C' = (C'_1, C'_2, \ldots, C'_p, z'_1, z'_2, \ldots, z'_n)$ be stable curves of genus $0$ with $n$ points.

$C$, $C'$ are called equivalent if $\exists$ iso. $f : C \to C'$: $f(z_i) = z'_i \ \forall\ i = 1, \ldots, n$.

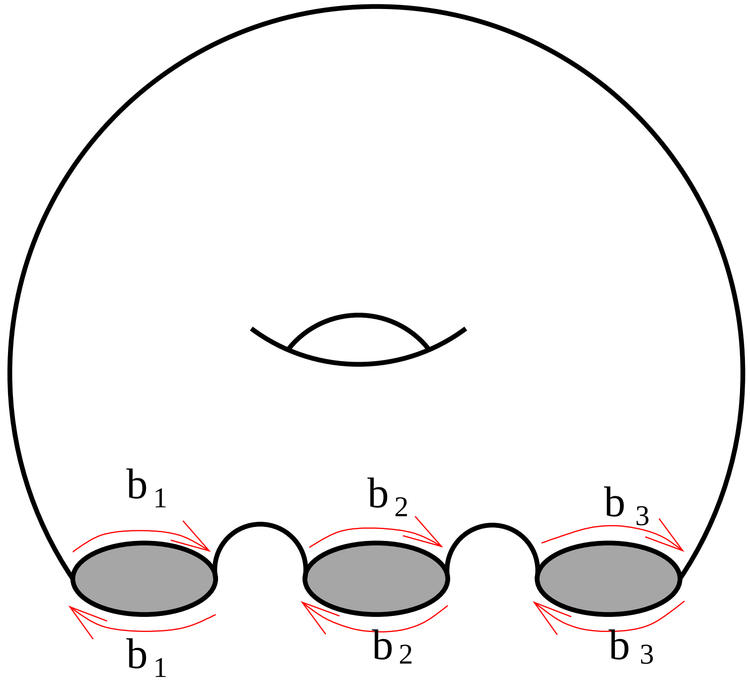
Definition. $n \geq 3$. Deligne-Mumford compactification of




Theorem. [S. Devadoss, 1999] Let $n > 4$. Then the space $\overline{\mathcal{M}_{0,n}^{\mathbb{R}}}$ is a non-orientable compact variety of real dimension $\dim(\overline{\mathcal{M}_{0,n}^{\mathbb{R}}}) = n - 3$.




$\overline{\mathcal{M}}^{\mathbb{R}}_{0,4}$




$\overline{\mathcal{M}^{\mathbb{R}}_{0,5}}$

## Cell decomposition of $\overline{\mathcal{M}^{\mathbb{R}}_{0,n}}$ [S. Devadoss, 1999]

Cells of $\overline{\mathcal{M}^{\mathbb{R}}_{0,n}}$ are marked by $n$-gons

Marked points — sides of $n$-gons

Intersection points — diagonals

cells of max dimension correspond to $n$-gons without diagonals

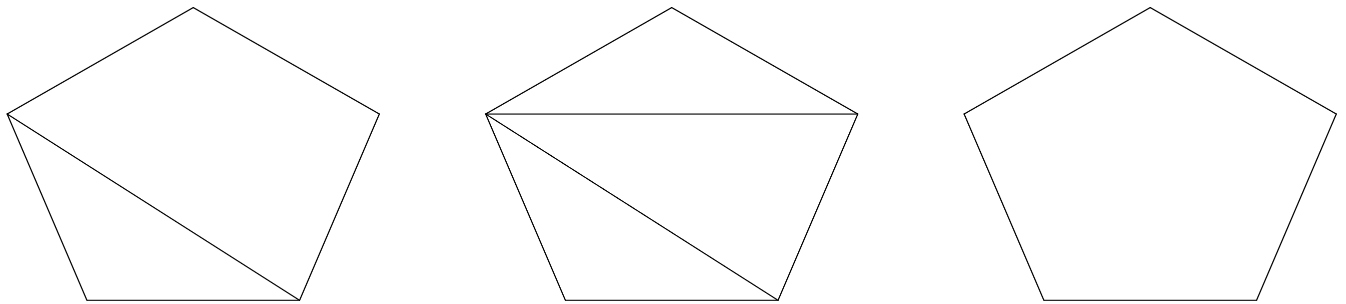
cells of codim 1 consist of 2-component s.c. and correspond to $n$-gons with 1 diagonal

cells of codim 2 — 3-component stable curves — $n$-gons with 2 diagonal

. . . . . . . . . .

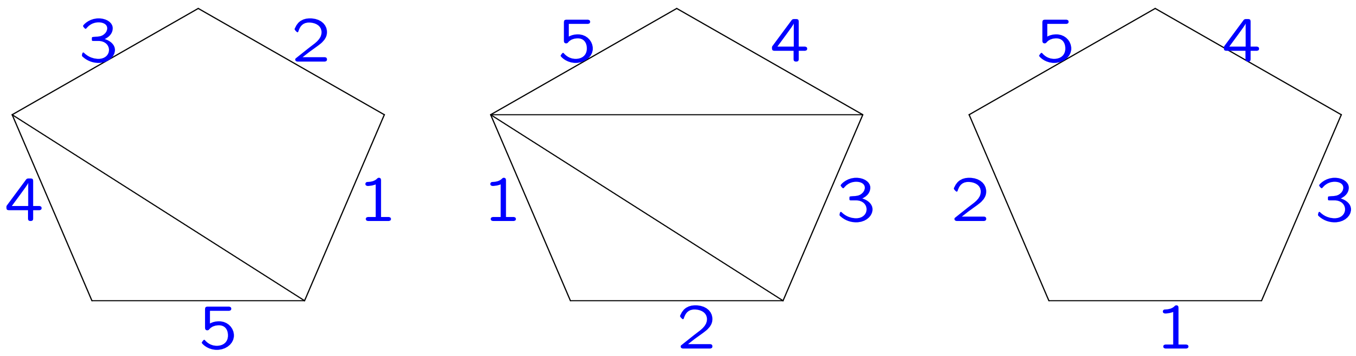
cells of codim $k$ — $k+1$-component stable curves — $n$-gons with $k$ diagonal

1. Consider right $n$-gons, possibly, with several non-intersecting outside the vertices diagonals:



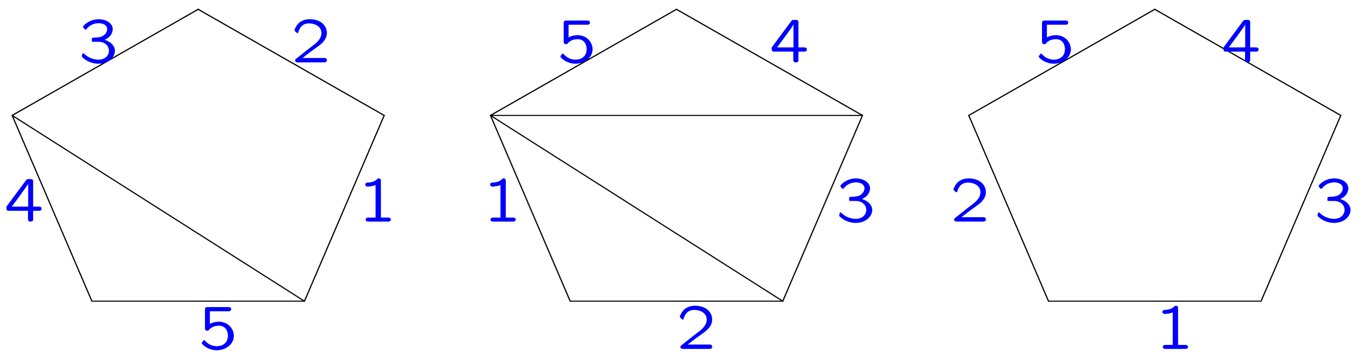
1. Consider right $n$-gons, possibly, with several non-intersecting outside the vertices diagonals:

2. Mark their edges by $1, 2, \ldots, n$



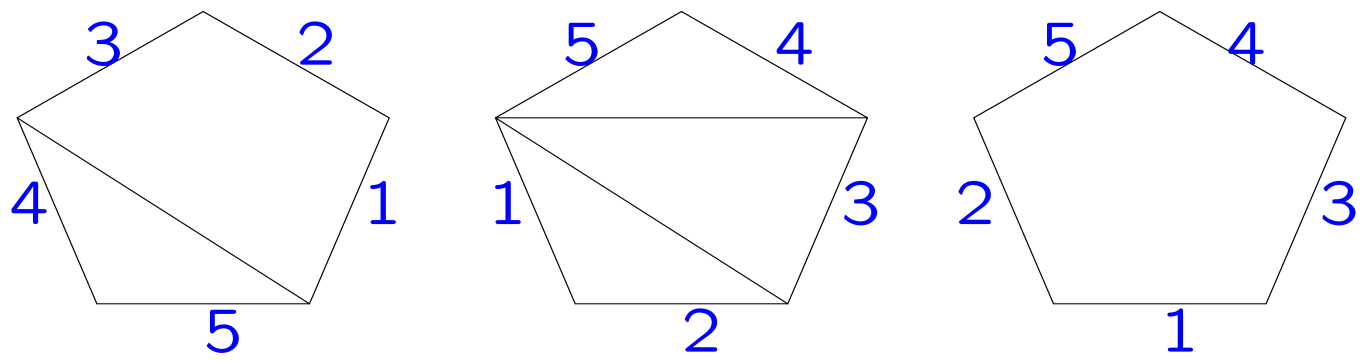
1. Consider right $n$-gons, possibly, with several non-intersecting outside the vertices diagonals:

