Trigonometric real form of the spin RS model of Krichever and Zabrodin

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The model introduced by Krichever and Zabrodin in 1995 deals with the dynamics of 'particle positions' x_i $(i = 1, \dots, n)$ and d-component, complex row vectors c_i, and column vectors a_i. The 'individual spins' enter the 'composite spin variables' $F_{ij} := c_i \cdot a_j := \sum_{\alpha=1}^{d} c_i^\alpha a_j^\alpha$, and the equations of motion read

$$\dot{x}_i = F_{ii}, \quad \dot{a}_i^\alpha = \lambda_i a_i^\alpha + \sum_{k \neq i} V(x_{ik}) a_k^\alpha F_{ki}, \quad \dot{c}_j^\alpha = -\lambda_j c_j^\alpha - \sum_{k \neq j} V(x_{kj}) c_k^\alpha F_{jk}$$

where $x_{ik} := x_i - x_k$. In the elliptic case the 'potential' is $V(x) = \zeta(x) - \zeta(x + \gamma)$ with the Weierstrass zeta-function and an arbitrary 'coupling constant' $\gamma \neq 0$. These imply the second order equations

$$\ddot{x}_i = \sum_{j \neq i} F_{ij} F_{ji} \left[V\left(x_{ij}\right) - V\left(x_{ji}\right) \right].$$

The parameters λ_i are arbitrary, and the 'physical observables' are invariant with respect to arbitrary rescalings $a_i \mapsto \Lambda_i^{-1} a_i$, $c_i \mapsto \Lambda_i c_i$.

Krichever and Zabrodin derived these equations from the dynamics of the poles of the elliptic solutions of the 2D non-Abelian Toda lattice, and asked about their Hamiltonian interpretation and integrability.

In the rational case, $V^{\mathrm{rat}}(x) = x^{-1} - (x+\gamma)^{-1}$, the answers were provided by Arutyunov and Frolov (1997), who re-derived the model via Hamiltonian reduction of a spin extension of the cotangent bundle of $\mathrm{GL}(n,\mathbb{C})$, $T^*\mathrm{GL}(n,\mathbb{C}) \times \mathbb{C}^{2nd}$. About twenty years later, the trigonometric/hyperbolic case was treated, first by Chalykh and Fairon and then by Arutyunov and Olivucci, applying quasi-Hamiltonian reduction and Hamiltonian reduction techniques, respectively. (The two methods led to different Hamiltonian structures for the model.)

Krichever (1998) proved the existence of Hamiltonian structure in the general case.

The pioneering papers on Calogero and Ruijsenaars type system were devoted to point particles moving along the **real** line or circle. However, all the above mentioned works deal with complex holomorphic systems. The real forms require separate attention, which poses open problems.

In this talk, we inquire about the trigonometric real form defined by taking $V(x) := \cot(x) - \cot(x - \mathrm{i}\gamma)$ with a real, positive γ, and setting $x_j := \frac{1}{2}q_j$ where the q_j are real and are regarded as angles, and also setting $c_j^\alpha = (a_j^\alpha)^* =: v(\alpha)_j$, for $j = 1, \ldots, n$ and $\alpha = 1, \ldots, d$.

If $d = 1$, then $F_{ij}F_{ji} = |F_{ij}|^2 = F_{ii}F_{jj}$ and there are no true internal degrees of freedom. In this case the gauge invariant content of the model is governed by the chiral Ruijsenaars–Schneider (RS) Hamiltonian

$$\mathcal{H}^+_{\mathrm{RS}} = \sum_i e^{2\theta_i} \prod_{j \neq i} \left[1 + \frac{\sinh^2 \gamma}{1 + \sin^2 \frac{q_i - q_j}{2}} \right]^{\frac{1}{2}}$$

via the change of variables $F_{jj} = e^{2\theta_j} \prod_{i \neq j} \left[1 + \frac{\sinh^2 \gamma}{1 + \sin^2 \frac{q_i - q_j}{2}} \right]^{\frac{1}{2}}$.

