Quadratic Stochastic Operators Generated by Measurable Finite Partitions of Infinite State Space

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Outline

Abstract

Today I would like to present some my results concerning theory of non-linear dynamical systems, namely, quadratic stochastic operators, received during Malaysian's period. Majority of these results were received jointly with my Malaysian PhD and Master students.

QSO with Finite State Space

Let

$$S^{m-1} = \{\mathbf{x} = (x_1, x_2, \cdots, x_m) \in R^m : \text{ for any } i \; x_i \geq 0, \text{ and } \sum_{i=1}^{m} x_i = 1\} \tag{1}$$

be the $(m-1)$- dimensional simplex. We consider the discrete time dynamical system is defined by the mapping $V : S^{m-1} \to S^{m-1}$ with

$$(V\mathbf{x})_k = \sum_{i,j=1}^{m} p_{ij,k} x_i x_j, \tag{2}$$

where

$$a) p_{ij,k} \geq 0, \quad b) p_{ij,k} = p_{ji,k} \text{ for all } i, j, k; \quad c) \sum_{k=1}^{m} p_{ij,k} = 1. \tag{3}$$

QSO with Finite State Space

A mapping (2 with conditions (3), called *quadratic stochastic operator* (qso), was first introduced by Bernstein. Such operators frequently arise in many models of mathematical genetics, namely, theory of heredity (Bernstein, Jenks, Kesten, Lyubich).

Note that the study of such operators in our country have been initiated by Rasul Ganikhodjaev in his PhD thesis in 1970.

At present the theory of non-linear dynamical systems is developing rather intensely.

Application QSO in Medicine

Blood groups are the distinguishing of blood by their antigenic properties. These properties are determined by the substances found on the surface of the red blood cells. There are approximately 200 blood group substances identified and categorized into 19 distinct systems. The most common system is the ABO system. The human ABO blood group was discovered by Karl Landsteiner in 1900[3], and its mode of inheritance as multiple alleles at a single generic locus was established by Felix Bernstein [1] a quarter century later. The ABO blood group antigens appear to have been important throughout our evolution because the frequencies of different ABO blood types vary among different populations, suggesting that a particular blood type conferred a selection advantage.

Application QSO in Medicine

The Rhesus system is the second most significant blood group system in human blood transfusion. Individuals either have, or do not have, the Rhesus factor (or RhD antigen) on the surface of their red blood cells. This is usually indicated by Rh^{+} (does have the RhD antigen) or Rh^{-} (does not have the antigen) suffix to the ABO blood type. A child inherits two rhesus genes, one from each parent, where gene D corresponds to positive rhesus factor and gene d corresponds to negative rhesus factor.

Offspring is rhesus negative if they have inherited a d gene from each parent (d, d) and offspring is rhesus positive if they inherited a D gene from both parents. If offspring have inherited a rhesus positive gene D and a rhesus negative gene d, they are most likely to be rhesus positive as the D gene is more dominant as compared to the d gene. Hence it is possible to have a rhesus negative child and a rhesus positive father.

Application QSO in Medicine

It is well known that blood groups and rhesus of parents do not determine unambiguously their offspring's blood group and rhesus. The transmission of blood groups and its rhesus from parents to their offspring is a random events. To study these transmissions we consider the following stochastic modeling generated by quadratic stochastic operators.

Application QSO in Medicine

We consider

$$(V\mathbf{x})_k = \sum_{i,j=1}^{m} p_{ij,k} x_i x_j, \tag{4}$$

for all $k = 1, \cdots, m$, with $P(ij, \{k\}) = p_{ij,k}$ where

$$a) p_{ij,k} \geq 0, \quad b) p_{ij,k} = p_{ji,k} \text{ for all } i, j, k; \quad c) \sum_{k=1}^{m} p_{ij,k} = 1.$$

Note that the condition $b) p_{ij,k} = p_{ji,k}$ is not overloaded, since otherwise one can determine new heredity coefficients $q_{ij,k} = \frac{p_{ij,k}+p_{ji,k}}{2}$ with preserving the operator V,

$$V : (V\mathbf{x})_k = \sum_{i,j=1}^{m} q_{ij,k} x_i x_j, \quad k = 1, \cdots, m \tag{5}$$

Application QSO in Medicine

Thus the transformation (4) or (5), which describes a model of heredity is a quadratic stochastic operator. A model of heredity is uniquely determined by heredity coefficients $p_{ij,k}$ or $q_{ij,k} = \frac{p_{ij,k}+p_{ji,k}}{2}$ for $i, j, k = 1, \cdots, m$.