2. Mark their edges by $1, 2, \ldots, n$



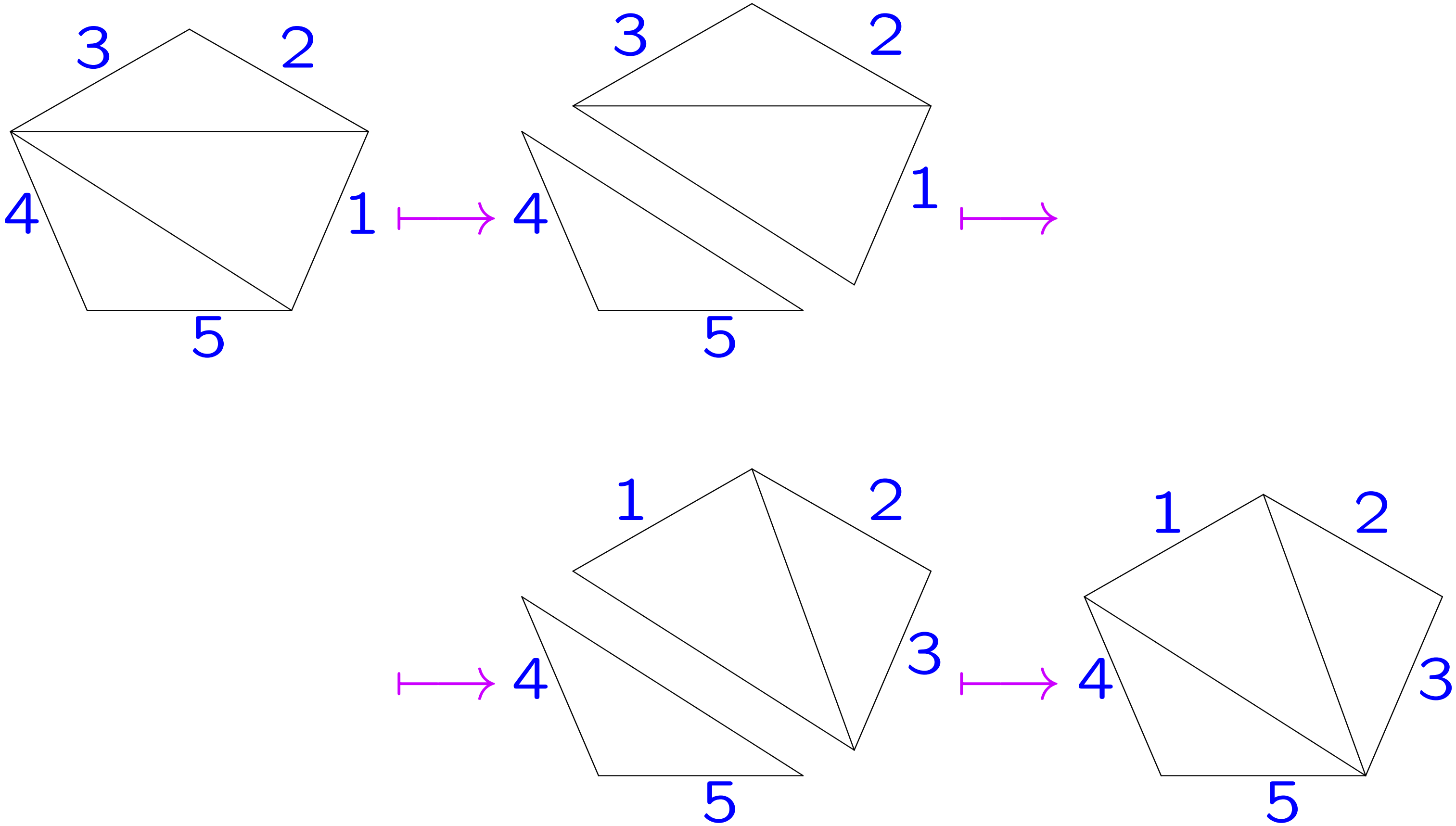

3. Identify polygons that be transformed to each other by the dihedral group action.

1. Right $n$-gons
2. Mark their edges by $1, 2, \ldots, n$



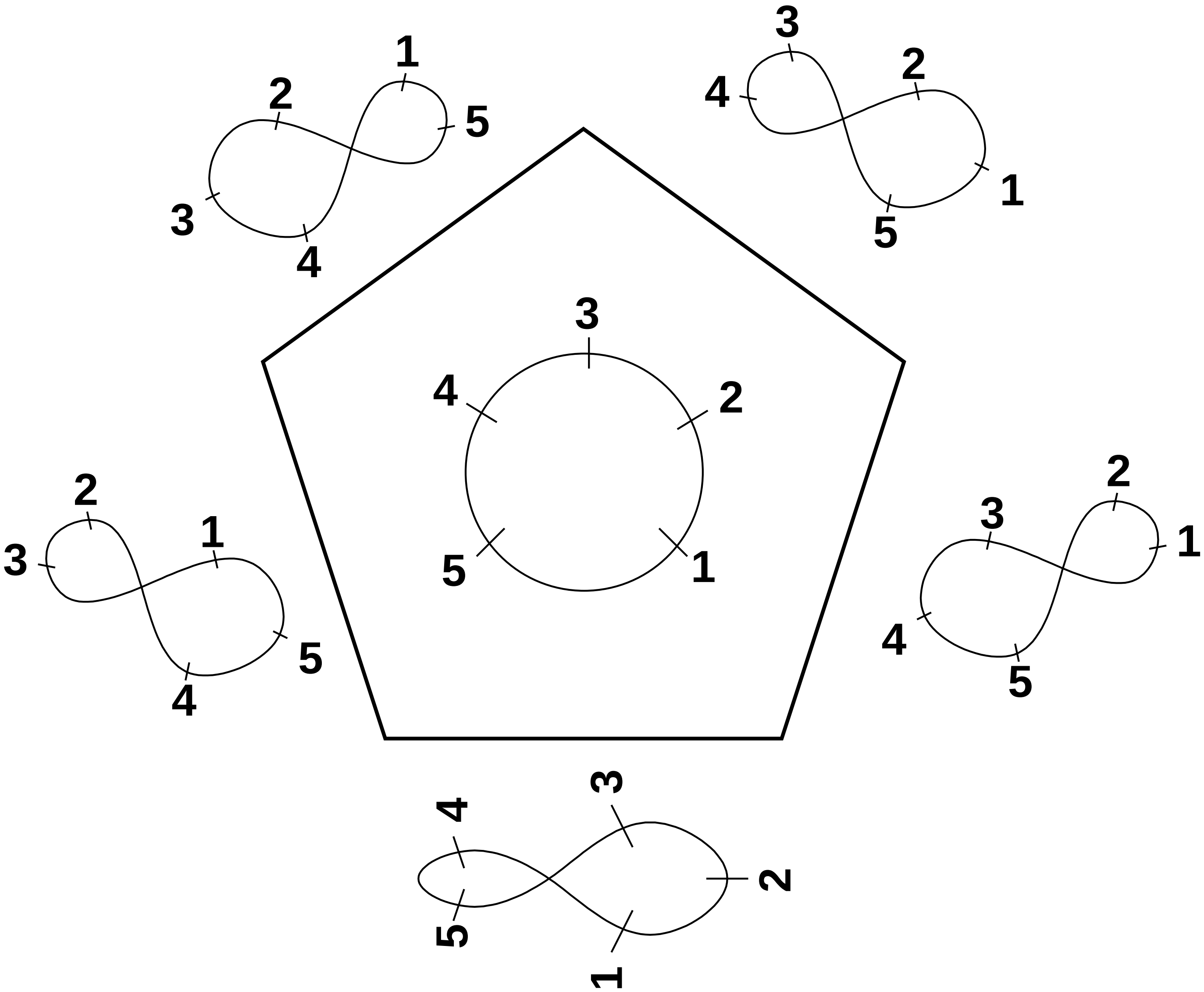

3. Action of the dihedral group — the same.
4. Identify polydons that be transformed to each other by the series of twist operations.

Twist operation:

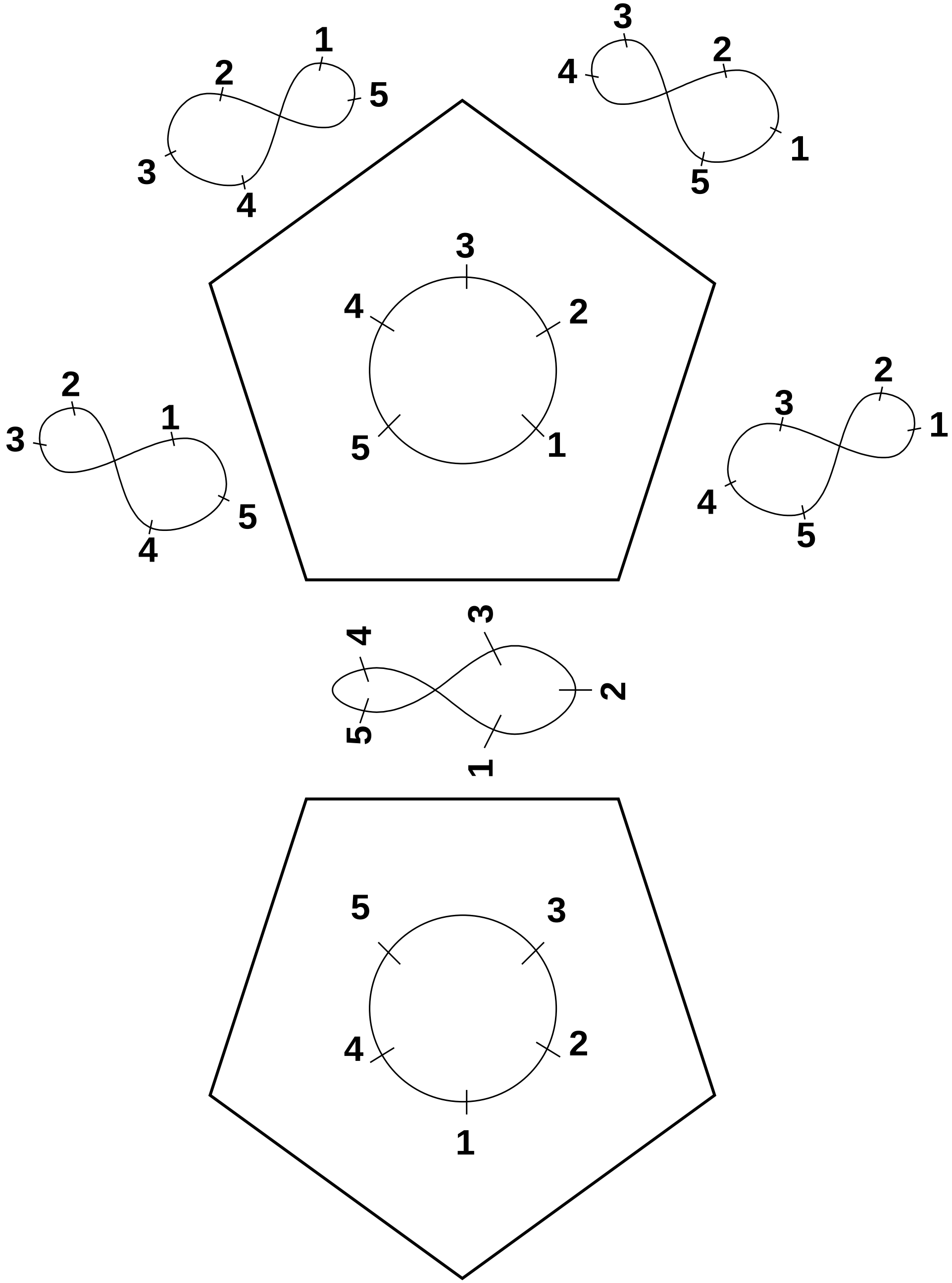


- The graph of $C$ is a tree — diagonals do not intersect inside the polygon

- The number of special points in $C_j \geq 3 \; \forall \, j = 1, \ldots, p$ — diagonals are diagonals, i.e., each part of the "big" polygon is at least a 3-gon

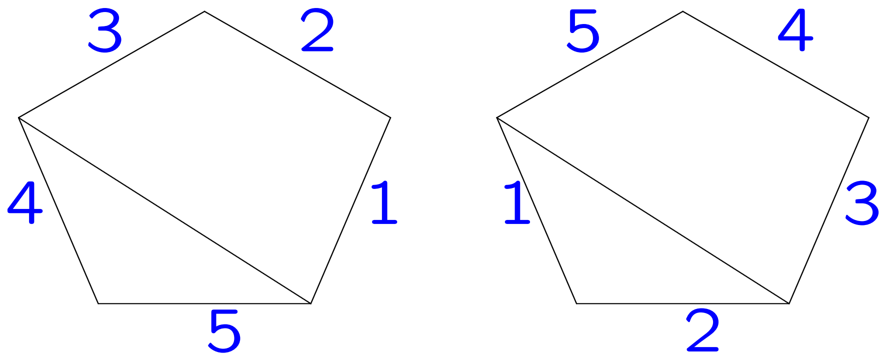



A cell of $\overline{\mathcal{M}^{\mathbb{R}}_{0,5}}$ with the boundary.



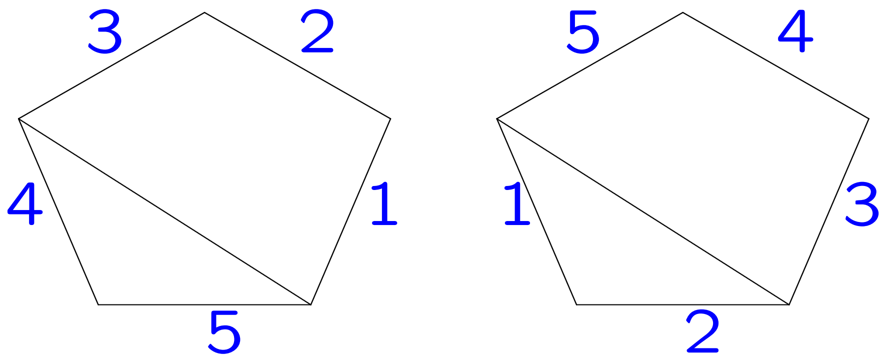

Two adjoint cells of $\overline{\mathcal{M}^{\mathbb{R}}_{0,5}}$.

