A quantum optomechanical system in a Mach-Zehnder interferometer

Alberto Barchielli

Dipartimento di Matematica, Politecnico di Milano
Istituto Nazionale di Fisica Nucleare, INFN
Istituto Nazionale d'Alta Matematica, INDAM-GNAMPA

November 30, 2020

Workshop on
New Trends in Mathematical Physics
November 9 – December 11, 2020, online
Steklov Mathematical Institute, Moscow

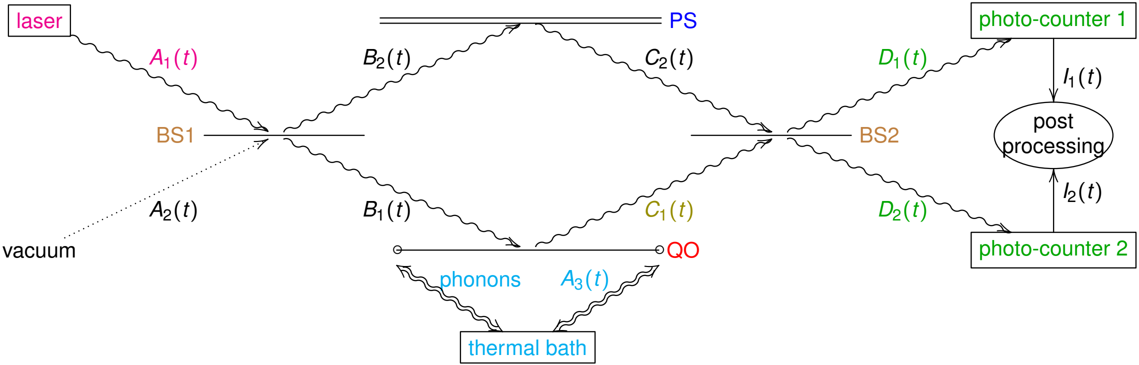
The aim: modelling of quantum optical devices — production of squeezed light

- The system: a quantum optomechanical system in an optical circuit
- Quantum optomechanics concerns the interaction of "quantum" oscillating micromirros with light, usually in a cavity.
  - Here: only travelling waves, no cavity.
  - The (ideal) optical circuit: the oscillating micromirror is inserted in a Mach-Zehnder interferometer (MZI).
  - The results: the input light is coherent, the output light is squeezed — typical quantum effects: sub-Poissonian statistics in direct detection, reduction of shot noise in spectra.
- The mathematical model is based on quantum stochastic calculus (QSC). QSC allows to describe
  - the interaction quantum micromirror/light, via Hudson-Parthasarathy equation,
  - the linear optical elements, via generalized Weyl operators,
  - the output light (the quantum fields of QSC in the Heisenberg picture) and its monitoring via photo-detectors and spectrum analyzers (direct and homodyne detection)
  - the quantum noise affecting the micromirror (even non-Markovian effects can be taken into account).

a Mach-Zehnder interferometer with a quantum subsystem inserted

Mach-Zehnder interferometer: 2 beam splitters (BS) + 2 mirrors

- QO: Quantum Oscillator (a quantum optomechanical micro-mirror)
- PS: fixed mirror and tunable Phase Shifter



- $A_1(t)$ – input: coherent light $C_1(t)$ – output: squeezed light
- Detection after interference at BS2: counting of photons or measurement of the spectrum of the "difference" current $I_-(t) = I_1(t) - I_2(t)\ldots$

The quantum fields and quantum stochastic calculus

- Symmetric Fock space: $\Gamma \equiv \Gamma\big(L^2(\mathbb{R};\mathbb{C}^d)\big) = \mathbb{C} \oplus \sum_{n=1}^{\infty} L^2(\mathbb{R};\mathbb{C}^d)^{\otimes_s n}$
- Coherent vectors, i.e. normalized exponential vectors,

$$e(f) = \mathrm{e}^{-\frac{1}{2}\|f\|^2}\Big(1, f, (2!)^{-1/2} f \otimes f, \ldots, (n!)^{-1/2} f^{\otimes n}, \ldots\Big) \qquad f \in L^2(\mathbb{R};\mathbb{C}^d)$$

- $A_j(t)$, $j = 1, \ldots, d$: quantum Bose fields in the Fock representation; heuristic definition:

$$A_j(t) = \int_0^t a_j(s)\mathrm{d}s \qquad [a_i(s), a_j(t)] = 0 \qquad [a_i(s), a_j^\dagger(t)] = \delta_{ij}\delta(t-s)$$

- Gauge process: $\Lambda_{ij}^A(t) = \int_0^t a_i^\dagger(s) a_j(s)\mathrm{d}s$ $\qquad$ $\Lambda_{jj}^A(t)$: Number operator in channel j
- Stochastic equations of Itô type. Itô table: (all the other possible products vanish)

$$\mathrm{d}A_k(t)\mathrm{d}A_l^\dagger(t) = \delta_{kl}\mathrm{d}t \qquad \mathrm{d}A_i(t)\mathrm{d}\Lambda_{kl}^A(t) = \delta_{ik}\mathrm{d}A_l(t)$$

$$\mathrm{d}\Lambda_{kl}^A(t)\mathrm{d}A_i^\dagger(t) = \delta_{li}\mathrm{d}A_k^\dagger(t) \qquad \mathrm{d}\Lambda_{kl}^A(t)\mathrm{d}\Lambda_{ij}^A(t) = \delta_{li}\mathrm{d}\Lambda_{kj}^A(t)$$