Assume $\{\mathbf{x}^{(n)} \in S^{m-1} : n = 0, 1, 2, \cdots\}$ is the trajectory of the initial point $\mathbf{x} \in S^{m-1}$, where $\mathbf{x}^{(n+1)} = V(\mathbf{x}^{(n)})$ for all $n = 0, 1, 2, \cdots$, with $\mathbf{x}^{(0)} = \mathbf{x}$.

A point $\mathbf{a} \in S^{m-1}$ is called a **fixed point** of a qso V if $V(\mathbf{a}) = \mathbf{a}$.

A qso V is called regular if for any initial point $\mathbf{x} \in S^{m-1}$ the limit

$$\lim_{n\to\infty} V^n(\mathbf{x}) \tag{6}$$

exists.

Application QSO in Medicine

Note that the limit point be a fixed point of a qso V. Thus the fixed points of qso describe limit or long run behavior the trajectories of any initial point. Limit behavior of trajectories and fixed points of qso play important role in many applied problems.
We introduce a nonlinear model of transmission Rhesus factor that is described by nonlinear transformation, namely, a quadratic stochastic operator.

Application QSO in Medicine

Let a set of species be a set of factor rhesus $\{1, 2\}$, where 1 denote positive rhesus and 2 denote negative rhesus. To describe the transmission of factor rhesus from parents to their offspring we need to find the probability $P_{XY,Z}$ that from a father with factor rhesus X and a mother with factor rhesus Y their child receives factor rhesus Z, where $X, Y, Z \in \{1, 2\}$. Let $N(F_X, M_Y)$ be the number of offspring of fathers F_X and mothers M_Y, that is fathers with factor rhesus X and mothers with factor rhesus Y and $N^Z(F_X, M_Y)$ be the number of offspring with factor rhesus Z of fathers F_X and mothers M_Y. Then the transmission probability $P_{XY,Z}$ is defined as

$$P_{XY,Z} = \frac{N^Z(F_X, M_Y)}{N(F_X, M_Y)}$$

Let $q_{ij,k} = \frac{p_{ij,k}+p_{ji,k}}{2}$ be the heredity coefficients for $i, j, k = 1, 2$.

Application QSO in Medicine

For collected data according definition we have following:

$$q_{11,1} = 0.985, q_{12,1} = 0.652, q_{22,1} = 0.092$$

$$q_{11,2} = 0.015, q_{12,2} = 0.348, q_{22,2} = 0.908$$

and corresponding quadratic stochastic operator has form

$$\begin{aligned} x_1' &= 0.985x_1^2 + 1.305x_1x_2 + 0.092x_2^2; \\ x_2' &= 0.015x_1^2 + 0.695x_1x_2 + 0.908x_2^2; \end{aligned} \quad (7)$$

where x_1 is the fraction of the population with positive rhesus factor and x_2 is the fraction of the population with negative rhesus factor.

Application QSO in Medicine

The transformation (7) has single fixed point

$$x_1^* = 0.954, x_2^* = 0.046$$

The Jacobian of the quadratic stochastic operator (7) at the fixed point has following form

$$J_V(x_1*, x_2*) = \begin{bmatrix} 1.939 & 1.253 \\ 0.061 & 0.746 \end{bmatrix}$$

with eigen values $\lambda_1 = 0.685$ and $\lambda_2 = 2$. The fixed point is stable and any trajectory of quadratic stochastic operator converge to fixed point. Thus the quadratic stochastic operator is a regular.

Application QSO in Medicine

In this case $q_{22,1} = 0.092$ means that a child of parents with negative rhesus factor receives positive rhesus factors with probability 0.092. In the case of a mutation, the factor rhesus typing may not hold true in the question of parentage.

QSO Generated by Finite Measurable Partitions of Infinite Countable State Space

Let $(X, \mathbb{F})$ be a measurable space and $S(X, \mathbb{F})$ be the set of all probability measures on $(X, \mathbb{F})$, where X is a state space and $\mathbb{F}$ is σ-algebra of subsets of X.