Boundary cells:




Bottom edge Right-lower edge

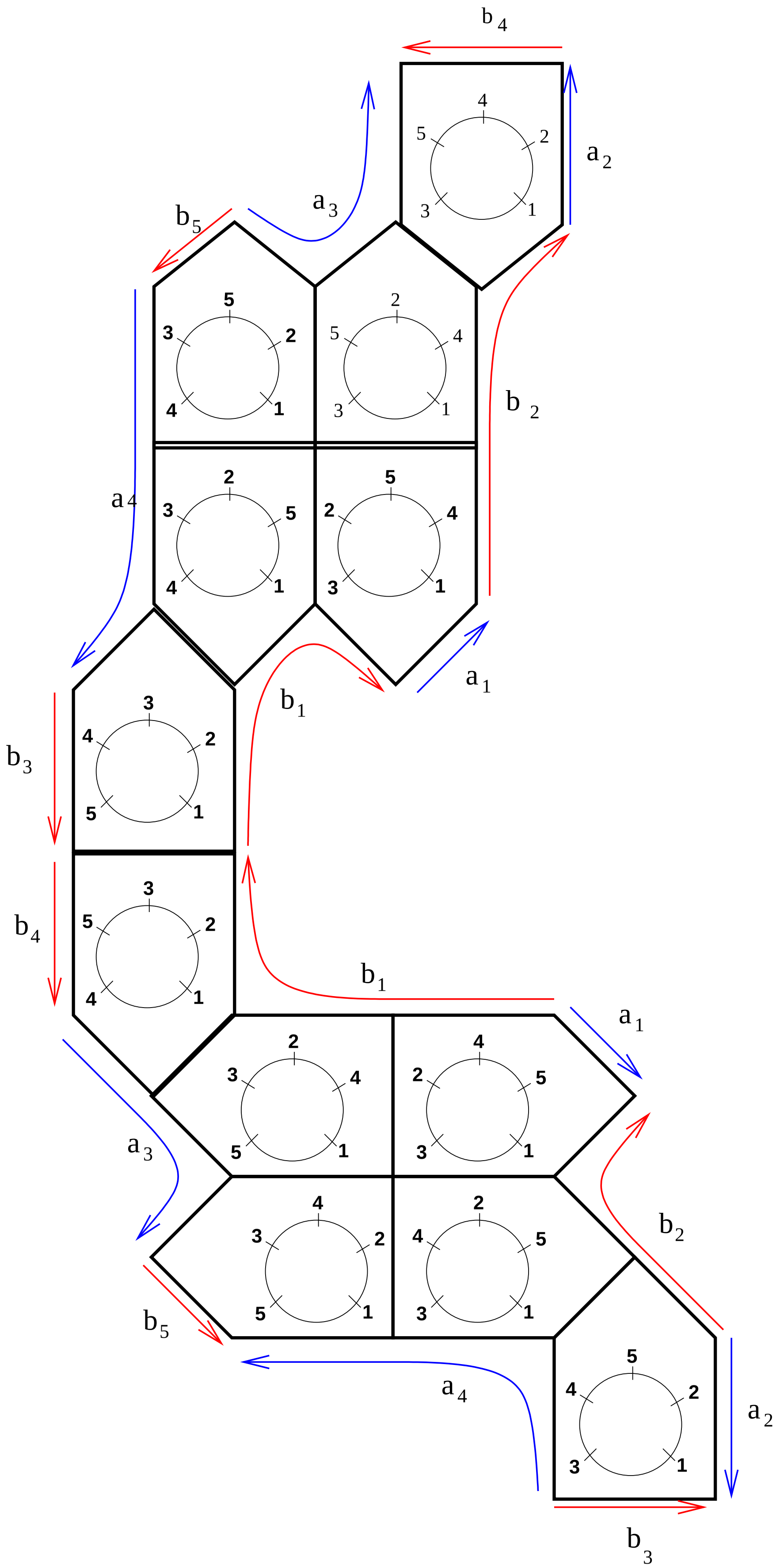
Boundary cells:



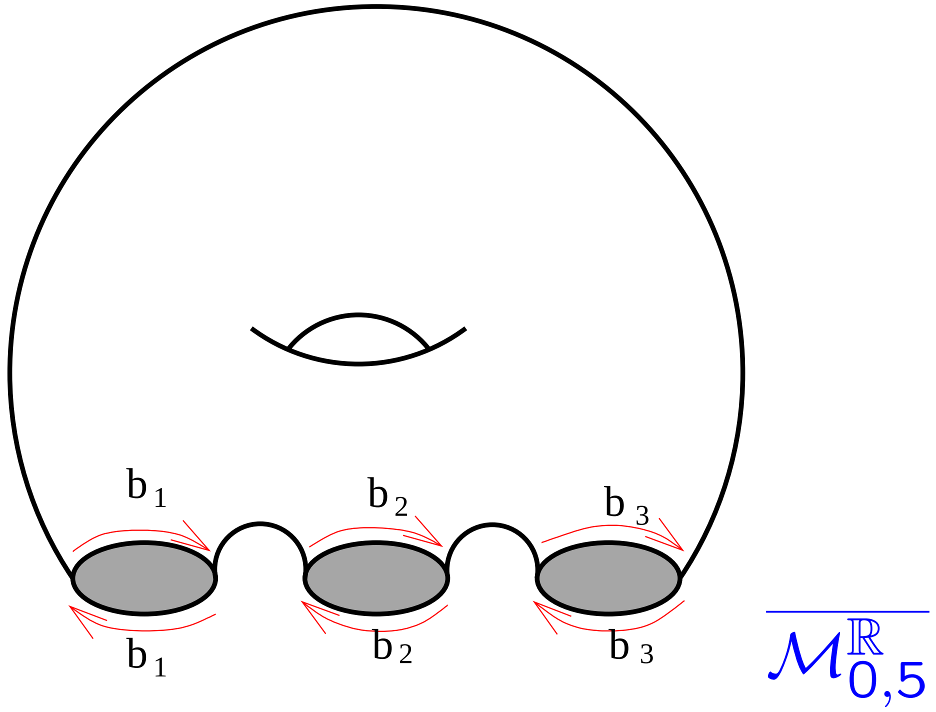

Bottom edge Right-lower edge

Proposition. Cell decomposition of $\overline{\mathcal{M}_{0,n}^{\mathbb{R}}}$ contains $\frac{(n-1)!}{2}$ cells of the maximal dimension $n-3$.

$n = 5 \Longrightarrow 12$ cells



b4
a2
a3
b5
b2
a4
a1
b1
b3
b4
b1
a1
a3
b2
b5
a4
a2
b3


b 1
b 2
b 3
b 1
b 2
b 3
$\overline{\mathcal{M}_{0,5}^{\mathbb{R}}}$

Stiefel-Whitney class of $\overline{\mathcal{M}^{\mathbb{R}}_{0,n}}$

StWh are topological invariants of a real vector bundles that describe the obstructions to constructing everywhere independent sets of sections of the vector bundle, is a $\mathbb{Z}/2\mathbb{Z}$-characteristic class associated to real vector bundles.

StWh are indexed from 0 to $d$ — the dimension of the vector space fiber of the vector bundle.

StWh$\neq 0$ for some $i \Rightarrow \nexists\ (n-i+1)$ everywhere linearly independent sections of the vector bundle.

$0 \neq n$'th StWh indicates that $\forall$ section of the bundle must vanish at some point.

$0 \neq 1$'st StWh indicates that the vector bundle is not orientable.

We consider the homological class $W_{n-4}$, which is Poincaré dual to the 1st StWh class of $\overline{\mathcal{M}^{\mathbb{R}}_{0,n}}$.

Theorem. [Milnor, Stasheff] Let $M$ be a smooth compact variety without a boundary, $K$ be a cell decomposition of $M$, $k_j \subset K$ denote the cells of the maximal dimension $d$. Let us fix the orientation on the cells of MAX dim $\overline{k_j}$. Then

$$W_{d-1}(M) = \left(\frac{1}{2}\sum \partial\overline{k_j}\right) \mod 2.$$

Theorem. [AK, 2014]

$n \geq 5$, $\mathcal{M}_{0,n}^{\mathbb{R}}$, points $\{1, 2, \ldots, n\}$.

$\overline{\mathcal{M}_{0,n}^{\mathbb{R}}}$ is Deligne-Mumford compactification.

Then $W_{n-4}(\overline{\mathcal{M}_{0,n}^{\mathbb{R}}})$ consists exactly from those cells of codim 1, that satisfy: irreducible component of the curve which contains $\leq 1$ point from the set $\{1, 2, 3\}$ contains an odd number of points from $\{1, 2, \ldots, n\}$.

Definition. A coordinate map on the space $\mathcal{M}_{0,n}^{\mathbb{R}}$ is

$$\varphi : \mathcal{M}_{0,n}^{\mathbb{R}} \to \mathbb{R}^{n-3}$$

Let $(\mathbb{P}_1(\mathbb{R}), z_1, \ldots, z_n) \in \mathcal{M}_{0,n}^{\mathbb{R}}$, $z_i \in \mathbb{P}_1(\mathbb{R})$, we fix

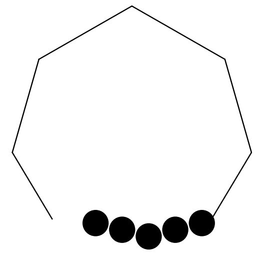
$$z_1 = 0, z_2 = 1, z_3 = \infty$$

Then

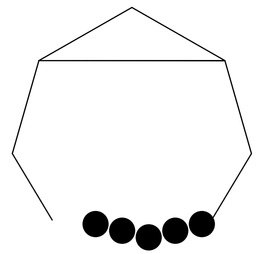
$$\varphi(\mathbb{P}_1(\mathbb{R}), z_1, \ldots, z_n) = (z_4, \ldots, z_n).$$

Standard orientation of $\mathbb{R}^{n-3} \Rightarrow$ orientation on cells of MAX dim

Cells of the maximal dimension:

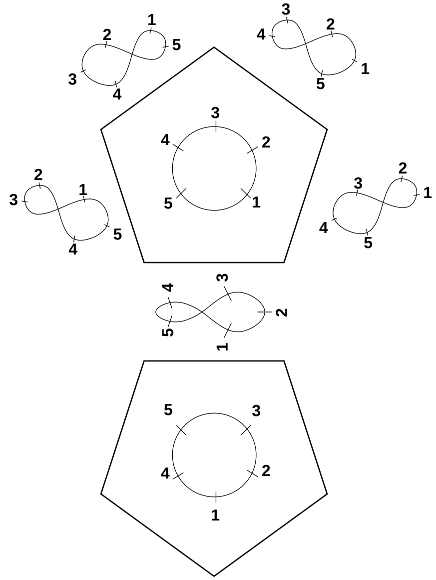


Boundaries by gluing $i$, $j$, cells $K_{ij|l_1 \ldots l_{n-2}}$:

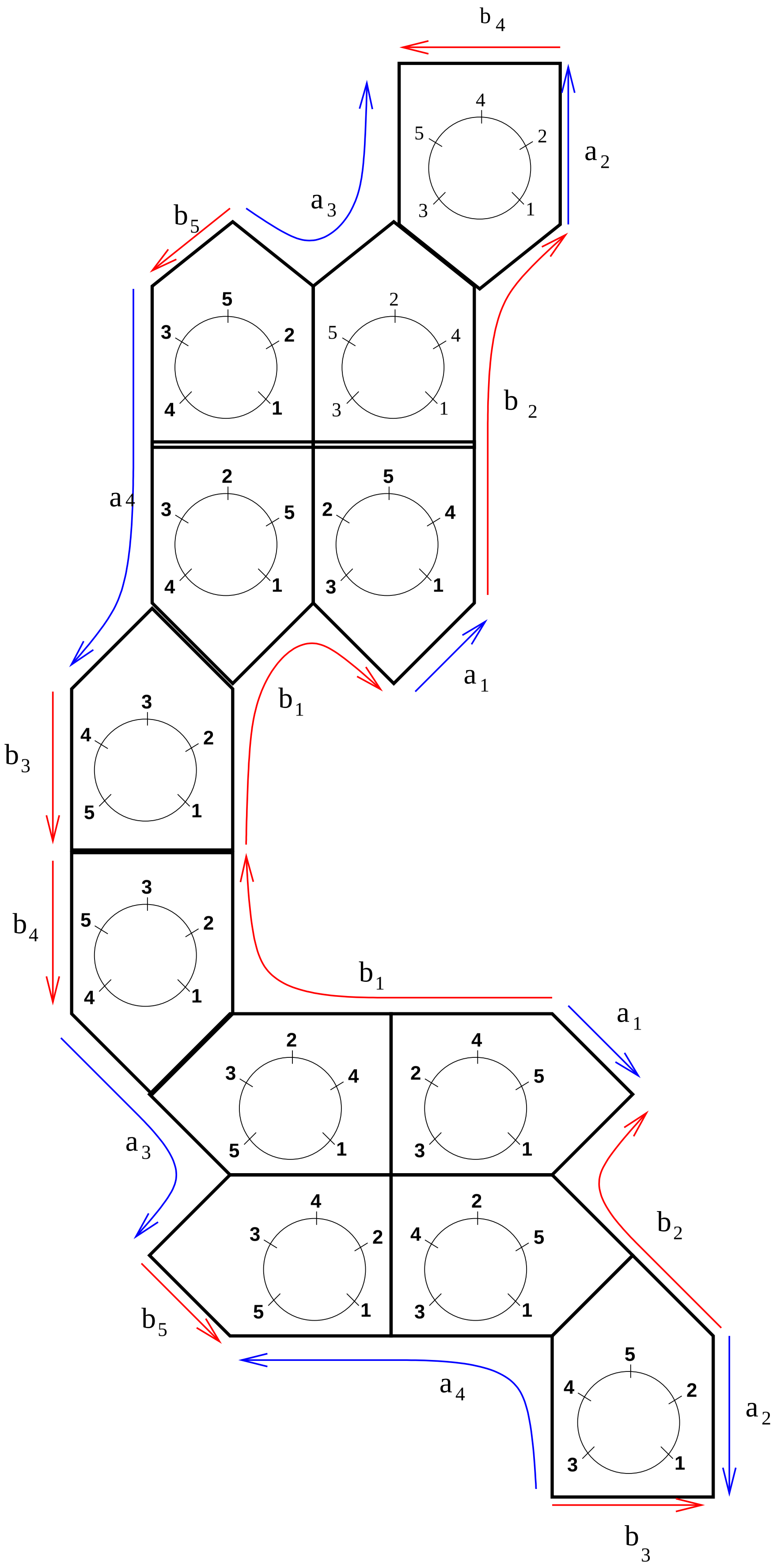


Lemma. $\forall\ n \geq 5$ and $\forall\ i, j,\ 1 \leq i \leq n,\ 4 \leq j \leq n$
the cells $K_{ij|l_1 \ldots l_{n-2}}$ are not in the class $W_{n-4}(\overline{\mathcal{M}^{\mathbb{R}}_{0,n}})$.

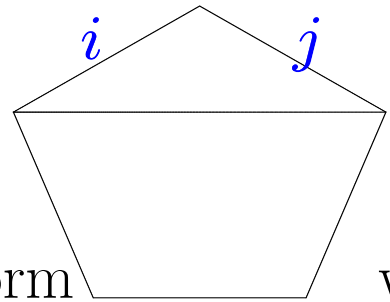
Proof. $\forall\ i, j,\ 1 \leq i \leq n,\ 4 \leq j \leq n$ the cells of MAX dim with common boundary $K_{ij|l_1 \ldots l_{n-2}}$ look like



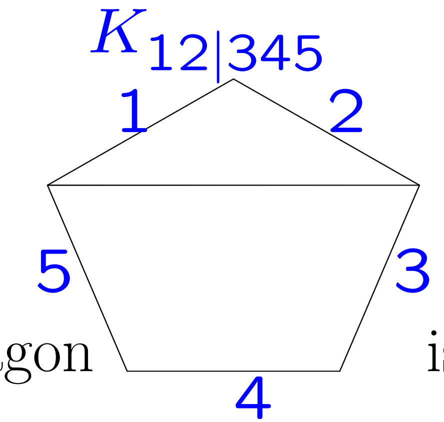

So, $K_{ij|l_1 \ldots l_{n-2}}$ is in the sum twice with the opposite signs, hence, it is not in the class $W_{n-4}(\overline{\mathcal{M}^{\mathbb{R}}_{0,n}})$. □




$n = 5$

By Lemma it remains to consider the cells of codim 1 of the form 

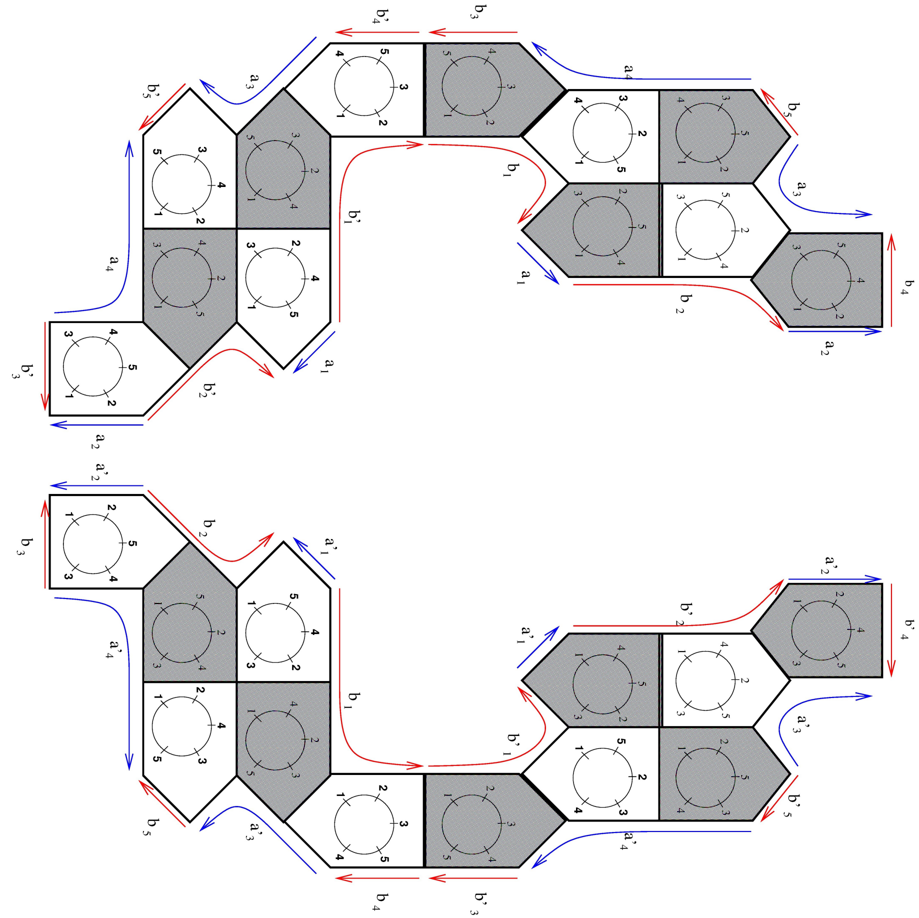
where $1 \leq i, j \leq 3$.

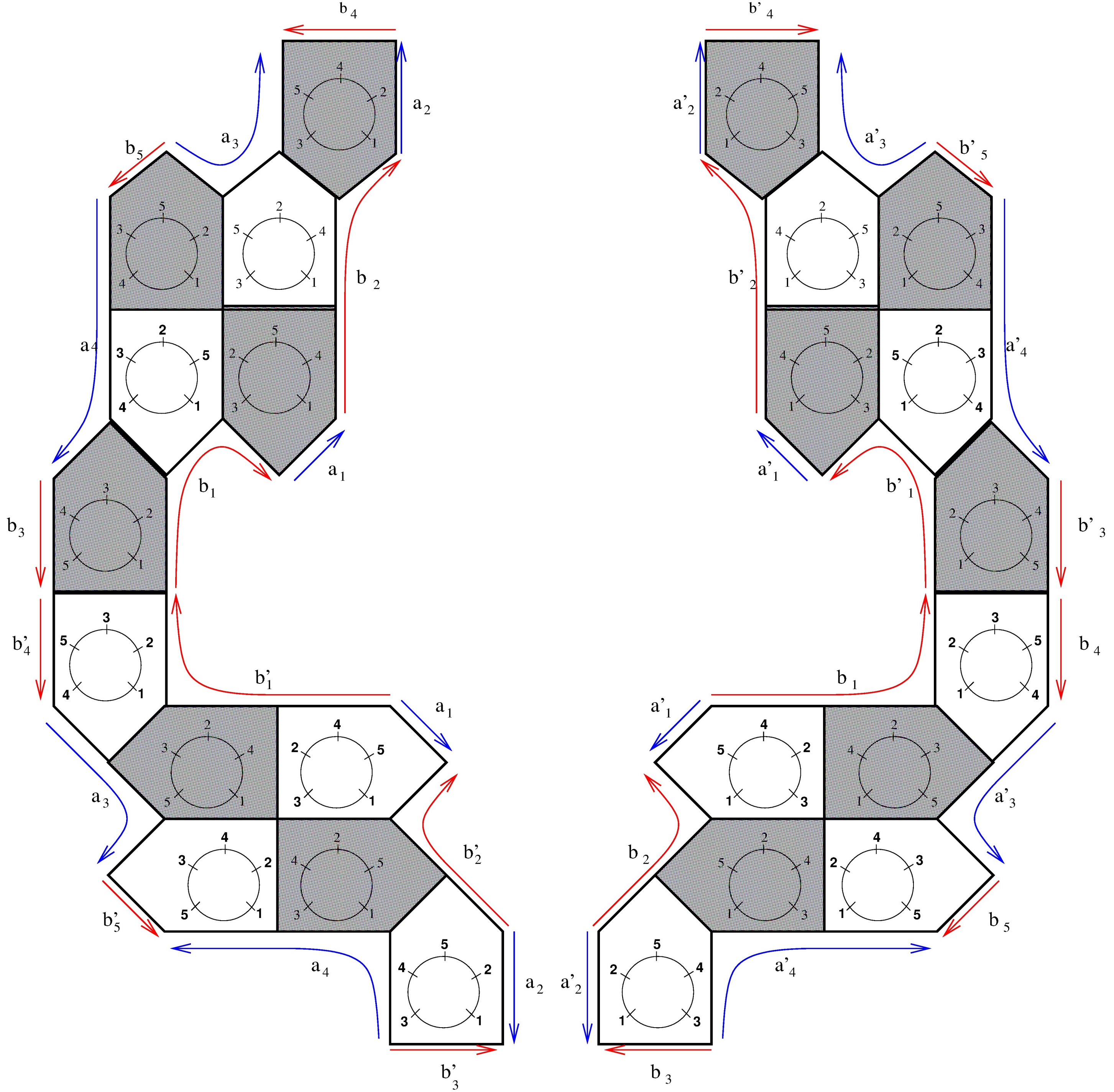
Lemma. The boundary cell labeled by the pentagon 

is in the class $W_1(\overline{\mathcal{M}^{\mathbb{R}}_{0,5}})$.