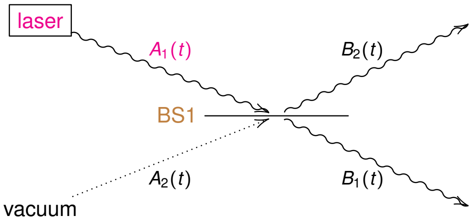
The rigorous definition of field and gauge operators is through their action on the exponential vectors $\qquad$ Our case: $d = 3$ (2 optical fields, 1 phonon field = noise)

Linear optical devices and Weyl operators

- Generalized Weyl operators: $\mathcal{W}(g;V)\in\mathcal{U}(\Gamma)\qquad g\in L^2(\mathbb{R};\mathbb{C}^d)\qquad V\in\mathcal{U}\big(L^2(\mathbb{R};\mathbb{C}^d)\big)$

$$\mathcal{W}(g;V)\,e(f)=\exp\left\{\mathrm{i}\,\mathrm{Im}\langle Vf|g\rangle\right\}e(Vf+g),\qquad \forall f\in L^2(\mathbb{R};\mathbb{C}^d).$$

- Composition rules $\mathcal{W}(h;U)\,\mathcal{W}(g;V)=\exp\left\{-\mathrm{i}\,\mathrm{Im}\langle h|Ug\rangle\right\}\mathcal{W}(h+Ug;UV)$
  In quantum optics $\mathcal{W}(g;\mathbb{1})$ is called a ***displacement operator***
- Linear optical devices: represented by $\mathcal{W}(0;V)$, $V\in\mathcal{U}(\mathbb{C}^d)$
  - $A_j(t)\quad\longmapsto\quad\mathcal{W}(0;V)^\dagger A_j(t)\mathcal{W}(0;V)=\sum_i V_{ji}A_i(t)$
  - Unitary transformation $\Rightarrow$ the CCRs are preserved.
  - A beam splitter of transmittance $\eta\in[0,1]$: $V\to V_\eta=\begin{pmatrix}\sqrt{\eta} & \mathrm{i}\sqrt{1-\eta}\\ \mathrm{i}\sqrt{1-\eta} & \sqrt{\eta}\end{pmatrix}$

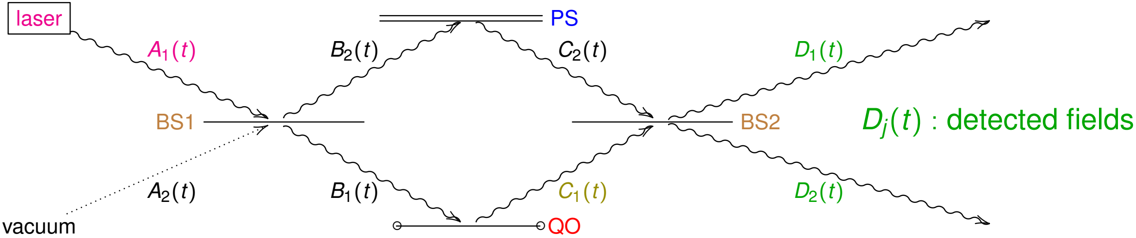


$$B_1(t)=\sqrt{\eta}\,A_1(t)+\mathrm{i}\sqrt{1-\eta}\,A_2(t)$$

$$B_2(t)=\mathrm{i}\sqrt{1-\eta}\,A_1(t)+\sqrt{\eta}\,A_2(t)$$

The Mach-Zehnder interferometer

- BS1: $B_1(t) = \sqrt{\eta}\, A_1(t) + \mathrm{i}\sqrt{1-\eta}\, A_2(t)$ $\quad B_2(t) = \mathrm{i}\sqrt{1-\eta}\, A_1(t) + \sqrt{\eta}\, A_2(t)$
- Tunable phase shift: $C_2(t) = \mathrm{e}^{\mathrm{i}\psi} B_2(t)$



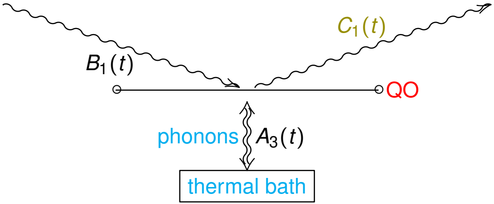
- Interaction light/oscillator: $C_1(t) = U(t)^\dagger B_1(t) U(t)$,
  $\mathrm{d}U(t) = \cdots$ (Hudson-Parthasarathy equation). Peculiar property:

$$U(T)^\dagger B_1(t) U(T) = U(t)^\dagger B_1(t) U(t), \qquad \forall 0 \le t \le T$$

$\Rightarrow$ also the output fields satisfy the CCRs.

- BS2 (transmittance $1/2$): $D_1(t) = \frac{1}{\sqrt{2}}\,[C_1(t) + \mathrm{i}C_2(t)]$ $\quad D_2(t) = \frac{1}{\sqrt{2}}\,[\mathrm{i}C_1(t) + C_2(t)]$