The real trigonometric RS model was derived by L.F. and Klimčík (2009) by Hamiltonian reduction of the Heisenberg double of the Poisson–Lie group $\mathrm{U}(n)$, which served as our starting point for the present work.

The rest of the talk

- The ‘free’ system to be reduced and the definition of the reduction
- A model of the reduced phase space and its symplectic form
- The reduced equations of motion and degenerate integrability
- Conclusion

We shall apply symplectic reduction to an 'obviously integrable' Hamiltonian system on the *real* symplectic manifold

$$\mathcal{M} = \mathrm{GL}(n, \mathbb{C}) \times \mathbb{C}^{n \times d}.$$

Consider the decomposition $\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{u}(n) + \mathfrak{b}(n)$, where $\mathfrak{b}(n)$ denotes the Lie algebra of upper triangular complex matrices with real diagonal. Let $\mathrm{U}(n)$ and $\mathrm{B}(n)$ denote the corresponding subgroups of $\mathrm{GL}(n, \mathbb{C})$. Using the r-matrix on $\mathfrak{gl}(n, \mathbb{C})$ given by $R := \frac{1}{2}\left(P_{\mathfrak{u}(n)} - P_{\mathfrak{b}(n)}\right)$, define two Poisson structures on $C^\infty(\mathrm{GL}(n, \mathbb{C}), \mathbb{R})$:

$$\{f, h\}_\pm := \langle \nabla f, R \nabla h \rangle \pm \langle \nabla' f, R \nabla' h \rangle,$$

where $\langle X, Y \rangle := \Im \mathrm{tr}(XY)$, $\forall X, Y \in \mathfrak{gl}(n, \mathbb{C})$, $\langle \nabla f(K), X \rangle := \frac{d}{dt}\Big|_{t=0} f(e^{tX} K)$ $\forall K \in \mathrm{GL}(n, \mathbb{C})$, and similar for the right-derivative $\nabla' f$.

The minus bracket makes $\mathrm{GL}(n, \mathbb{C})$ into a real Poisson–Lie group, while the plus one gives a symplectic structure. The former is called the Drinfeld double Poisson bracket and the latter the Heisenberg double Poisson bracket. $\mathrm{U}(n)$ and $\mathrm{B}(n)$ are Poisson submanifolds w.r.t. the minus bracket, and thus become Poisson–Lie groups, equipped with the inherited Poisson structures denoted $\{\ ,\ \}_U$ and $\{\ ,\ \}_B$.

The Heisenberg double goes back to Semenov-Tian-Shansky (1985). Its symplectic form, Ω_{GL}, was found by Alekseev and Malkin (1994). For any element $K \in \mathrm{GL}(n,\mathbb{C})$, use the Iwasawa decompositions

$$K = b_L g_R^{-1} = g_L b_R^{-1} \quad \text{with} \quad b_L, b_R \in \mathrm{B}(n),\ g_L, g_R \in \mathrm{U}(n),$$

and define the maps Λ_L, Λ_R into $\mathrm{B}(n)$ and Ξ_L, Ξ_R into $\mathrm{U}(n)$ by

$$\Lambda_L(K) := b_L, \quad \Lambda_R(K) := b_R, \quad \Xi_L(K) := g_L, \quad \Xi_R(K) := g_R.$$

Then $\displaystyle \Omega_{GL} = \frac{1}{2}\Im \mathrm{tr}(d\Lambda_L \Lambda_L^{-1} \wedge d\Xi_L \Xi_L^{-1}) + \frac{1}{2}\Im \mathrm{tr}(d\Lambda_R \Lambda_R^{-1} \wedge d\Xi_R \Xi_R^{-1}).$