Let $\{P(x, y, A) : x, y \in X, A \in \mathbb{F}\}$ be a family of functions on $X \times X \times \mathbb{F}$ that satisfy the following conditions:

i) $P(x, y, \cdot) \in S(X, \mathbb{F})$, for any fixed $x, y \in X$, that is, $P(x, y, \cdot) : \mathbb{F} \rightarrow [0,1]$ is the probability measure on $\mathbb{F}$;

ii) $P(x, y, A)$ regarded as a function of two variables x and y with fixed $A \in \mathbb{F}$ is measurable function on $(X \times X, \mathbb{F} \otimes \mathbb{F})$;

iii) $P(x, y, A) = P(y, x, A)$ for any $x, y \in X, A \in \mathbb{F}$.

QSO Generated by Finite Measurable Partitions of Infinite Countable State Space

We consider a nonlinear transformation called quadratic stochastic operator (qso) $V : S(X, \mathbb{F}) \to S(X, \mathbb{F})$ defined by

$$(V\lambda)(A) = \int_X \int_X P(x, y, A) d\lambda(x) d\lambda(y), \tag{8}$$

where $\lambda \in S(X, \mathbb{F})$ is an arbitrary initial probability measure and $A \in \mathbb{F}$ is an arbitrary measurable set.

QSO Generated by Finite Measurable Partitions of Infinite Countable State Space

Let $\xi = \{A_1, A_2, \cdots, A_m\}$ be a measurable m-partition of the set X and $\zeta = \{B_{ij} : i, j = 1, \cdots, m\}$ be a corresponding partition of the cartesian product $X \times X$, where $B_{ii} = A_i \times A_i$ and $B_{ij} = (A_i \times A_j) \cup (A_j \times A_i)$ for $i < j$. We select a family $\{\mu_{ij} : i, j = 1, 2, \cdots, m\}$ of probability measures on $(X, \mathbb{F})$ with $\mu_{ij} = \mu_{ji}$, and define probability measure $P(x, y, A)$ as follows:

$$P(x, y, A) = \mu_{ij}(A) \text{ if } (x, y) \in B_{ij} \tag{9}$$

for arbitrary $A \in \mathbb{F}$.

QSO Generated by Finite Measurable Partitions of Infinite Countable State Space

Then for arbitrary $\lambda \in S(X, \mathbb{F})$ we have

$$\begin{aligned} (V\lambda)(A) &= \int_X \int_X P(x, y, A) d\lambda(x) d\lambda(y) \\ &= \sum_{i,j=1}^{m} \int_{A_i} \int_{A_j} \mu_{ij}(A) \cdot d\lambda(x) d\lambda(y) \\ &= \sum_{i,j=1}^{m} \mu_{ij}(A) \lambda(A_i) \lambda(A_j), \end{aligned} \tag{10}$$

where $A \in \mathbb{F}$ is an arbitrary measurable set.

QSO Generated by Finite Measurable Partitions of Infinite Countable State Space

Assume $\{V^n\lambda : n = 0, 1, 2, \cdots\}$ is the trajectory of the initial point $\lambda \in S(X, \mathbb{F})$, where $V^{n+1}\lambda = V(V^n\lambda)$ for all $n = 0, 1, 2, \cdots$, with $V^0\lambda = \lambda$. Then

$$\begin{aligned}
(V^{n+1}\lambda)(A) &= \int_X \int_X P(x, y, A) dV^n\lambda(x) dV^n\lambda(y) \\
&= \sum_{i,j=1}^{m} \int_{A_i} \int_{A_j} \mu_{ij}(A) \cdot dV^n\lambda(x) dV^n\lambda(y) \\
&= \sum_{i,j=1}^{m} \mu_{ij}(A)(V^n\lambda)(A_i)(V^n\lambda)(A_j), \qquad (11)
\end{aligned}$$

$$(V^{n+1}\lambda)(A_k) = \sum_{i,j=1}^{m} \mu_{ij}(A_k)(V^n\lambda)(A_i)(V^n\lambda)(A_j), \qquad (12)$$

with $k = 1, \cdots, m$.