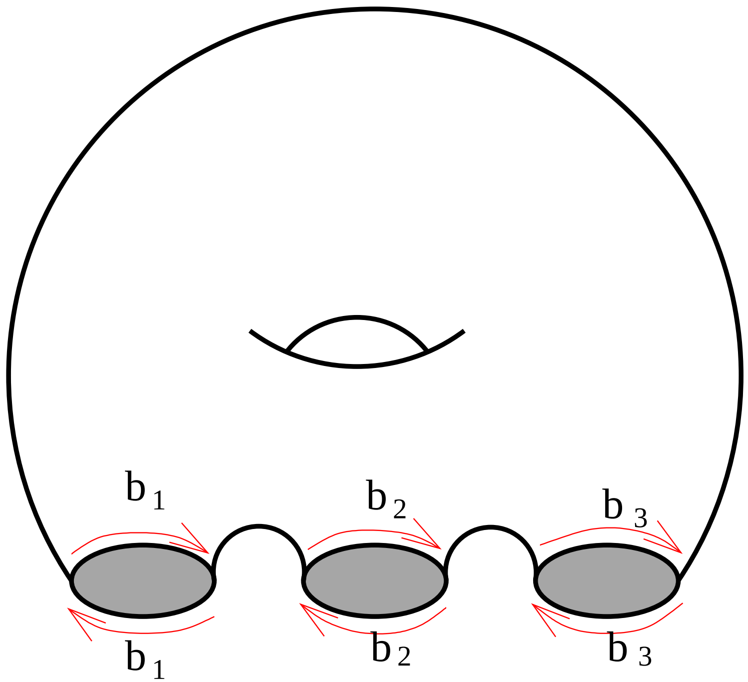
Corollary. $W_1(\overline{\mathcal{M}^{\mathbb{R}}_{0,5}}) = \{K_{12|345}, K_{12|435}, K_{12|354}, K_{13|245}, K_{13|254}, K_{13|425}, K_{23|145}, K_{23|415}, K_{23|154}\}$.

$f$=24
$e$=60
$v$=30
$g$=4, $S_5$



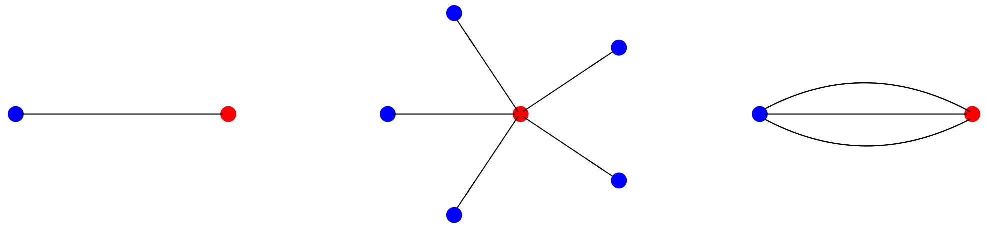


b 4
a 2
a 3
b 5
b 2
a 4
a 1
b 1
b 3
b'4
b'1
a 1
a 3
b'2
b'5
a 4
a 2
b'3
b' 4
a'2
a'3
b' 5
b' 2
a'4
a'1
b' 1
b' 3
b 4
b 1
a'1
a'3
b 2
b 5
a'2
a'4
b 3



b 1
b 2
b 3
b 1
b 2
b 3

Definition. A dessin d'enfant $D$ is a compact connected smooth oriented surface $M$ together with a bicolored graph $\Gamma$ on it such that the complement $M \setminus \Gamma$ is homeomorphic to a disjoint union of open discs. Such a disc is called a face of the dessin. Vertices and edges of the dessin are vertices and edges of the graph.



Definition. A Belyi pair $(\mathcal{X}, \beta)$ is an algebraic curve $\mathcal{X}$ together with a non-constant rational function $\beta : \mathcal{X} \to \mathbb{CP}^1$, which has at most three critical values. $\beta$ is a Belyi function.

Up to the linear-fractional transformation of $\mathbb{CP}^1$ we may and we do fix the critical values of $\beta$, $crit(\beta) \subseteq \{0, 1, \infty\}$.

Example. $(\mathbb{CP}^1, x)$

Definition. A Belyi pair $(\mathcal{X}, \beta)$ is an algebraic curve $\mathcal{X}$ together with a non-constant rational function $\beta : \mathcal{X} \to \mathbb{CP}^1$, which has at most three critical values. $\beta$ is a Belyi function.

Up to the linear-fractional transformation of $\mathbb{CP}^1$ we may and we do fix the critical values of $\beta$, $crit(\beta) \subseteq \{0, 1, \infty\}$.

Example. $(\mathbb{CP}^1, x)$,

$(\mathbb{CP}^1, x^n)$

Definition. A Belyi pair $(\mathcal{X}, \beta)$ is an algebraic curve $\mathcal{X}$ together with a non-constant rational function $\beta : \mathcal{X} \to \mathbb{CP}^1$, which has at most three critical values. $\beta$ is a Belyi function.

Up to the linear-fractional transformation of $\mathbb{CP}^1$ we may and we do fix the critical values of $\beta$, $crit(\beta) \subseteq \{0, 1, \infty\}$.

Example. $(\mathbb{CP}^1, x)$,

$(\mathbb{CP}^1, x^n)$,

$(\mathbb{CP}^1, T_n(x) = \cos(n \arccos x))$ — Chebyshev polynomials

Definition. A Belyi pair $(\mathcal{X}, \beta)$ is an algebraic curve $\mathcal{X}$ together with a non-constant rational function $\beta : \mathcal{X} \to \mathbb{CP}^1$, which has at most three critical values. $\beta$ is a Belyi function.

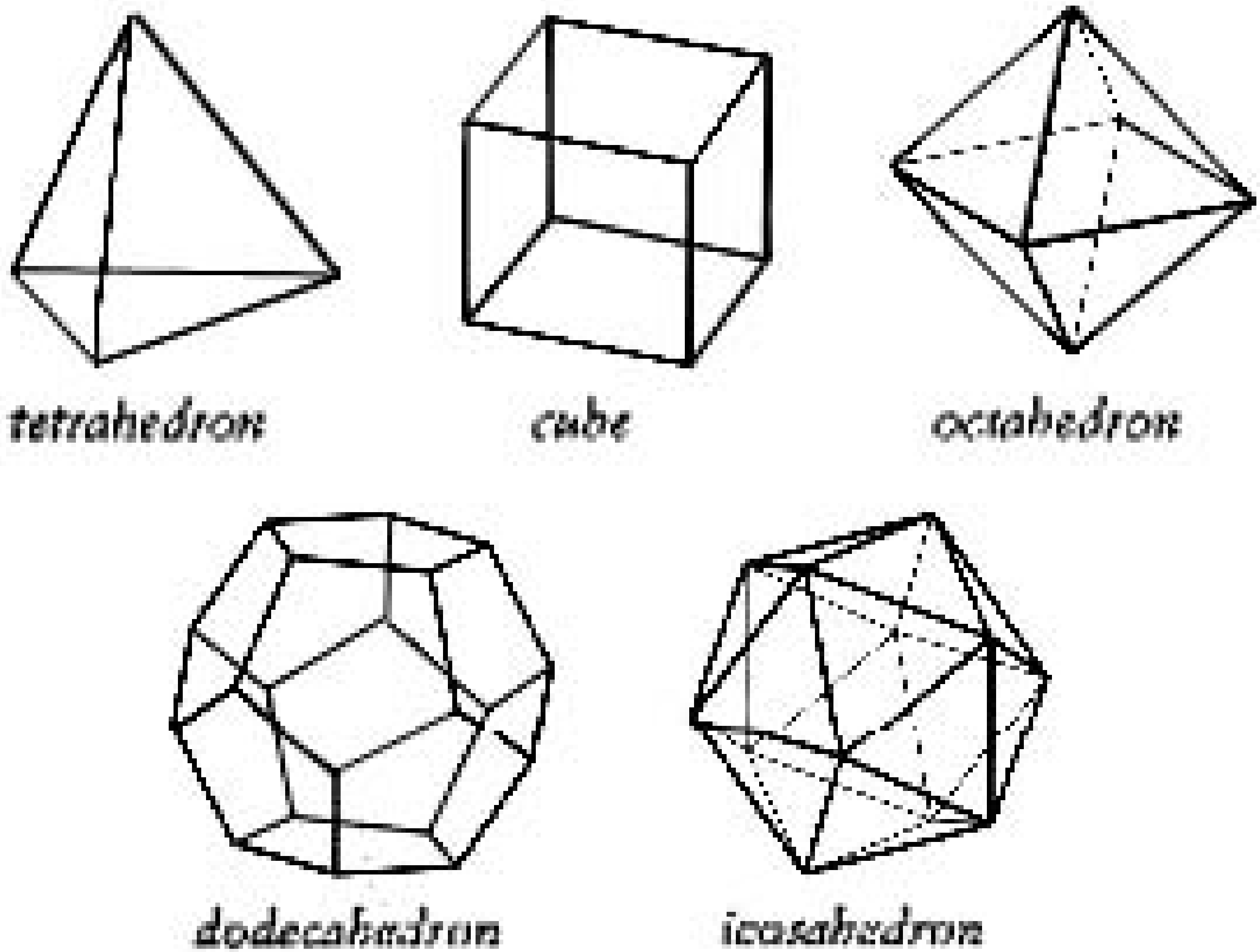
Up to the linear-fractional transformation of $\mathbb{CP}^1$ we may and we do fix the critical values of $\beta$, $crit(\beta) \subseteq \{0, 1, \infty\}$.