To continue the preparations, equip $\mathbb{C}^n = \mathbb{R}^{2n}$ with the symplectic form

$$\Omega_{\mathbb{C}^n} = \frac{\mathrm{i}}{2}\sum_{k=1}^{n} \frac{1}{\mathcal{G}_k} dw_k \wedge d\overline{w}_k + \frac{\mathrm{i}}{4}\sum_{k=1}^{n-1} \frac{1}{\mathcal{G}_k \mathcal{G}_{k+1}} d\mathcal{G}_{k+1} \wedge (\overline{w}_k dw_k - w_k d\overline{w}_k)$$

where $\mathcal{G}_j = 1 + \sum_{k=j}^n |w_k|^2$ $(j = 1, \dots, n)$ and $\mathcal{G}_{n+1} := 1$. This yields a $\mathrm{U}(n)$ covariant Poisson structure on $\mathbb{C}^n$, due to Zakrzewski (1996). We then take $d > 1$ independent, $\mathbb{C}^n$-valued variables, $w^1, \dots, w^d$, called primary spins, which give $W := (w^1, \dots, w^d) \in \mathbb{C}^{n\times d}$. The so obtained symplectic form is denoted $\Omega_{\mathbb{C}^{n\times d}}$.

The phase space to be reduced is $\mathcal{M} := \mathrm{GL}(n,\mathbb{C}) \times \mathbb{C}^{n\times d}$ endowed with the symplectic form $\Omega_{\mathcal{M}} = \Omega_{GL} + \Omega_{\mathbb{C}^{n\times d}}$. $\mathcal{M}$ carries the Abelian Poisson algebra $\mathfrak{H}$ generated by the 'free' Hamiltonians H_k:

$$H_k(K,W) := \frac{1}{2k}\mathrm{tr}(L^k) \quad \text{with} \quad L := b_R b_R^\dagger = (K^\dagger K)^{-1}, \quad k = 1,\ldots,n.$$

Along the Hamiltonian flow of H_k, we have $g_R(t) = \exp(\mathrm{i}L(0)^k t) g_R(0)$, while b_R, b_L (in $K = b_L g_R^{-1} = g_L b_R^{-1}$) and W do not change. The functions of b_L, b_R and W form the Poisson algebra $\mathfrak{C}$ of constants of motion. The functional dimensions of $\mathfrak{H}$ and $\mathfrak{C}$ add up to $\dim(\mathcal{M})$, and thus $\mathfrak{H}$ represents a degenerate integrable Hamiltonian system on $\mathcal{M}$.

We reduce by imposing the constraint $\Lambda = e^{\gamma}\mathbf{1}_n$ (with $\gamma > 0$), where $\Lambda : \mathcal{M} \to \mathrm{B}(n)$ is a certain Poisson–Lie moment map, which generates a Poisson–Lie action of $\mathrm{U}(n)$ on $\mathcal{M}$. We obtain that

$$\mathcal{M}_{\mathrm{red}} = \Lambda^{-1}(e^{\gamma}\mathbf{1}_n)/\mathrm{U}(n)$$

is a smooth (actually real analytic) symplectic manifold. The free Hamiltonians descend to $\mathcal{M}_{\mathrm{red}}$, and form a degenerate integrable system since the $\mathrm{U}(n)$ invariant constants of motion also descend to $\mathcal{M}_{\mathrm{red}}$.

Suppose that we have a Poisson map, Λ, from a symplectic manifold $\mathcal{M}$ into the Poisson–Lie group $(\mathrm{B}(n), \{\ ,\ \}_B)$. Then, for any $X \in \mathfrak{u}(n)$ the following formula defines a vector field $X_{\mathcal{M}}$ on $\mathcal{M}$:

$$\mathcal{L}_{X_{\mathcal{M}}}(\mathcal{F}) \equiv X_{\mathcal{M}}[\mathcal{F}] := \left\langle X, \{\mathcal{F}, \Lambda\}_{\mathcal{M}} \Lambda^{-1} \right\rangle, \qquad \forall \mathcal{F} \in C^{\infty}(\mathcal{M}).$$

This generates an infinitesimal left action of $\mathrm{U}(n)$. If it integrates to a global action of $\mathrm{U}(n)$, then the resulting action is Poisson, i.e., the action map $\mathcal{A} : \mathrm{U}(n) \times \mathcal{M} \to \mathcal{M}$ is Poisson. Then Λ is called the (Poisson–Lie) **moment map** for the corresponding Poisson action.