The Hudson-Parthasarathy equation for the mechanical oscillator



mechanical mode: $[a_\mathrm{m}, a_\mathrm{m}^\dagger] = 1$
position and momentum: $[q, p] = \mathrm{i}$

$$(q, p) \overset{?}{\leftrightarrow} (a_\mathrm{m}, a_\mathrm{m}^\dagger)$$

$$H_\mathrm{m} = H_\mathrm{m}(q, p) \text{ (quadratic)}$$

- The choice of $H_\mathrm{m}(q, p)$ and the connection between the mode operator a_m and the position and momentum operators q, p must give rise to the classical equations of motion for the mean values: $\langle p \rangle$ must be proportional to the mean velocity.
- Absorption/emission of phonons and scattering of photons:

$$\mathrm{d}U(t) = \Big\{ -\frac{\mathrm{i}}{\hbar} H_\mathrm{m} \mathrm{d}t + \sqrt{\gamma_\mathrm{m}} \left(a_\mathrm{m} \mathrm{d}A_3^\dagger(t) - a_\mathrm{m}^\dagger \mathrm{d}A_3(t) \right) \qquad \gamma_\mathrm{m} > 0 \qquad S = \mathrm{e}^{\mathrm{i}(vq+\phi)}$$

$$- \frac{\gamma_\mathrm{m}}{2} a_\mathrm{m}^\dagger a_\mathrm{m} \mathrm{d}t + (S - \mathbb{1}) \mathrm{d}\Lambda_{11}^B(t) \Big\} U(t) \qquad v \in \mathbb{R}, \quad \phi \in [0, 2\pi).$$

- $-\frac{\gamma_\mathrm{m}}{2} a_\mathrm{m}^\dagger a_\mathrm{m} \mathrm{d}t$ is an Itô correction.
- $U(t)$ is the unitary evolution in the interaction picture with respect to the free field dynamics. In the Schrödinger picture it becomes a strongly continuous unitary group.

Position/momentum $\leftrightarrow$ mode operator

We take:

- $H_{\rm m} = H_0 + H_1 \qquad H_0 = \frac{\hbar\Omega_{\rm m}}{2}\left(p^2 + q^2\right) \qquad H_1 = \frac{\hbar\gamma_{\rm m}}{4}\{q, p\}$
  the free mechanical Hamiltonian H_0 is modified by the interaction with the bath and H_1 is added
- $a_{\rm m} = \sqrt{\frac{\Omega_{\rm m}}{2\omega_{\rm m}}}(q + {\rm i}\tau p) \qquad \tau = \frac{\omega_{\rm m}}{\Omega_{\rm m}} - \frac{\rm i}{2}\frac{\gamma_{\rm m}}{\Omega_{\rm m}} \qquad \Omega_{\rm m}^2 = \omega_{\rm m}^2 + \frac{\gamma_{\rm m}^2}{4}$

  The mechanical mode operator $a_{\rm m}$ and q, p are connected in an unusual way: the extra-phase τ appears. $\quad [q, p] = {\rm i} \Leftrightarrow [a_{\rm m}, a_{\rm m}^\dagger] = 1$

Consequences:

(a) $H_{\rm m} = \hbar\omega_{\rm m}\left(a_{\rm m}^\dagger a_{\rm m} + \frac{1}{2}\right)$ ($a_{\rm m}$ diagonalizes $H_{\rm m}$)

(b) Consider the quantum Langevin equations for position and momentum, i.e. ${\rm d}q(t) = \cdots,\ {\rm d}p(t) = \cdots,$ where $q(t) = U(t)^\dagger q U(t),\ p(t) = U(t)^\dagger p U(t)$:
the damping force and the radiation pressure force appear only in the equation for the momentum, as in the classical case.

The quantum Langevin equations for position and momentum

- Quantum Langevin equations (Heisenberg equations of motion)

$$\mathrm{d}q(t) = \Omega_{\mathrm{m}} p(t)\mathrm{d}t + \mathrm{d}\hat{W}_q(t)$$
$$\mathrm{d}p(t) = -\big(\Omega_{\mathrm{m}} q(t) + \gamma_{\mathrm{m}} p(t)\big)\mathrm{d}t + v\mathrm{d}\Lambda_{11}^B(t) + \mathrm{d}\hat{W}_p(t)$$

- Damping force: $-\gamma_{\mathrm{m}} p(t)$
- Radiation pressure force: $v\mathrm{d}\Lambda_{11}^B(t)/\mathrm{d}t$
- Thermal noises:

$$\hat{W}_q(t) = -\sqrt{\frac{\gamma_{\mathrm{m}}\Omega_{\mathrm{m}}}{2\omega_{\mathrm{m}}}}\left(\mp A_3(t) + \tau A_3^\dagger(t)\right), \quad \hat{W}_p(t) = \mathrm{i}\sqrt{\frac{\gamma_{\mathrm{m}}\Omega_{\mathrm{m}}}{2\omega_{\mathrm{m}}}}\left(A_3(t) - A_3^\dagger(t)\right).$$

The means of the quantum noises are zero and the equations for the means of position and momentum turn out to be the classical ones, with
damping constant $\gamma_{\mathrm{m}} > 0$,
bare frequency $\Omega_{\mathrm{m}} > 0$,
effective frequency $\omega_{\mathrm{m}} = \sqrt{\Omega_{\mathrm{m}}^2 - \gamma_{\mathrm{m}}^2/4}$ (no overdamped case)

The state of the fields

- The field state: $\rho^T_{\text{field}} = \rho^T_{\text{em}} \otimes \rho^T_{\text{th}}$ $\rho^T_{\text{em}} = \mathbb{E}\left[|e(f_T)\rangle\langle e(f_T)|\right] \otimes |e(0)\rangle\langle e(0)|$
- $f_T(t) = f(t)\mathbb{1}_{(0,T)}(t)$ $f(t) = \lambda \mathrm{e}^{-\mathrm{i}\omega_0 t}$, $\lambda \in \mathbb{C}$, $\omega_0 > 0$,
  a coherent monochromatic laser & vacuum for the optical fields
- For the thermal field: A field analog of the P-representation of quantum optics
  - Let u be a stationary Gaussian complex random process with
    $\mathbb{E}[u(t)] = 0$ $\mathbb{E}[u(t)\,u(s)] = 0$ $\mathbb{E}[\overline{u(t)}\,u(s)] = F(t-s)$