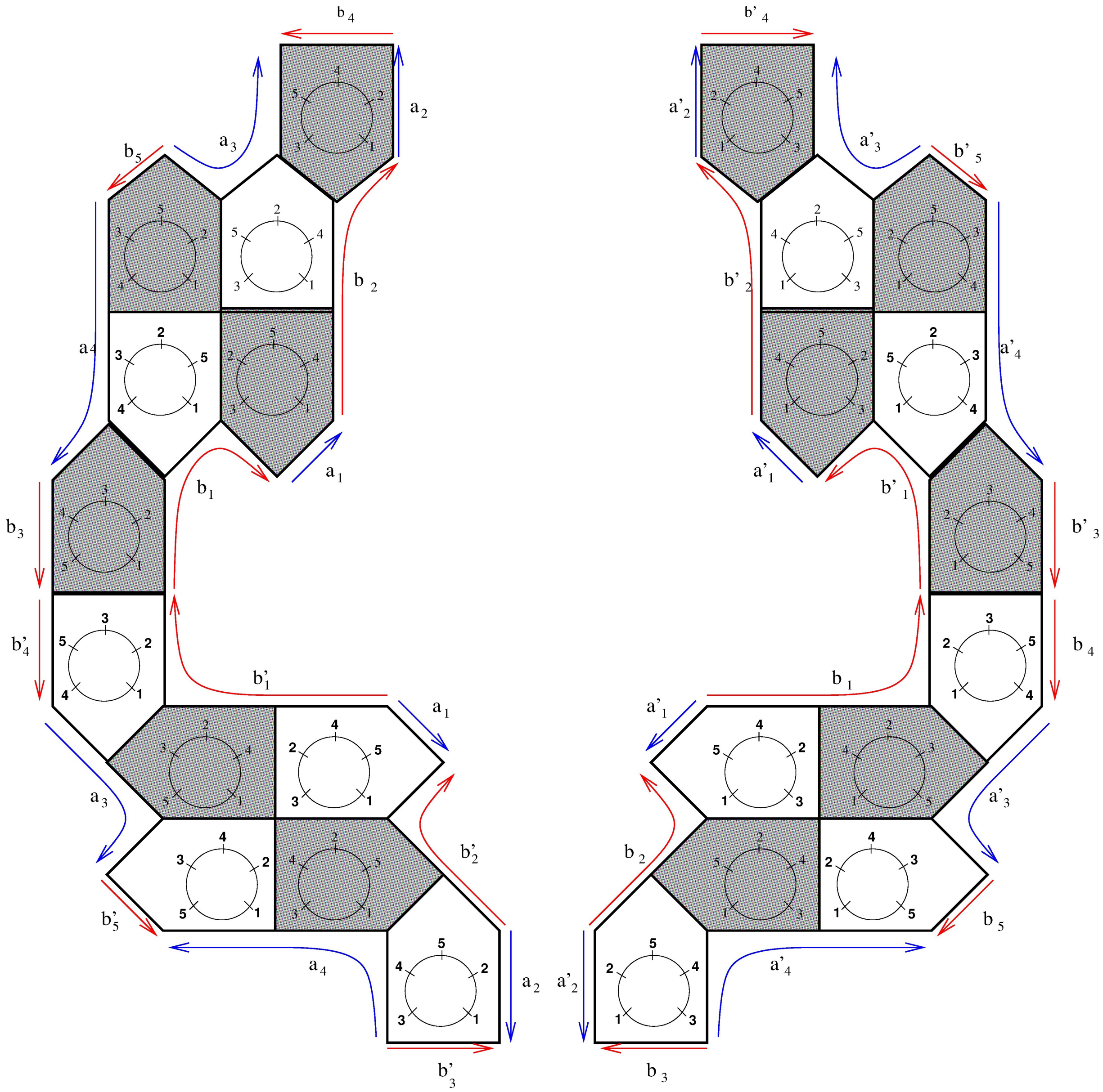
Example. $(\mathbb{CP}^1, x)$,

$(\mathbb{CP}^1, x^n)$,

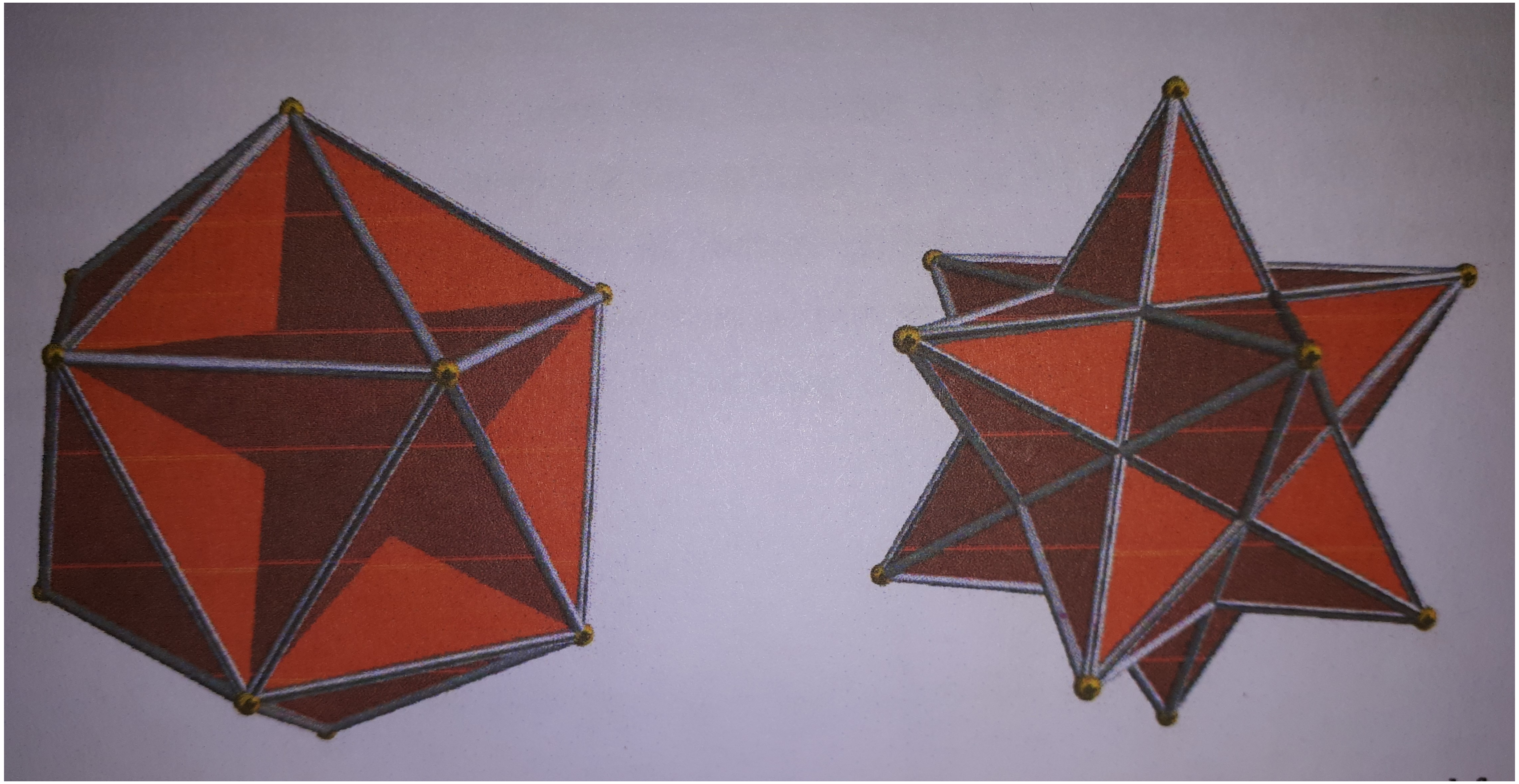
$(\mathbb{CP}^1, T_n(x) = \cos(n \arccos x))$,

$(\mathbb{T} : y^2 = x^3 - x, x^2)$


tetrahedron
cube
octahedron
dodecahedron
icosahedron



b 4
a 2
a 3
b 5
b 2
a 4
a 1
b 1
b 3
b'4
b'1
a 1
a 3
b'2
b'5
a 4
a 2
b'3
b' 4
a'2
a'3
b' 5
b' 2
a'4
a'1
b' 1
b' 3
b 4
b 1
a'1
a'3
b 2
b 5
a'2
a'4
b 3



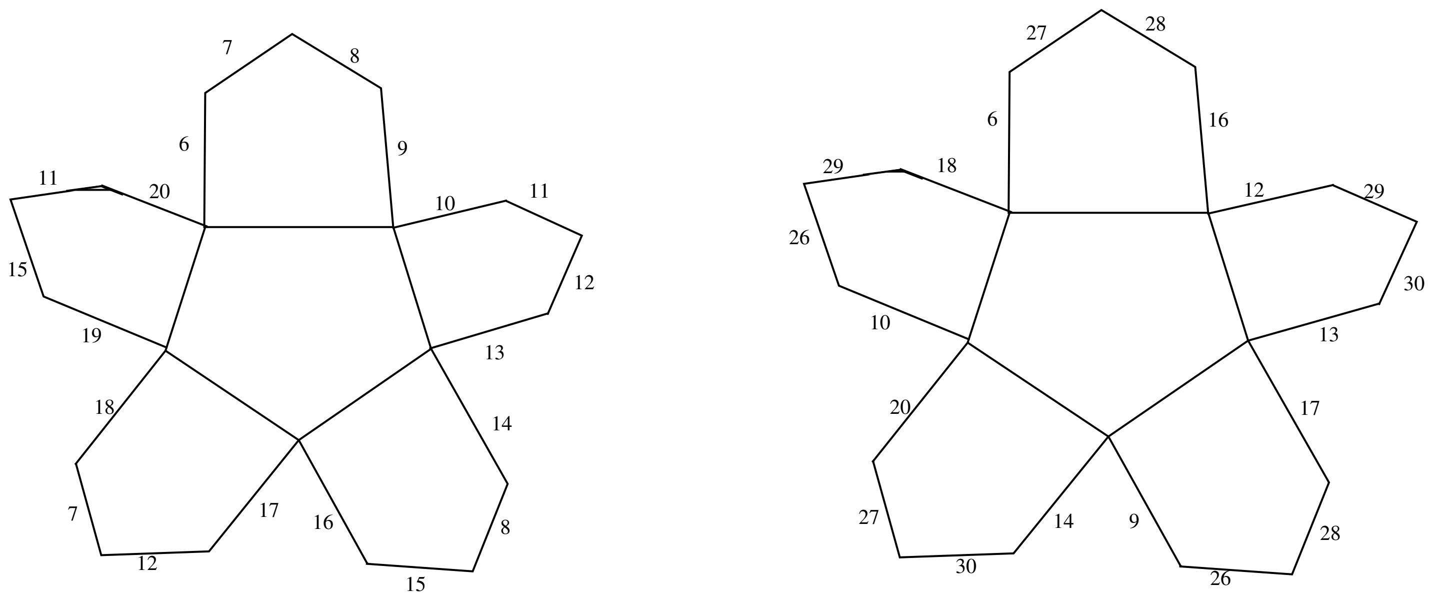



Figure 1: Scanning of $I_4$

Definition. Let $(X, G)$ be a dessin d'enfant. A dessin $(X, G^*)$ is called dual to $(X, G^*)$ iff the sets of blue vertices of $G$ and $G^*$ coincide, the set of red vertices of $G^*$ coincides with the set of centers of faces of $G$, and the edges connect centers of faces of $G$ with all blue vetrices incident to this face.

Definition. Let $(X, G)$ be a dessin d'enfant. A union $(X, G) \cup (X, G^*)$ is the dessin on $X$. Its blue vertices are the common blue verices of $G$ and $G^*$, its red vertices are the unions of red vertices of $G$ and red vertices of $G^*$, and its edges are the union of edges of red vertices of $G$ and red vertices of $G^*$.

Remark. Let $(X, G)$ be a dessin d'enfant, and $\beta$ be its Belyi function. Then the Belyi function of $(X, G^*)$ is $1/\beta$. Also the Belyi function of the union $(X, G) \cup (X, G^*)$ is $\frac{4\beta}{(\beta+1)^2}$.

Theorem. [A.K. Zvonkin, Functional composition is a generalized symmetry, Symmetry: Culture and Science, 21, 1-4 (2010) 333-368]

Belyi pair for $I_4 \cup I_4^*$ is Bring curve — an algebraic curve in dim=4 complex projective space with coordinates $x_1 : \ldots : x_5$ defined by

$$B_5 : \quad \begin{cases} \sum_{i=1}^{5} x_i = 0 \\ \sum_{i=1}^{5} x_i^2 = 0 \\ \sum_{i=1}^{5} x_i^3 = 0 \end{cases}$$

and the function

$$f_{B_5}(x_1 : \ldots : x_5) = \frac{256\, a^5}{256\, a^5 + 3125\, b^4}$$

where $b = x_1 \cdots x_5$, $a = \sum_{i=1}^{5} x_1 \cdots \widehat{x_i} \cdots x_5$ on it.

Bring curve was introduced and firstly investigated by Felix Klein in 1884 after in 1786 Erland Bring, a history professor at Lund, found a change of variables which reduces a generic quintic equation to the form $x^5 + ax + b$.

Theorem. [AK, 2020] Belyi pair for $\mathcal{L}(\overline{\mathcal{M}_{0,5}^{\mathbb{R}}})$ is the Bring curve and the function

$$1 - \frac{1}{f_{B_5}(x_1 : \ldots : x_5)} = -\frac{3125b^4}{256a^5} \ .$$

Proof.