Picking a moment map value, μ, one obtains the reduced phase space

$$\mathcal{M}_{\mathrm{red}} := \Lambda^{-1}(\mu)/\mathrm{U}(n)_{\mu}$$

where $\mathrm{U}(n)_{\mu}$ is the isotropy group of μ w.r.t. dressing action of $\mathrm{U}(n)$ on $\mathrm{B}(n)$, given by $\mathrm{Dress}_g(\mu) := \Lambda_L(g\mu)$. If the action of $\mathrm{U}(n)_{\mu}$ is free, then $\mathcal{M}_{\mathrm{red}}$ is a smooth symplectic manifold. Letting $\iota_{\mu} : \Lambda^{-1}(\mu) \to \mathcal{M}$ and $\pi_{\mu} : \Lambda^{-1}(\mu) \to \mathcal{M}_{\mathrm{red}}$ denote the natural maps, one has

$$\pi_{\mu}^* \Omega_{\mathrm{red}} = \iota_{\mu}^* \Omega_{\mathcal{M}}, \ \pi_{\mu}^* \mathcal{F}_{\mathrm{red}} = \iota_{\mu}^* \mathcal{F}, \ \{\mathcal{F}, \mathcal{H}\}_{\mathcal{M}} \circ \iota_{\mu} = \{\mathcal{F}_{\mathrm{red}}, \mathcal{H}_{\mathrm{red}}\}_{\mathrm{red}} \circ \pi_{\mu}$$

for $\mathrm{U}(n)$-invariant functions on $\mathcal{M}$, with reduced symplectic form Ω_{red} and corresponding Poisson structure. (This generalization of Marsden–Weinstein reduction is due to J.–H. Lu (1990).)

We have a Poisson map $\mathbf{b} : \mathbb{C}^n \to \mathrm{B}(n)$ that satisfies

$$\mathbf{1}_n + ww^\dagger =: \mathbf{b}(w)\mathbf{b}(w)^\dagger$$

and generates the natural left action of $\mathrm{U}(n)$ on $\mathbb{C}^n$. With $(K, W) \in \mathcal{M} \equiv \mathrm{GL}(n, \mathbb{C}) \times \mathbb{C}^{n\times d}$, we define the moment map $\Lambda : \mathcal{M} \to \mathrm{B}(n)$ by

$$\Lambda(K, W) := \Lambda_L(K)\Lambda_R(K)\mathbf{b}(w^1)\mathbf{b}(w^2)\cdots\mathbf{b}(w^d).$$

The $\mathrm{U}(n)$ action generated by Λ takes a simple form if instead of $K, w^1, \ldots, w^d$ we use the new variables

$$g_R, b_R \quad \text{and} \quad v(\alpha) := b_R\mathbf{b}(w^1)\cdots\mathbf{b}(w^{\alpha-1})w^\alpha \quad \text{for} \quad 1 \leq \alpha \leq d.$$

The $v(\alpha)$ are called 'dressed spins'. The $\mathrm{U}(n)$ action is orbit-equivalent to the 'obvious action', where $\eta \in \mathrm{U}(n)$ acts as

$$(g_R, b_R, v(1), \ldots, v(d)) \mapsto \left(\eta g_R \eta^{-1}, \mathrm{Dress}_\eta(b_R), \eta v(1), \ldots, \eta v(d)\right).$$

Here, $b_R \mapsto \mathrm{Dress}_\eta(b_R) := \Lambda_L(\eta b_R)$ is equivalent to $L := b_R b_R^\dagger \mapsto \eta L \eta^{-1}$.

