On matrix Painlevé II equations

Vladimir V. Sokolov[1]

L.D. Landau Institute for Theoretical Physics,
Chernogolovka, Russia,

and UFABC, Sao Paulo, Brazil

Moscow, 17.12.2020

[1] **Based on a joint paper with V.Adler**

- **S.P. Balandin, V.V. Sokolov**, *On the Painlevé test for non-Abelian equations*, Phys. Lett. A., 1998, **246**, no.**3-4**, 267–272.
- **V. Retakh, V. Rubtsov**, *Noncommutative Toda chain, Hankel quasideterminants and Painlevé II equation*, J. Phys. A, 2010, **43**, 505204.
- **V.A. Marchenko**, *Nonlinear equations and operator algebras*, Kiev, Naukova dumka, 1986.
- **P.J. Olver, V.V. Sokolov**, *Integrable evolution equations on associative algebras*, Comm. in Math. Phys., 1998, **193**, no.**2**, 245–268.
- **F.A. Khalilov, E.Ya. Khruslov**, *Matrix generalization of the modified Korteweg–de Vries equation*, Inverse Problems, 1990, **6**, no.**2**, 193–204.
- **V.E. Adler, V.V. Sokolov**, *On matrix Painlevé II equations*, submitted to Theor. and Math. Phys. arXiv nlin, 2012.05639

Introduction

The second of the six famous Painlevé equations reads

$$y'' = 2y^3 + zy + a, \qquad (1)$$

where $'$ denotes the derivative with respect to z and a is an arbitrary complex constant. A matrix version of the Painlevé –Kovalevskaya test was proposed in [1], where it was proved that it holds for (1) in the case when $y(z)$ is a matrix of arbitrary size $n \times n$ and a is a scalar matrix.

One of possible generalizations of the matrix equation (1) is as follows. It is clear that in the non-commutative case one can change the principal differential-homogeneous part $y'' = 2y^3$ of this equation by adding the term of the same weight $\kappa[y, y']$, where $\kappa \in \mathbb{C}$. It turns out that the equation

$$y'' = \kappa[y, y'] + 2y^3 + zy + a, \qquad a \in \mathbb{C}$$

satisfies the matrix Painlevé–Kovalevskaya test if and only if $\kappa = 0, \pm 1, \pm 2$.

Another direction for generalizations was suggested by the results of [2], where the equation P_2 with matrix coefficients appeared. More precisely, in this paper the term linear in y was written as $\frac{1}{2}(zy + yz)$ where z denoted a non-commutative dependent variable such that $z' = 1$. For our purposes it is more convenient to replace this variable with $z + 2b$, where z is a commutative independent variable and b is a matrix constant.

Consider matrix generalizations of the P_2 equation of the general form

$$y'' = \kappa[y, y'] + 2y^3 + zy + b_1 y + y b_2 + a, \qquad (2)$$

where a, b_1 and b_2 are matrix constants and κ is a scalar constant.

Remark 1. Equation (2) is invariant under the change

$$b_1 \to b_1 + \beta_1 I, \qquad b_2 \to b_2 + \beta_2 I, \qquad z \to z - \beta_1 - \beta_2, \qquad (3)$$

where $\beta_i \in \mathbb{C}$ and I is the identity matrix.

Using the matrix version of Painlevé–Kovalevskaya test, we examine the sets of κ, a, b_1 and b_2 which are candidates for integrable cases and prove the following statement.

Theorem. The equation (2) satisfies the Painlevé –Kovalevskaya test if and only if it is reduced by the transformation (3) to one of the following cases:

$$y'' = 2y^3 + zy + by + yb + \alpha I, \qquad \alpha \in \mathbb{C}, \quad b \in \mathrm{Mat}_n,$$

$$y'' = \pm[y, y'] + 2y^3 + zy + a, \qquad a \in \mathrm{Mat}_n,$$

$$y'' = \pm 2[y, y'] + 2y^3 + zy + by + yb + a, \qquad a, b \in \mathrm{Mat}_n, \quad [b, a] = \pm 2b.$$

The integrability of the found cases is substantiated by construction of the isomonodromic Lax pairs

$$A' = B_\zeta + [B, A], \tag{4}$$

where ζ is a spectral parameter, $A(z, \zeta)$ and $B(z, \zeta)$ are 2×2 matrices with non-commutative entries.