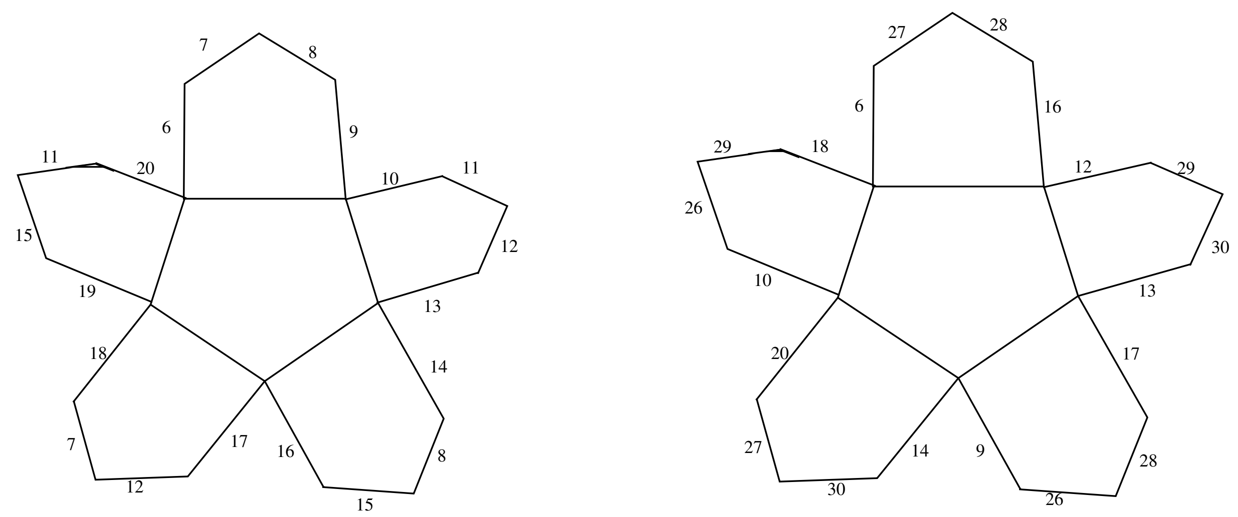



Figure 1: Scanning of $I_4$

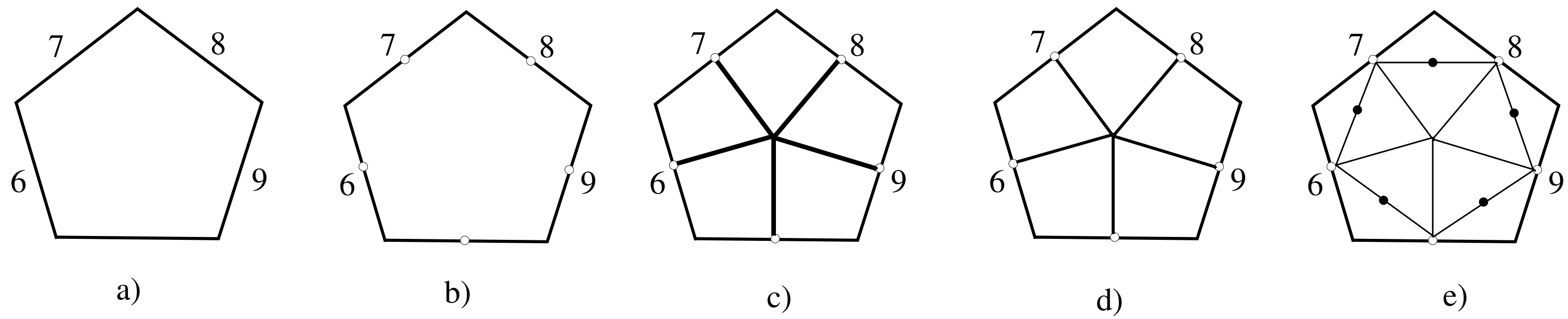



Figure 2: Transition from a face of $I$ to a face of $\mathcal{J}$

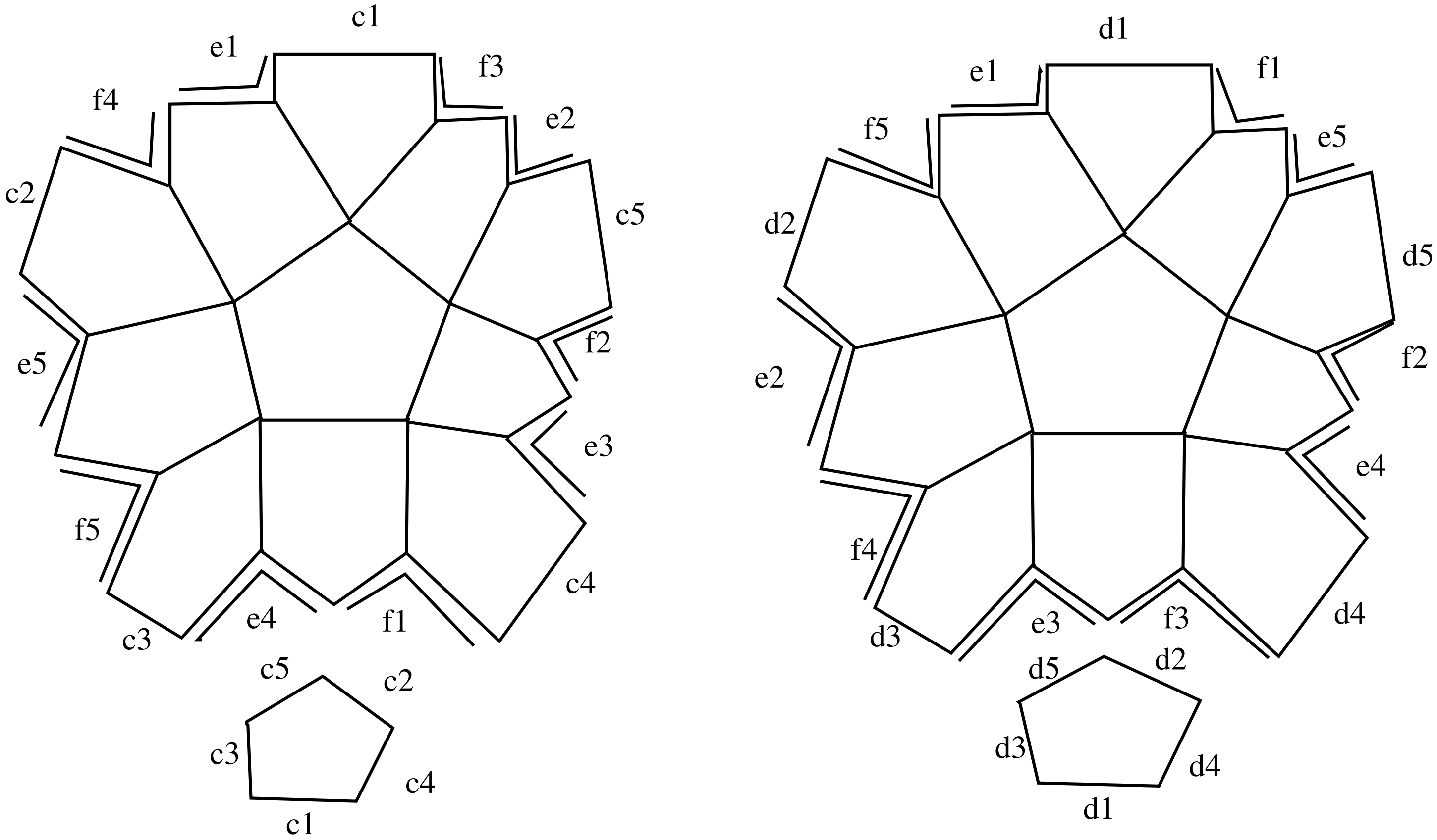



Figure 3: Scanning of the dessin $\mathcal{J}$

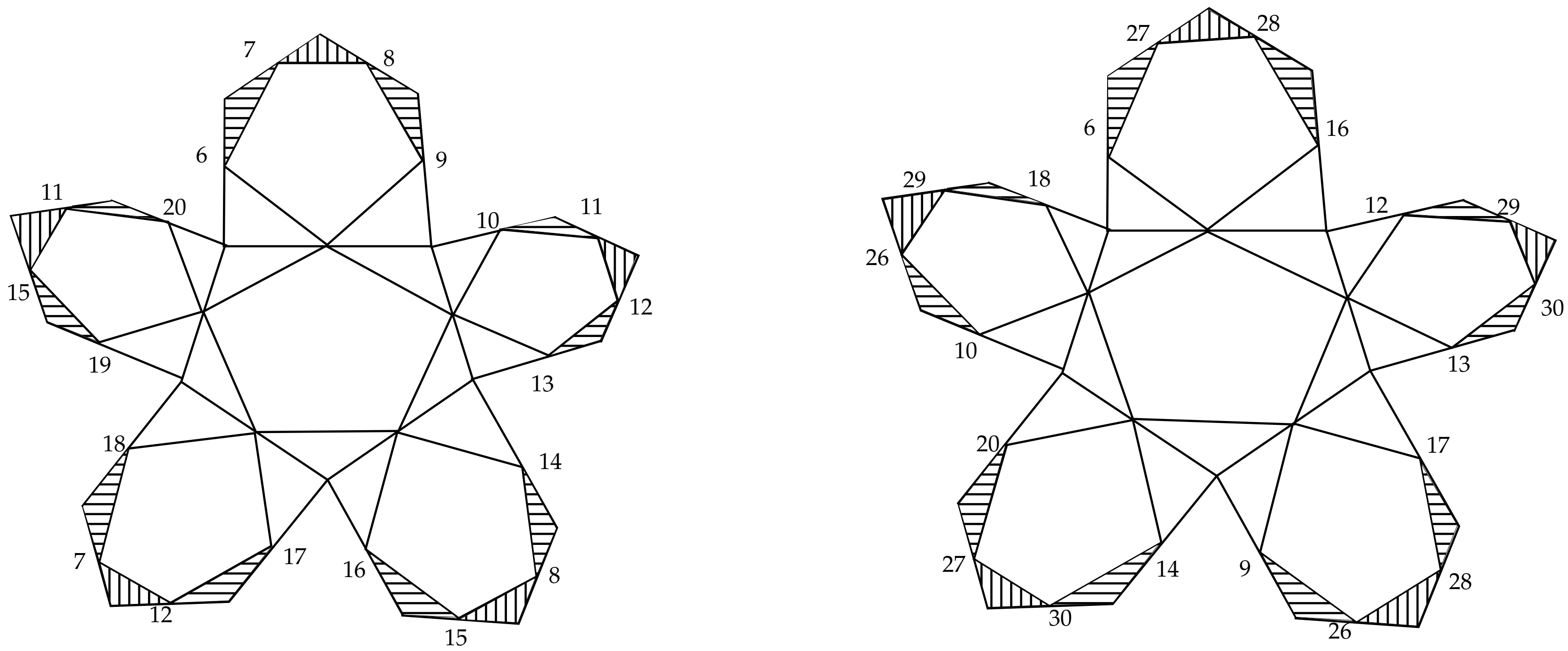



Figure 4: The triangles for cutting off at the dessin $\mathcal{J}$

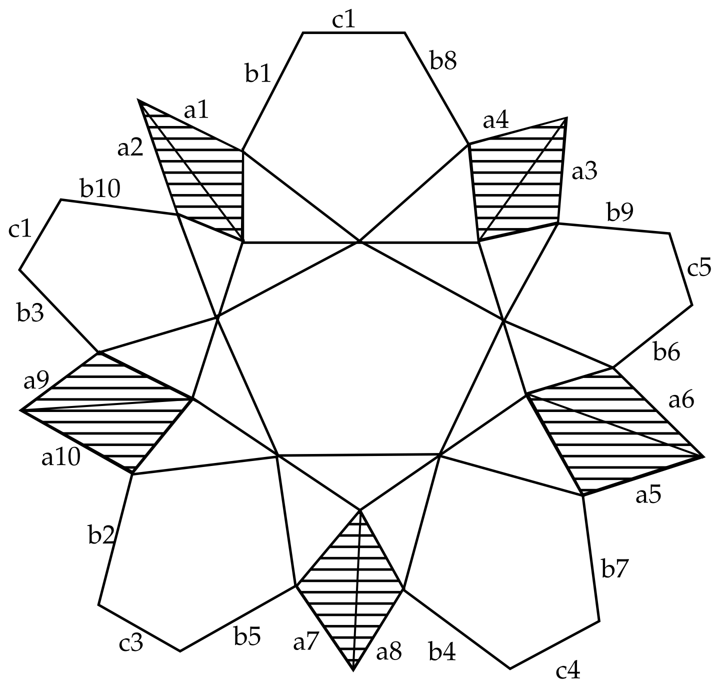



Figure 5: Gluing of horizontally hatched parts of $\mathcal{J}$

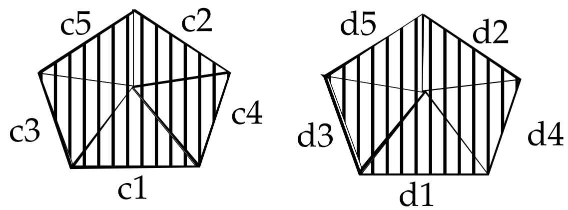