QSO Generated by Finite Measurable Partitions of Infinite Countable State Space

Assume $x_k^{(n)} = (V^n\lambda)(A_k)$ and $p_{ij,k} = \mu_{ij}(A_k)$. Then $(x_1^{(n)}, x_2^{(n)}, \cdots, x_m^{(n)}) \in S^{m-1}$, where

$$S^{m-1} = \{\mathbf{x} = (x_1, x_2, \cdots, x_m) \in R^m : \text{ for any } i \; x_i \geq 0, \text{ and } \sum_{i=1}^{m} x_i = 1\} \tag{13}$$

be the $(m-1)$-dimensional simplex, and the system of equations (1.5) one can rewrite as follows

$$(W\mathbf{x})_k = \sum_{i,j=1}^{m} p_{ij,k} x_i x_j, \tag{14}$$

for all $k = 1, \cdots, m$, with

$$a) p_{ij,k} \geq 0, \quad b) p_{ij,k} = p_{ji,k} \text{ for all } i, j, k; \quad c) \sum_{k=1}^{m} p_{ij,k} = 1.$$

QSO Generated by Finite Measurable Partitions of Infinite Countable State Space

Thus for fixed measurable $m-$partition $\xi = \{A_1, A_2, \cdots, A_m\}$ and selected probability measures $\{\mu_{ij} : i, j = 1, \cdots, m\}$ one can approximate qso V (8) by finite-dimensional qso W (14)

Geometric QSO

Let $X = \{0, 1, 2, \cdots\}$ be countable set of nonnegative integers. On X we consider Geometric μ^G distribution, where $\mu^G(k) = (1-r)r^k$ for arbitrary $k \in X$ with $r \in (0, 1)$.
Let $\xi = \{A_1, A_2\}$ be a measurable 2-partition of the set X and $\zeta = \{B_{11}, B_{22}, B_{12}\}$ be a corresponding partition of the cartesian product $X \times X$, where $B_{ii} = A_i \times A_i$ for $i = 1, 2$ and $B_{12} = (A_1 \times A_2) \cup (A_2 \times A_1)$. We select a family $\{\mu^G_{i,j} : i, j = 1, 2\}$ of Geometric distributions with parameters $r_{11} = r_1; r_{22} = r_2; r_{12} = r_3$ and define probability measure $P(x, y, A)$ as follows:

$$P(x, y, A) = \mu^G_{ij}(A) \text{ if } (x, y) \in B_{ij}, \quad i, j = 1, 2 \tag{15}$$

for arbitrary $A \in \mathbb{F}$.

According $p_{ij,k} = \mu_{ij}(A_k)$, we have

$$p^G_{11,1} = \mu_{11}(A_1); \;\; p^G_{22,1} = \mu_{22}(A_1); \;\; p^G_{12,1} = \mu_{12}(A_1)$$

and

$$p^G_{11,2} = \mu_{11}(A_2); \;\; p^G_{22,2} = \mu_{22}(A_2); \;\; p^G_{12,2} = \mu_{12}(A_2).$$

A quadratic stochastic operator W on S^1 has the following form

$$\begin{aligned}(W\mathbf{x})_1 &= ax_1^2 + 2bx_1x_2 + cx_2^2 \\ (W\mathbf{x})_2 &= (1-a)x_1^2 + 2(1-b)x_1x_2 + (1-c)x_2^2\end{aligned} \tag{16}$$

where $a = p^G_{11,1}, b = p^G_{12,1} = p^G_{21,1}, c = p^G_{22,1}$ are arbitrary coefficients with $0 \leq a, b, c \leq 1$.

Geometric QSO

It is evident that the parameters a, b, c depend on 2-partition $\xi = \{A_1, A_2\}$.

First case Two parameters: $r_1 = r_2 \neq r_3$. Then $a = c$.

$$
\begin{aligned}
(W\mathbf{x})_1 &= ax_1^2 + 2bx_1x_2 + ax_2^2 \\
(W\mathbf{x})_2 &= (1-a)x_1^2 + 2(1-b)x_1x_2 + (1-a)x_2^2
\end{aligned}
\tag{17}
$$

According Lyubich Theorem corresponding qso be regular.

Second case Three parameters: $r_1 \neq r_2 \neq r_3$. Then $a \neq b \neq c$. Apply Lyubich theorem.

Poisson qso

Let $X = \{0, 1, 2, \cdots\}$ be countable set of nonnegative integers. On X we consider Poisson μ distribution, where $\mu(k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$ for arbitrary $k \in X$ with $\lambda > 0$ Let $\xi = \{A_1, A_2\}$ be a measurable 2-partition of the set X and $\zeta = \{B_{11}, B_{22}, B_{12}\}$ be a corresponding partition of the cartesian product $X \times X$, where $B_{ii} = A_i \times A_i$ for $i = 1, 2$ and $B_{12} = (A_1 \times A_2) \cup (A_2 \times A_1)$. We select a family $\{\mu_{i,j} : i, j = 1, 2\}$ of Poisson distributions with parameters $\lambda_{11} = \lambda_1; \lambda_{22} = \lambda_2; \lambda_{12} = \lambda_3$ and define probability measure $P(x, y, A)$ as follows:

$$P(x, y, A) = \mu_{ij}(A) \text{ if } (x, y) \in B_{ij}, \quad i, j = 1, 2 \tag{18}$$

for arbitrary $A \in \mathbb{F}$.