$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\nu t} N(\nu)\,\mathrm{d}\nu \qquad N(\nu) \geq 0 \quad N(\nu) \in L^1(\mathbb{R})$$

  - Take the state of the thermal field to be the mixture of coherent states (for $A_3(t)$)
    $\rho^T_{\text{th}} = \mathbb{E}\left[|e_{\text{th}}(u_T)\rangle\langle e_{\text{th}}(u_T)|\right]$, $u_T(t) := 1_{[0,T]}(t)u(t)$
  - the current time t is always smaller than T, but in the final physical formulae $T \to +\infty$
- First consequences:
  - the reduced state for the mechanical oscillator does not satisfy a simple closed master equation
  - the quantity $N(\nu)$ will play the role of noise spectral density

Equilibrium state of the quantum oscillator

$$\mathrm{d}q(t) = \Omega_\mathrm{m} p(t)\mathrm{d}t + \mathrm{d}\hat{W}_q(t) \qquad \mathrm{d}p(t) = -\big(\Omega_\mathrm{m} q(t) + \gamma_\mathrm{m} p(t)\big)\mathrm{d}t + v\mathrm{d}\Lambda_{11}^B(t) + \mathrm{d}\hat{W}_p(t)$$

Non homogeneous, linear equations $\Rightarrow$ ***explicit solution for*** $q(t)$, $p(t)$

Explicit solution + quantum correlations of the fields $\Rightarrow$ in principle all the properties of the mechanical oscillator can be computed (without relying on a master equation).

In particular, the reduced equilibrium state of the quantum oscillator is

$$\sigma_\mathrm{eq} = \lim_{t\to+\infty} \lim_{T\to+\infty} \mathrm{Tr}_\Gamma \left\{ U(t) \left(\sigma_0 \otimes \rho_\mathrm{field}^T \right) U(t)^\dagger \right\} \qquad (\mathrm{Tr}_\Gamma : \text{partial trace over the fields})$$

and it turns out to be a Gaussian state with

$$\langle p \rangle_\mathrm{eq} = 0, \qquad \langle q \rangle_\mathrm{eq} = \frac{v\eta\,|\lambda|^2}{\Omega_\mathrm{m}},$$

$$\langle q^2 \rangle_\mathrm{eq} - \langle q \rangle_\mathrm{eq}^2 = \langle p^2 \rangle_\mathrm{eq} = \frac{\Omega_\mathrm{m}}{\omega_\mathrm{m}} \left(N_\mathrm{eff} + \frac{1}{2} \right) + \frac{\eta\,|\lambda|^2\, v^2}{2\gamma_\mathrm{m}} \qquad \text{equipartition in the mean}$$

$$\langle \{q, p\} \rangle_\mathrm{eq} = -\frac{\gamma_\mathrm{m}}{\omega_\mathrm{m}} \left(N_\mathrm{eff} + \frac{1}{2} \right), \qquad N_\mathrm{eff} := \frac{\gamma_\mathrm{m}}{2\pi} \int_\mathbb{R} \frac{N(\nu)}{\frac{\gamma_\mathrm{m}^2}{4} + (\omega_\mathrm{m} - \nu)^2} \mathrm{d}\nu$$

The mean of the Hamiltonian turns out to be $\langle H_\mathrm{m} \rangle_\mathrm{eq} = \hbar\omega_\mathrm{m} \left(N_\mathrm{eff} + \frac{1}{2} \right)$

The output field

By the explicit solutions of the quantum Langevin equations:

$$C_1(t) = U(t)^\dagger B_1(t) U(t) \quad \Rightarrow \quad \mathrm{d}C_1(t) = \mathrm{e}^{\mathrm{i}vq(t)+\mathrm{i}\phi}\mathrm{d}B_1(t),$$

Also: the number operator commutes with $U(t)$, $\quad \Lambda^C_{11}(t) = U(t)^\dagger \Lambda^B_{11}(t) U(t) = \Lambda^B_{11}(t)$

$$\mathrm{e}^{\mathrm{i}vq(t)} = S_0(q,p,t)\mathcal{W}_{\mathrm{th}}(\ell_t; \mathbb{1})\mathcal{W}_{\mathrm{em}}(0; V_t), \qquad S_0(q,p,t) \overset{t\to+\infty}{\longrightarrow} \mathbb{1},$$

$$\mathcal{W}_{\mathrm{th}}(\ell_t; \mathbb{1}) = \exp\left\{\int_0^t \ell_t(s)\mathrm{d}A_3^\dagger(s) - \text{h.c.}\right\}, \qquad \mathcal{W}_{\mathrm{em}}(0; V_t) = \exp\left\{\int_0^t V_t(s)\,\mathrm{d}\Lambda^B_{11}(s)\right\},$$

$$\ell_t(s) = -\mathrm{i}v\tau\sqrt{\frac{\Omega_\mathrm{m}\gamma_\mathrm{m}}{2\omega_\mathrm{m}}}\,\mathrm{e}^{\left(\mathrm{i}\omega_\mathrm{m}-\frac{\gamma_\mathrm{m}}{2}\right)(t-s)}, \qquad V_t(s) = \exp\left\{\mathrm{i}\frac{\Omega_\mathrm{m}v^2}{\omega_\mathrm{m}}\mathrm{e}^{-\frac{\gamma_\mathrm{m}}{2}(t-s)}\sin\omega_\mathrm{m}(t-s)\right\}.$$