Figure 6: Gluing of vertically hatched parts of $\mathcal{J}$

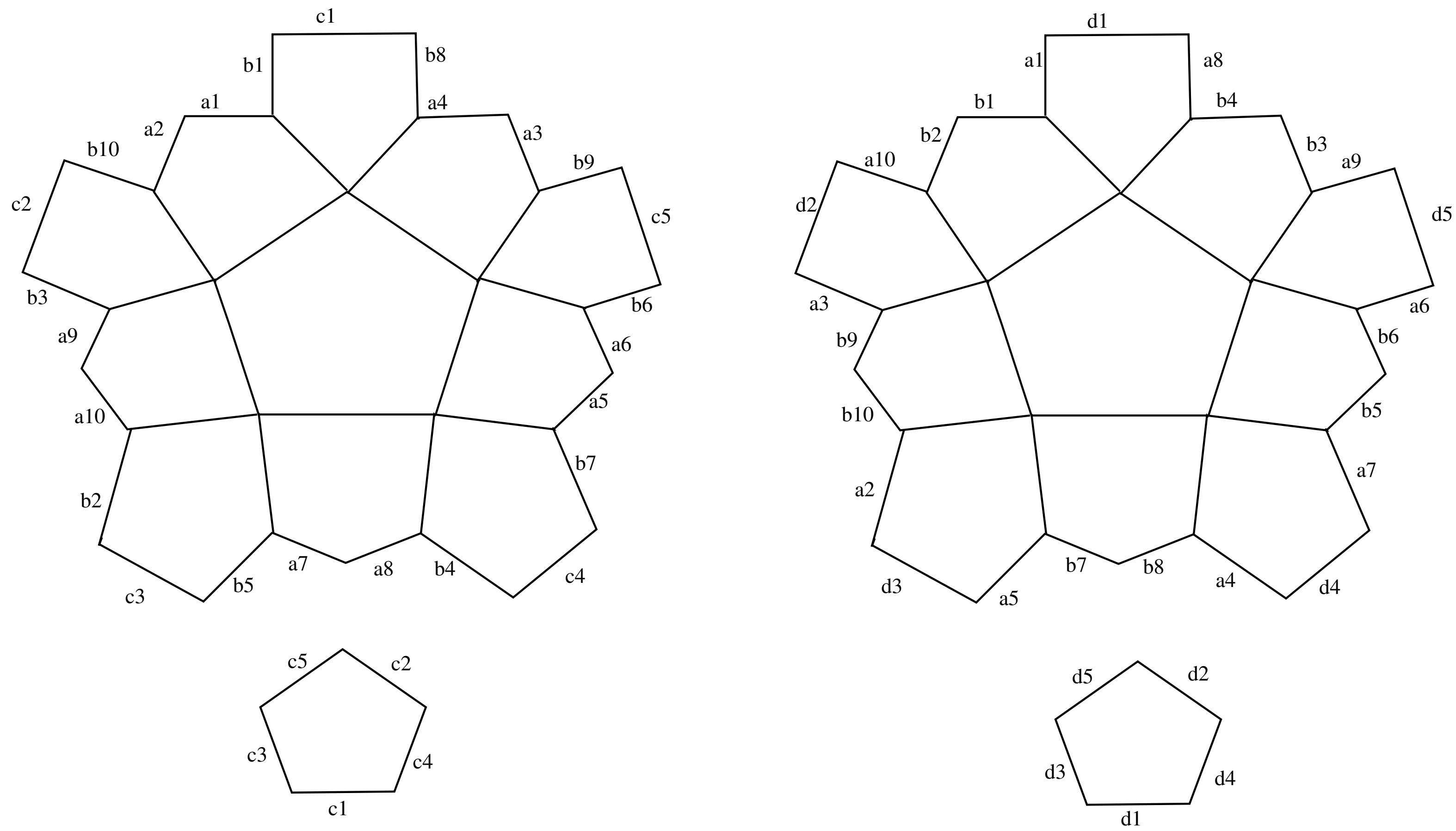



Figure 7: Another gluing of $\mathcal{J}$

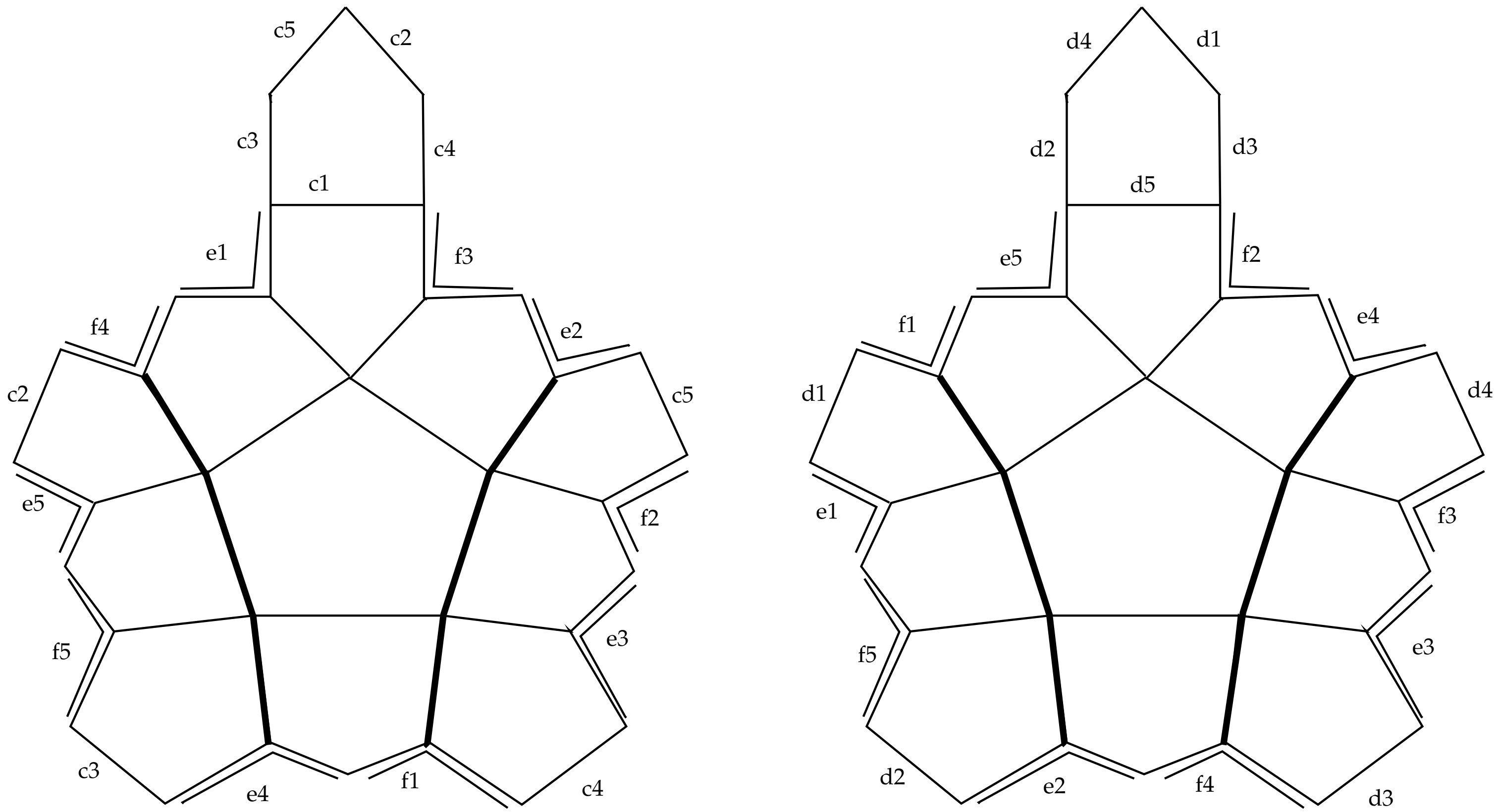
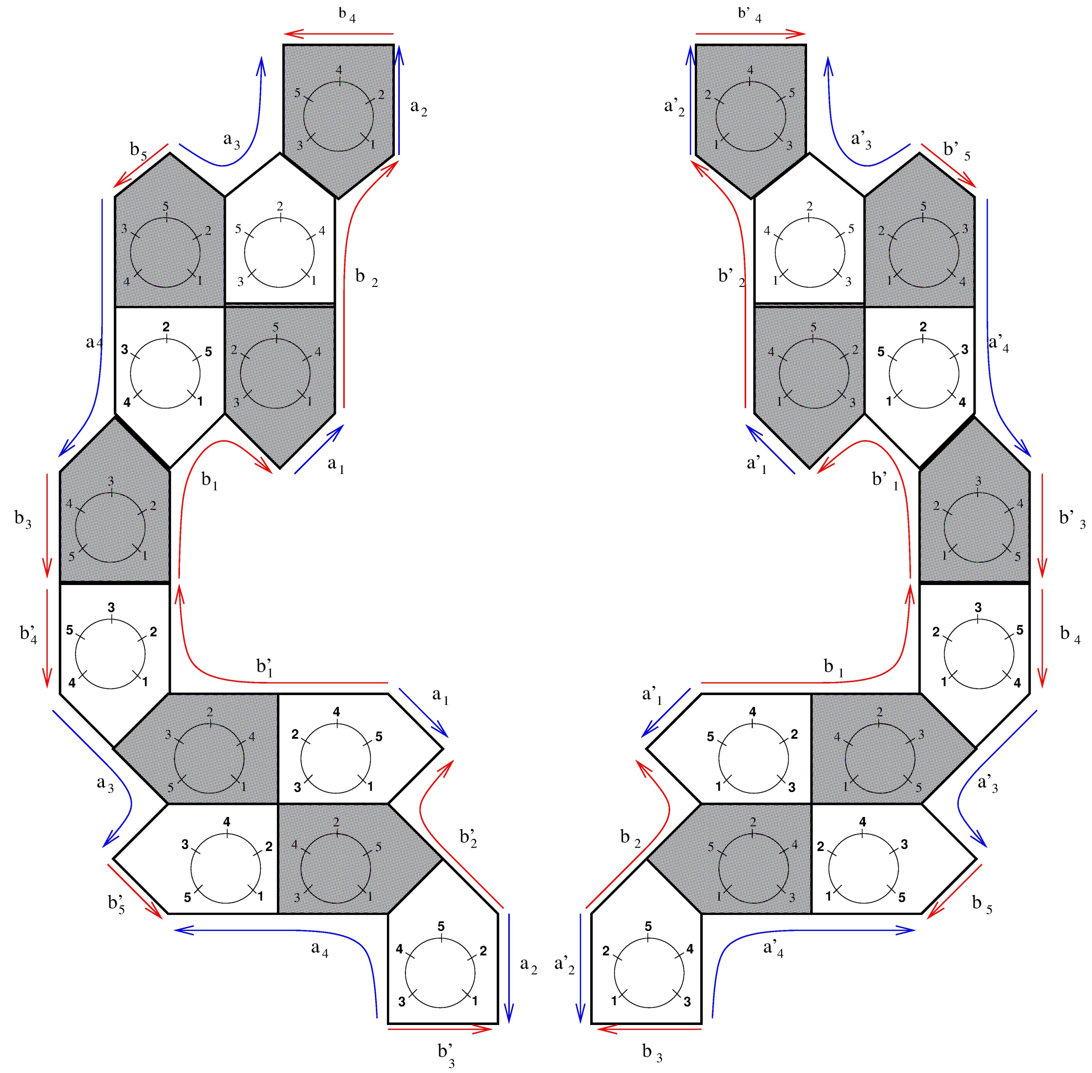



Figure 8: Another scanning of the dessin $\mathcal{J}$ with the lines to cut off.



b4
a2
a3
b5
b2
a4
b1
a1
b3
b'4
b'1
a1
a3
b'2
b'5
a4
a2
b'3
b'4
a'2
a'3
b'5
b'2
a'4
a'1
b'1
b'3
b4
b1
a'1
a'3
b2
b5
a'4
a'2
b3

Icosahedron $g = 4$ appeared as a mosaic by Paolo Uccello on the floor of San Marco cathedral, Venice, 1430.

THANK YOU!!!