Since $g_R \in \mathrm{U}(n)$ can be diagonalized by gauge transformations, every gauge orbit has representatives in

$$\mathcal{M}_0 := \{(Q, b_R, W) \in \Lambda^{-1}(e^{\gamma}\mathbf{1}_n) \mid Q \in \mathbb{T}^n\}.$$

We focus on the dense subset $\mathcal{M}_0^{\mathrm{reg}}$ where Q is regular. Using the (diagonal $\times$ strictly upper-triangular) decomposition $\mathrm{B}(n) = \mathrm{B}(n)_0 \mathrm{B}_+(n)$, we can write

$$b_R = b_0 b_+ \quad \text{and} \quad \mathbf{b}(w^1)\mathbf{b}(w^2)\cdots\mathbf{b}(w^d) =: S(W) =: S_0(W) S_+(W).$$

Then the moment map constraint becomes equivalent to

$$S_0(W) = e^{\gamma}\mathbf{1}_n \quad \text{and} \quad b_+ S_+(W) = Q^{-1} b_+ Q.$$

The first equation constraints W only, while the second one permits us to express b_+ in terms of $Q = e^{\mathrm{i}q} \in \mathbb{T}^n_{\mathrm{reg}}$ and W. The explicit formula of $b_+(Q, W)$ is given in our paper.

Note that $Q \in \mathbb{T}^n_{\mathrm{reg}}$ and $b_0 \equiv \exp(p)$ ($p = \mathrm{diag}(p_1, \ldots, p_n)$) are arbitrary. The reduced phase space can be parametrized Q, p and the constrained primary spins, W, up to residual gauge transformations.

The map $\phi : \mathbb{C}^{n\times d} \to \mathfrak{b}(n)_0 \simeq \mathbb{R}^n$ defined by writing $S_0(W) := \exp(\phi(W))$ is the moment map for $\mathbb{T}^n$ action given by $\tau\cdot(w^1,\dots,w^d) = (\tau w^1,\dots,\tau w^d)$. The 'reduced space of primary spins' $\mathbb{C}^{n\times d}_{\mathrm{red}} := \phi^{-1}(\gamma \mathbf{1}_n)/\mathbb{T}^n$, is a smooth, compact and connected symplectic manifold of dimension $2n(d-1)$.

With the normalizer $\mathcal{N}(n)$ of $\mathbb{T}^n$, consider the regular part of $\mathcal{M}_{\mathrm{red}}$:

$$\mathcal{M}^{\mathrm{reg}}_{\mathrm{red}} = \mathcal{M}^{\mathrm{reg}}_0/\mathcal{N}(n) = (\mathcal{M}^{\mathrm{reg}}_0/\mathbb{T}^n)/S_n, \quad (S_n = \mathcal{N}(n)/\mathbb{T}^n).$$

Theorem. The covering space $\mathcal{M}^{\mathrm{reg}}_0/\mathbb{T}^n$ of the regular part of the reduced phase space can be identified with the symplectic manifold

$$T^*\mathbb{T}^n_{\mathrm{reg}} \times \mathbb{C}^{n\times d}_{\mathrm{red}} = \{(Q, p, [W])\}$$

equipped with its natural product symplectic structure. The dense open submanifold $\mathcal{M}^{\mathrm{reg}}_{\mathrm{red}} \subseteq \mathcal{M}_{\mathrm{red}}$, and thus also $\mathcal{M}_{\mathrm{red}}$, is connected.

The explicit form of the main reduced Hamiltonian, $\mathrm{tr}(b_R b_R^\dagger)$ as a function of $(Q, p, [W])$, can be used to connect our reduced system to the spin Sutherland model of Gibbons and Hermsen (1984).