- $\frac{\mathrm{d}\Lambda^B_{11}(s)}{\mathrm{d}s}$ is the rate of arrival of photons —
- $\mathcal{W}_{\mathrm{em}}(0; V_t)$, which appears in the transformation $b_1(t) \to c_1(t) = \mathrm{e}^{\mathrm{i}vq(t)+\mathrm{i}\phi}b_1(t)$, introduces an intensity dependent phase shift
  — a typical situation known in quantum optics to produce squeezed light

Direct detection of the output fields

A general property of output fields: $C_1(t) = U(t)^\dagger B_1(t) U(t) = U(T)^\dagger B_1(t) U(T)$ for $t \leq T$
Moreover, by construction $U(t)$ and $C_2(s)$ commute

$\Rightarrow$ $C_1(\bullet)$ and $C_2(\bullet)$ satisfy the CCRs as the free Bose fields

$\Rightarrow$ $D_1(\bullet)$ and $D_2(\bullet)$ satisfy the CCRs as the free Bose fields

$\Rightarrow$ the number operators $\left\{\Lambda^D_{11}(t), \Lambda^D_{22}(s)\right\}_{t,s\in[0,T]}$ form a family of commuting self-adjoint operators $\Rightarrow$ The associated observables, $N_1(\bullet)$, $N_2(\bullet)$, form a couple of counting processes, whose probability law P is given by the "usual" rules of quantum mechanics (from the joint projection valued measure and the system state).

Some notations:

- $\mathbb{E}_P[\bullet]$, expectation of a random variable with respect to the probability P.
- $\langle \bullet \rangle_T = \operatorname{Tr}\left\{\bullet \rho_{\rm osc} \otimes \rho^T_{\rm field}\right\}$, quantum expectation of an operator with respect to the initial state of oscillator and fields.
- The laser state is the coherent state $e(f_T)$ with $f_T(t) = \lambda \mathrm{e}^{\mathrm{i}\omega_0 t} 1\!\!1_{(0,T)}(t)$; $|\lambda|^2$ is the intensity of the laser; the final time T is the largest one.
- By using the field densities we write $\Lambda^D_{jj}(t) = \int_0^t d_j^\dagger(s) d_j(s) \mathrm{d}s$

Example: $\mathbb{E}_P[N_j(t)] = \int_0^t \mathrm{d}s \, \langle d_j^\dagger(s) d_j(s) \rangle_T$

Mean of the counts

For large t: $\mathbb{E}_P[N_j(t+\Delta t) - N_j(t)] \simeq n_j \Delta t$, $\qquad n_j = \lim_{t\to+\infty} \lim_{T\to+\infty} \langle d_j^\dagger(t) d_j(t)\rangle_T$

By using the HP-equation and the various transformations of the fields it is possible to compute n_j:

$$n_j = \frac{|\lambda|^2}{2}\left[1 + (-1)^j \chi \mathrm{e}^{-(K+M)} \cos(\psi - \phi - \theta)\right], \qquad \chi := 2\sqrt{\eta(1-\eta)} \in [0,1]$$

ψ is the tunable phase shift; η is the transmittance of the first beam splitter

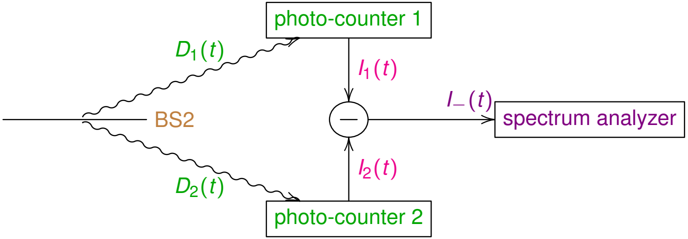
- The constants $K > 0$, $M > 0$ and θ depend on the oscillator dynamics $(\omega_\mathrm{m}, \gamma_\mathrm{m})$ and on the intensity of the optomechanical interaction (v^2); moreover, K depends also on the temperature, while M and θ depend on the laser intensity $|\lambda|^2$.
- Case of no interaction, $v = 0$, and balanced beam splitter, $\eta = 1/2$: $\Rightarrow \chi = 1$, $K = M = 0$, $\theta = 0$. Then: (a) $\psi = \phi \Rightarrow n_1 = 0$, i.e. no light from port 1; (b) $\psi = \phi + \pi \Rightarrow n_2 = 0$, i.e. no light from port 2. This is a classical result for a MZI.
- For $v \neq 0$, there is always some light at the two output ports due to the factor $\mathrm{e}^{-(K+M)}$.

Mandel Q-parameter

$\mathrm{Var}[N_j(T)] = \langle \Lambda_{jj}^D(T)^2 \rangle_T - (\langle \Lambda_{jj}^D(T) \rangle_T)^2$ an long analytical expression can be obtained

- $\lim_{T \to +\infty} \frac{\mathrm{Var}[N_j(T)]}{T} = n_j \left(1 + Q_j\right)$ Q_j: (asymptotic) Mandel parameter
- Poisson distribution $\Rightarrow Q_j = 0$
- Squeezed light $\Rightarrow$ sub-Poissonian statistics: $-1 \leq Q_j < 0$
- We find $Q_j < 0$ for certain choices of the parameters
- moreover, if $Q_j\big|_{\psi=\psi^*} < 0$, then, $Q_j\big|_{\psi=\psi^*+\pi/2} > 0$, as it must be for squeezed light: "when a quadrature is squeezed, the orthogonal quadrature is anti-squeezed"
- For any choice of the parameters:
  $\mathrm{Var}[N_1(T) + N_2(T)] = \mathbb{E}_P[N_1(T)] + \mathbb{E}_P[N_2(T)] = |\lambda|^2 T.$
  To recombine in this way the two rays gives the same result as to count the photons in the initial coherent laser field.