As above one can define and investigate QSO W with finite state space generated by selected family of measures.

QSO Generated by Finite Measurable Partitions of Continual State Space

Below we consider QSO defined on continual sets $X = (-\infty, \infty)$ of real numbers and $X = [0, 1)$.

Let $X = \mathbb{R}$ be the set of real numbers, $\mathbb{B}$ be the $\sigma-$ algebra of Borel subsets of $\mathbb{R}$ and let $S(\mathbb{R}, \mathbb{B})$ be the set of all probability measures on $\mathbb{R}$. Remind that the distribution

$$\mu(A) = \frac{1}{\sigma\sqrt{2\pi}} \int_A e^{-\frac{1}{2}\frac{(u-m)^2}{\sigma^2}} du \tag{19}$$

where $A \in \mathbb{B}$, is called a Gaussian measure with mean m and standard deviation σ.

As before let $\xi = \{A_1, A_2, \cdots, A_m\}$ be a measurable m-partition of the set X and $\zeta = \{B_{ij} : i, j = 1, \cdots, m\}$ be a corresponding partition of the cartesian product $X \times X$, where $B_{ii} = A_i \times A_i$ and $B_{ij} = (A_i \times A_j) \cup (A_j \times A_i)$ for $i < j$. We select a family $\{\mu_{ij} : i, j = 1, \cdots, m\}$ of Gaussian distributions with mean m_{ij} and standard deviation σ_{ij} and define probability measure $P(x, y, A)$ as follows:

$$P(x, y, A) = \mu_{ij}(A) \text{ if } (x, y) \in B_{ij}, \quad i, j = 1, \cdots, m \tag{20}$$

for arbitrary $A \in \mathbb{F}$.

Definition

An operator W generated by this family of functions is called Gaussian quadratic stochastic operator.

This operator we can interpret as evolutionary operator of free population with continuum genotypes.

A transformation V is called a Lebesgue qso, if $X = [0, 1)$ and $\mathbb{F}$ is a Borel σ-algebra $\mathbb{B}$ on $[0, 1)$. Let $\xi = \{A_1, A_2, \cdots, A_m\}$ be a measurable m-partition of the set X and $\zeta = \{B_{ij} : i, j = 1, \cdots, m\}$ be a corresponding partition of the cartesian product $X \times X$, where $B_{ii} = A_i \times A_i$ and $B_{ij} = (A_i \times A_j) \cup (A_j \times A_i)$ for $i < j$. We select a family $\{\mu_{ij} : i, j = 1, \cdots, m\}$ of Lebesgue distributions and define probability measure $P(x, y, A)$ as follows:

$$P(x, y, A) = \mu_{ij}(A) \text{ if } (x, y) \in B_{ij}, \quad i, j = 1, \cdots, m \qquad (21)$$

for arbitrary $A \in \mathbb{F}$.

We define a discrete distributions $P(x, y, \cdot)$ as follows
(i) for $x < y$ assume $P(x, y, \{x\}) = p$ and $P(x, y, \{y\}) = q$,
(ii) for $x = y$ assume $P(x, x, \{x\}) = 1$,
(iii) for $x > y$ assume $P(y, x, \cdot) = P(x, y, \cdot)$.

We define a family $\{f(x,y,z) \ : x,y,z \in [0,1]\}$ of simple density functions as follows: for $x,y \in B_k$ with $k = 1,2,3$,

$$f(x,y,z) = \begin{cases} p_k & \text{if } (x,y) \in B_k \text{ and } z \in A_1 \\ q_k & \text{if } (x,y) \in B_k \text{ and } z \in A_2 \end{cases}$$

Consider a measure μ_{ij} as follows

$$\mu_{ij}(a, b) = (k_{ij} + 1) \int_a^b x^{k_{ij}} dx. \tag{22}$$

For any element $(x, y) \in B_i$ we define a continuous probability $P(x, y, \cdot)$ as follows

$$P(x, y, A) = \mu_{ij}(A) \text{ if } (x, y) \in B_{ij}, \quad i, j = 1, \cdots, m \tag{23}$$

for measurable set A.

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THANK YOU!