Let us connect our reduced system with the spin Sutherland model of Gibbons and Hermsen (1984). For this, we introduce a positive 'scaling parameter' ϵ and make the replacements

$$p \to \epsilon p, \quad W \to \epsilon^{\frac{1}{2}} W, \quad Q \to Q, \quad \Omega_{\mathcal{M}} \to \epsilon^{-1} \Omega_{\mathcal{M}}, \quad \gamma \to \epsilon\gamma$$

With $L := b_R b_R^\dagger$ and $b_R = e^{\epsilon p} b_+(Q, \epsilon^{\frac{1}{2}} W)$, we find

$$\mathrm{tr}(L^{\pm 1}) = n \pm 2\epsilon\, \mathrm{tr}(p) + 2\epsilon^2 \mathrm{tr}(p^2) + \epsilon^2 \sum_{i<j} \frac{|(w_i^\bullet, w_j^\bullet)|^2}{|Q_j Q_i^{-1} - 1|^2} + \mathrm{o}(\epsilon^2)$$

where $w_i^\bullet \in \mathbb{C}^d$ with components w_i^α, and $(w_i^\bullet, w_j^\bullet) := \sum_{\alpha=1}^d w_i^\alpha \overline{w}_j^\alpha$. Writing $Q_j = e^{\mathrm{i} q_j}$, we obtain on $\mathcal{M}_0^{\mathrm{reg}}$

$$\lim_{\epsilon \to 0} \frac{1}{8\epsilon^2} (\mathrm{tr}(L) + \mathrm{tr}(L^{-1}) - 2n) = \frac{1}{2} \mathrm{tr}(p^2) + \frac{1}{32} \sum_{i \neq j} \frac{|(w_i^\bullet, w_j^\bullet)|^2}{\sin^2 \frac{q_i - q_j}{2}},$$

which reproduces the Hamiltonian of the (real, trigonometric) Gibbons–Hermsen model. Moreover,

$$\lim_{\epsilon \to 0} \epsilon^{-1} (\Omega_{\mathcal{M}}) = \sum_{j=1}^n dp_j \wedge dq_j + \frac{\mathrm{i}}{2} \sum_{j=1}^n \sum_{\alpha=1}^d dw_j^\alpha \wedge d\overline{w}_j^\alpha$$

reproduces the symplectic form of the Gibbons–Hermsen model.

Next, we describe the evolution equations generated by the Hamiltonians $H_m := \frac{1}{2m}\mathrm{tr}(L^m)$ for $m \in \mathbb{N}$. Before reduction, we now parametrize the phase space by the variables (g_R, L, v), where $L = b_R b_R^\dagger$ and $v = (v(1), v(2), \dots, v(d)) \in \mathbb{C}^{n\times d}$ denotes the 'dressed spins'.

The Hamiltonian vector field X_{H_m} reads

$$X_{H_m}[g_R] = \mathrm{i}L^m g_R, \quad X_{H_m}[v(\alpha)] = 0, \quad X_{H_m}[L] = 0.$$

Its projection onto $\mathcal{M}_{\mathrm{red}}$ does not change if we add any infinitesimal gauge transformation, i.e., consider Y_{H_m} given by

$$\begin{aligned} Y_{H_m}[g_R] &= \mathrm{i}L^m g_R + [Z(g_R, L, v), g_R], \\ Y_{H_m}[v(\alpha)] &= Z(g_R, L, v)v(\alpha), \\ Y_{H_m}[L] &= [Z(g_R, L, v), L], \end{aligned}$$

with arbitrary $Z(g_R, L, v) \in \mathfrak{u}(n)$. To determine the projection, one may use the restriction of Y_{H_m} to the 'diagonal gauge'

$$\mathcal{M}_0 \equiv \{(Q, L, v) \in \Lambda^{-1}(e^\gamma \mathbf{1}_n) \mid Q = e^{\mathrm{i}q} \in \mathbb{T}^n\}.$$

Proposition 1. If $(Q, L, v) \in \mathcal{M}_0$, then L can be expressed in terms of Q and v as follows:

$$L_{ij} = \frac{F_{ij}}{e^{2\gamma} Q_j Q_i^{-1} - 1} \quad \text{with} \quad F := \sum_{\alpha=1}^{d} v(\alpha) v(\alpha)^\dagger.$$

Conversely, if the Hermitian matrix L given by the above formula is positive definite, then $(Q, L, v) \in \mathcal{M}_0$. Thus, $\mathcal{M}_0$ is identified with an open subset of $\mathbb{T}^n \times \mathbb{C}^{n\times d}$.