Post-processing



$$I_j(t) = c\varkappa \int_0^t \mathrm{e}^{-\varkappa(t-r)}\,\mathrm{d}N_j(r),$$
$j = 1, 2 \qquad 0 < t \leq T$

$c\varkappa \mathrm{e}^{-\varkappa(t-r)}$ represents the ***response function*** of the detector

- Quantum observables: $\hat{I}_j(t) = c\varkappa \int_0^t \mathrm{e}^{-\varkappa(t-r)}\,\mathrm{d}\Lambda_{jj}(r)$

  $\Rightarrow \quad [\hat{I}_j(t), \hat{I}_i(s)] = 0 \quad$ (from the commutation property of the number operators)
- The "difference" current $\hat{I}_-(t) = \hat{I}_1(t) - \hat{I}_2(t) \qquad \Rightarrow \qquad [\hat{I}_-(t), \hat{I}_-(s)] = 0$
- $I_-(t) = I_1(t) - I_2(t)$ is a stochastic process; its law P can be computed, in principle.
- The intensity spectrum: $S_{I_-}(\mu) = \lim_{T\to+\infty} \frac{1}{T}\,\mathbb{E}_P\left[\left|\int_0^T \mathrm{e}^{\mathrm{i}\mu t} I_-(t)\mathrm{d}t\right|^2\right]$

  This is the usual definition of spectrum of an asymptotically stationary stochastic process.

The idea of studying the difference current comes from ***balanced homodyne detection***, which has analogies with our MZI scheme

The structure of the spectrum

In terms of the quantum observables $\mathrm{d}\Lambda^D_{jj}(t) = d_j^\dagger(t) d_j(t) \mathrm{d}t$, by a few steps, we get

$$S_{I_-}(\mu) = \lim_{T\to+\infty} \frac{c^2\varkappa^2}{(\mu^2+\varkappa^2)T} \sum_{i,j=1}^{2} (-1)^{i+j} \int_0^T \mathrm{d}t \int_0^T \mathrm{d}s\, \mathrm{e}^{\mathrm{i}\mu(t-s)} \langle d_j^\dagger(t) d_j(t) d_i^\dagger(s) d_i(s) \rangle_T$$

CCRs for the D-fields: $\quad [d_i(s), d_j(t)] = 0, \quad [d_i(s), d_j^\dagger(t)] = \delta_{ij}\delta(t-s)$

$$\langle d_j^\dagger(t) d_j(t) d_i^\dagger(s) d_i(s) \rangle_T = \langle d_j^\dagger(t) d_j(t) \rangle_T \langle d_i^\dagger(s) d_i(s) \rangle_T + \delta_{ij}\delta(t-s) \langle d_j(t)^\dagger d_j(t) \rangle_T + \Big(\langle d_j^\dagger(t) d_i^\dagger(s) d_i(s) d_j(t) \rangle_T - \langle d_j^\dagger(t) d_j(t) \rangle_T \langle d_i^\dagger(s) d_i(s) \rangle_T \Big)$$

$$S_{I_-}(\mu) = 2\pi c^2 (n_1 - n_2)^2 \delta(\mu) + \frac{c^2\varkappa^2}{\mu^2+\varkappa^2} \Big[n_1 + n_2 + |\lambda|^2 (1-\eta) \Sigma_-(\mu) \Big]$$

The first term is the contribution of the constant part of I_-;

$$n_2 - n_1 = |\lambda|^2 \chi \mathrm{e}^{-(K+M)} \cos(\psi - \phi - \theta)$$

Fourier transform of the detector response function: $\frac{c^2\varkappa^2}{\mu^2+\varkappa^2}$

Shot noise: $n_1 + n_2 = |\lambda|^2$ it comes out from normal ordering the d's.

The reduced spectrum

Reduced spectrum: $\Sigma_-(\mu) = \lim_{T\to+\infty} \sum_{i,j=1}^{2} (-1)^{i+j} \frac{1}{|\lambda|^2 (1-\eta)\, T} \int_0^T \mathrm{d}t \int_0^T \mathrm{d}s\, \mathrm{e}^{\mathrm{i}\mu(t-s)}$
$\times \left(\langle d_j^\dagger(t) d_i^\dagger(s) d_i(s) d_j(t) \rangle_T - \langle d_j^\dagger(t) d_j(t)\rangle_T \langle d_i^\dagger(s) d_i(s)\rangle_T \right)$

We can express $\Sigma_-(\mu)$ in terms of the output field $c_1(t)$, and of $a_1(t)$, $a_2(t)$. We use the fact that the initial state is a coherent state for the a_j-fields: $e(f_T) \otimes e(0)$. We obtain:

- $1 + \Sigma_-(\mu) = \lim_{T\to+\infty} \langle \Delta Q_T(\mu;\psi)^\dagger \Delta Q_T(\mu;\psi)\rangle_T \geq 0, \qquad \Rightarrow \qquad \Sigma_-(\mu) \geq -1$

$$\Delta Q_T(\mu;\psi) := Q_T(\mu;\psi) - \langle Q_T(\mu;\psi)\rangle_T, \qquad Q_T(\mu;\psi) := \mathrm{e}^{\mathrm{i}\psi} c_T(\mu) + \mathrm{e}^{-\mathrm{i}\psi} c_T^\dagger(-\mu),$$

$$c_T(\mu) := \frac{1}{|\lambda| \sqrt{T}} \int_0^T \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\mu t}\, \overline{f(t)}\, c_1(t) \qquad \Rightarrow \qquad [c_T(\mu), c_T^\dagger(\mu)] = 1$$

- $c_T(\mu)$ is a "mode operator".
- A Heisenberg-like relation holds for the "quadrature" operators $Q_T(\mu;\psi)$:

$$\langle \Delta Q_T(\mu;\psi)^\dagger \Delta Q_T(\mu;\psi)\rangle_T \,\langle \Delta Q_T(\mu;\psi \pm \pi/2)^\dagger \Delta Q_T(\mu;\psi \pm \pi/2)\rangle_T \geq 1.$$

$$\Rightarrow \qquad \left(1 + \Sigma_-(\mu)\big|_\psi\right)\left(1 + \Sigma_-(\mu)\big|_{\psi+\pi/2}\right) \geq 1$$

Squeezing

Consider $\mu = 0$. $\quad c_T(0) = \dfrac{1}{|\lambda|\sqrt{T}} \displaystyle\int_0^T \overline{f(t)}\, c_1(t)\,\mathrm{d}t \qquad [c_T(0), c_T^\dagger(0)] = 1$

$Q_T(0;\psi) = \mathrm{e}^{\mathrm{i}\psi} c_T(0) + \mathrm{e}^{-\mathrm{i}\psi} c_T^\dagger(0) = Q_T(0;\psi)^\dagger \qquad \langle \Delta Q_T(0;\psi)^2\rangle_T \, \langle \Delta Q_T(0;\psi \pm \pi/2)^2\rangle_T \geq 1$

$$1 + \Sigma_-(0) = \langle \Delta Q_T(0;\psi)^2\rangle_T$$

On a coherent vector for $c_T(0)$ we have $\langle \Delta Q_T(0;\psi)^2\rangle_T = 1$, $\forall\psi$.
When $\langle \Delta Q_T(0;\psi)^2\rangle_T < 1$ for a certain ψ (and, so, $\langle \Delta Q_T(0;\psi \pm \pi/2)^2\rangle_T > 1$) one says that the reduced state of the mode $c_T(0)$ is ***squeezed***.

- When $\Sigma_-(0) \in (-1, 0)$, the light in the channel $C_1(t)$ is squeezed.

Use $c_1(t) = \mathrm{e}^{\mathrm{i}vq(t)+\mathrm{i}\phi} b_1(t)$ and the decomposition of the scattering operator in terms of Weyl operators: $\quad \mathrm{e}^{\mathrm{i}vq(t)} = S_0(q,p,t)\mathcal{W}_{\mathrm{th}}(\ell_t;\mathbb{1})\mathcal{W}_{\mathrm{em}}(0;V_t), \qquad S_0(q,p,t) \xrightarrow{t\to+\infty} \mathbb{1}$

$\Rightarrow \quad \Sigma_-(\mu) =$ an involved analytical expression

Approximations are needed to get explicit expressions for $\Sigma_-(\mu)$

Some conditions for squeezing

Assumptions: small temperature, strong laser intensity, $|\lambda|^2 \uparrow +\infty$, weak interaction, $v^2 \downarrow 0$ – (ideally the parameter v^2 can be changed by changing the incidence angle in the MZI); precisely, we ask

$$\left(N_{\text{eff}} + \frac{1}{2}\right) \frac{v^2 \Omega_{\mathrm{m}}}{2\omega_{\mathrm{m}}} \ll 1, \qquad \frac{\eta |\lambda|^2 v^4}{4\gamma_{\mathrm{m}}} \ll 1, \qquad \frac{\Omega_{\mathrm{m}}}{\eta |\lambda|^2 v^2} \ll 1$$

Take $\psi = \psi_1$ such that

$$\sin 2(\psi_1 - \phi - \theta) \simeq -\frac{\Omega_{\mathrm{m}}}{\eta |\lambda|^2 v^2} \ll 1, \qquad 1 - \cos 2(\psi_1 - \phi - \theta) \simeq \frac{1}{2}\left(\frac{\Omega_{\mathrm{m}}}{\eta |\lambda|^2 v^2}\right)^2 \ll 1$$

$$\Sigma_-(\mu)\big|_{\psi_1} \simeq \frac{\Omega_{\mathrm{m}}^2 \left(2\mu^2 - \Omega_{\mathrm{m}}^2\right)}{\left[\frac{\gamma_{\mathrm{m}}^2}{4} + (\mu + \omega_{\mathrm{m}})^2\right]\left[\frac{\gamma_{\mathrm{m}}^2}{4} + (\mu - \omega_{\mathrm{m}})^2\right]}$$

$$\Sigma_-(0)\big|_{\psi_1} \simeq -1 \qquad \Sigma_-(\mu)\big|_{\psi_1} < 0 \text{ for } \mu \in (-\Omega_{\mathrm{m}}/\sqrt{2}, \Omega_{\mathrm{m}}/\sqrt{2}) \qquad \lim_{\mu \to \pm\infty} \Sigma_-(\mu) = 0$$

$$S_{I_-}(\mu) = 2\pi c^2 (n_1 - n_2)^2 \delta(\mu) + \frac{c^2 \varkappa^2 |\lambda|^2}{\mu^2 + \varkappa^2}\left[1 + (1-\eta)\Sigma_-(\mu)\right]$$

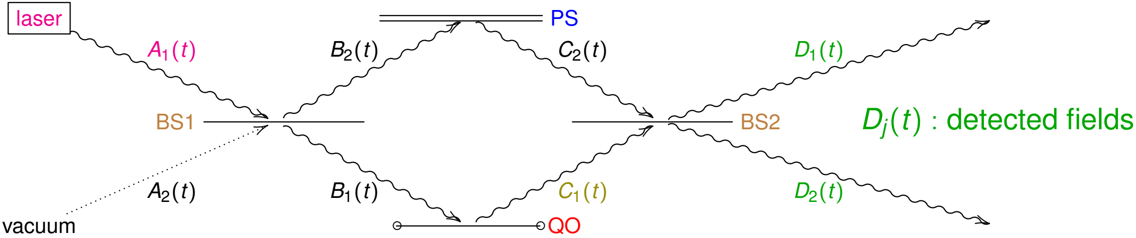
η can be small, so $\Sigma_-(0)\big|_{\psi_1}$ can nearly cancel the whole shot noise.

Back to the Mandel parameters

- The spectra of the two output currents: $S_{I_j}(\mu) = \lim_{T\to+\infty} \frac{1}{T}\,\mathbb{E}_P\left[\left|\int_0^T \mathrm{e}^{\mathrm{i}\mu t} I_j(t)\mathrm{d}t\right|^2\right] =$
$$= 2\pi c^2 n_j^2 \delta(\mu) + \frac{c^2\varkappa^2}{\mu^2+\varkappa^2}\left[n_j + \frac{|\lambda|^2}{4}\Big((1-\eta)\,\Sigma_-(\mu) + (-1)^j\Sigma_0(\mu)\Big)\right], \qquad j=1,2,$$
$\Sigma_0(\mu) = \cdots$; when $\Sigma_-(0) < 0$, at least one of the to light beams presents reduction of the shot noise.
- The two Mandel Q-parameters for the counting of photons:
$$Q_j = \lim_{T\to+\infty} \frac{\mathrm{Var}[N_j(T)] - \mathbb{E}_P[N_j(T)]}{\mathbb{E}_P[N_j(T)]} = \frac{|\lambda|^2}{4n_j}\left[(-1)^j\Sigma_0(0) + (1-\eta)\,\Sigma_-(0)\right],$$
When $\Sigma_-(0)$ at least in one channel we get $Q_j < 0$
- However, under the conditions which give $(1-\eta)\,\Sigma_-(0) \simeq -1$, we get $\Sigma_0(0) \simeq 0$, $n_j \simeq \frac{|\lambda|^2}{2}$, and, so $Q_1 \simeq -\frac{1}{2}$, $Q_2 \simeq -\frac{1}{2}$; we see squeezing in both rays, but the extreme value -1 is not reached.

Conclusion

BS1, BS2: two beam splitters
QO: Quantum Oscillator (a quantum optomechanical micro-mirror)
PS: fixed mirror and tunable Phase Shifter



The input light, in field $A_1(t)$, is coherent, "classical" light.
The output light, in field $C_1(t)$ is squeezed,"quantum" light (under some choices of the free parameters).
Squeezing is detected only after the interference with the reference beam $C_2(t)$.

References

- A. Barchielli, *Quantum stochastic equations for an opto-mechanical oscillator with radiation pressure interaction and non-Markovian effects*, Rep. Math. Phys. **77** (2016) 315–333.
- A. Barchielli, B. Vacchini, *Quantum Langevin equations for optomechanical systems*, New J. Phys. **17** (2015) 083004.
- A. Santamato, *A quantum theory of photodetection and other optical devices*, master thesis, University of Milan (2010). DOI 10.13140/RG.2.2.36655.48801
- A. Barchielli, M. Gregoratti, *Quantum continuous measurements: The stochastic Schrödinger equations and the spectrum of the output*, Quantum Measurements and Quantum Metrology, **1** (2013) 34–56.
- A. Barchielli, *Continual Measurements in Quantum Mechanics and Quantum Stochastic Calculus*. In S. Attal, A. Joye, C.-A. Pillet (eds.), *Open Quantum Systems III*, Lect. Notes Math. **1882** (Springer, Berlin, 2006), pp. 207–291.

- W.P. Bowen, G.J. Milburn, *Quantum Optomechanics* (CRC, Taylor & Francis, 2016)
- R.L. Hudson and K.R. Parthasarathy, *Quantum Itô's formula and stochastic evolutions*, Commun. Math. Phys. **93**, 301–323 (1984).
- C.W. Gardiner and M.J. Collet, *Input and output in damped quantum systems: Quantum stochastic differential equations and the master equation*, Phys. Rev. A **31**, 3761–3774 (1985).
- J.E. Gough, *Scattering processes in quantum optics*, Phys. Rev. A **91** (2015) 013802.
- A. Barchielli, M. Gregoratti, *Quantum measurements in continuous time, non-Markovian evolutions and feedback*, Phil. Trans. R. Soc. A **370** no. 1979 (2012) 5364–5385.
- M. Gregoratti, *The Hamiltonian operator associated to some quantum stochastic evolutions*, Commun. Math. Phys. **222** (2001) 181–200.