We focus on the regular part, and choose $Z(Q, L, v) := \mathcal{K}_m(Q, L)$ with $\mathcal{K}_m(Q, L)_{kk} = 0$ and

$$\mathcal{K}_m(Q, L)_{kl} = -\frac{1}{2}\mathrm{i}(L^m)_{kl} + \frac{1}{2}(L^m)_{kl} \cot\left(\frac{q_k - q_l}{2}\right). \quad \forall k \neq l.$$

This guarantees tangency of the restricted vector field, $Y^0_{H_m}$, to $\mathcal{M}_0^{\text{reg}}$.

One may add an arbitrary Lie($\mathbb{T}^n$)-valued function $\lambda(Q, v)$ to $\mathcal{K}_m$, expressing the residual infinitesimal gauge transformations.

Proposition 2. $H_m := \frac{1}{2m}\mathrm{tr}(L^m)$ induces the vector field $Y^0_{H_m}$ on $\mathcal{M}_0^{\mathrm{reg}}$,

$$Y^0_{H_m}[Q] = \mathrm{i}(L^m)_{\mathrm{diag}} Q$$
$$Y^0_{H_m}[v(\alpha)] = \mathcal{K}_m(Q,L)v(\alpha),$$

which descends to the Hamiltonian vector field of the corresponding reduced Hamiltonian on $\mathcal{M}_{\mathrm{red}}^{\mathrm{reg}} \subset \mathcal{M}_{\mathrm{red}}$.

Corollary. Consider $H := (e^{2\gamma}-1)\mathrm{tr}(L)$. Then the evolution equation on $\mathcal{M}_0^{\mathrm{reg}}$ corresponding to the vector field Y^0_H can be written as follows:

$$\frac{1}{2}\dot{q}_j := \frac{1}{2\mathrm{i}} Y^0_H[Q_j]Q_j^{-1} = F_{jj},$$
$$\dot{v}(\alpha)_i := Y^0_H[v(\alpha)_i] = -\sum_{j\neq i} F_{ij} v(\alpha)_j V\left(\frac{q_j - q_i}{2}\right)$$

with the 'potential function' $V(x) = \cot x - \cot(x - \mathrm{i}\gamma)$. This reproduces the spin RS equations of motion of Krichever and Zabrodin by setting $x_i = q_i/2$ and imposing the reality conditions $c_i^\alpha = (a_i^\alpha)^* \equiv v(\alpha)_i$.

We have also calculated the reduced Poisson bracket in terms of the variables $Q = e^{\mathrm{i}q}, v$ that appear in the reduced equations on motion.

Consider the following U(n) invariant complex functions on $\mathcal{M}$:

$$I^k_{\alpha\beta} := v(\beta)^\dagger L^k v(\alpha)\,, \quad 1 \leq \alpha, \beta \leq d,\ k \geq 0.$$

They belong to the commutant of the 'free' Hamiltonians H_m. Their real and imaginary parts generate a polynomial Poisson algebra. This descends to $\mathcal{M}_{\text{red}}$ and underlies the degenerate integrability of $\{H_m^{\text{red}}\}$.

In conclusion, we have shown that the trigonometric real form of the spin RS system of Krichever and Zabrodin arises from Hamiltonian reduction of a 'free system' on a spin extended Heisenberg double of U(n). Moreover, we have proved its degenerate integrability by displaying the required constants of motion in explicit form.

However, a global model of the reduced phase space was not obtained. What about the hyperbolic real form? Quantization? Elliptic systems?

For details and references, see our paper, arXiv:2007.08388 by Fairon, Fehér and Marshall, which just appeared in Annales Henri Poincaré.