One of the methods for obtaining representations (4) is based on the observation that group-invariant solutions of evolution equations admitting zero curvature representations satisfy ordinary differential equations of the Painlevé type. In this case, the isomonodromic representation is obtained from the zero curvature representation by a standard procedure.

In particular, it is well known that the P_2 equation corresponds to Galilean-invariant solutions of the nonlinear Schrödinger equation, as well as to scale-invariant solutions of the mKdV equation.

We generalize these reductions to the matrix case.

One source of non-commutative coefficients in (2) is the arbitrary matrices contained in the symmetry groups of non-Abelian [3,4] evolutionary equations. For example, the matrix nonlinear Schrödinger equation

$$u_t = u_{xx} + 2uvu, \qquad v_t = -v_{xx} - 2vuv \tag{5}$$

admits transformations of the form $u \to b_1 u b_2$, $v \to b_2^{-1} v b_1^{-1}$, where $b_i \in \mathrm{Mat}_n$. Two well-known matrix generalizations of the mKdV equation [4,5]

$$u_t = u_{xxx} - 3u^2 u_x - 3u_x u^2,$$

$$u_t = u_{xxx} + 3[u, u_{xx}] - 6uu_x u$$

obviously admit the transformation group $u \to bub^{-1}$.

Another source of matrix coefficients in (2) is related with matrix coefficients in integrable evolution equations themselves. We use the following integrable version of the matrix mKdV equation:

$$u_t = u_{xxx} + 3[u, u_{xx}] - 6uu_xu + (u_x + u^2)b + b(u_x - u^2), \qquad b \in \mathrm{Mat}_n\,.$$

The origin of the constant b is related with the Miura map for the matrix KdV equation which is constructed by solution of the linear Schrödinger equation

$$\psi'' + v\psi + \psi b = 0, \qquad b \in \mathrm{Mat}_n\,.$$

We do not know whether this generalization appeared in the literature.

We show that equations P_2^i, $i = 1, 2, 3$ from Theorem 1 are obtained from the above matrix evolution equations by some group-invariant reductions. This made possible to find the isomonodromic Lax pairs (4) for the matrix Painlevé II equations.

Part 1. Matrix Painlevé–Kovalevskaya test

Consider the scalar Painlevé–2 equation

$$y'' = 2y^3 + zy + a.$$

Let

$$y(z) = \frac{p}{z - z_0} + c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots .$$

Substituting this into equation, we get

$$p^3 = p, \qquad (j+2)(j-3)c_j = f_j(c_0, \ldots, c_{j-1}), \quad j = 0, 1, ...$$

Let $p = 1$. Then $c_0 = 0,\ c_1 = -\frac{z_0}{6},\ c_2 = -\frac{a}{4}$. The coefficient c_3 is arbitrary!

The Painlevé–Kovalevskaya matrix test [1] for equation (2) is based on counting of arbitrary scalar constants in a formal solution of the form

$$y = \frac{p}{z - z_0} + c_0 + c_1(z - z_0) + \cdots, \qquad p, c_j \in \mathrm{Mat}_n, \quad z_0 \in \mathbb{C}. \quad (6)$$

Here z_0 is one of these arbitrary constants. In order for the series y to represent a generic solution, it is necessary that the matrices p and c_j contain additionally $2n^2 - 1$ arbitrary constants. We assume that the Painlevé–Kovalevskaya test is fulfilled if such matrices exist.

Remark 2. For any nondegenerate matrix T, the series TyT^{-1} satisfies equation (2), where $b_i \to \bar{b}_i \stackrel{def}{=} Tb_iT^{-1}$ and $a \to \bar{a} \stackrel{def}{=} TaT^{-1}$. Hence, equation (2) satisfies the Painlevé–Kovalevskaya test simultaneously with the equation corresponding to the coefficients $\bar{b}_i$ and $\bar{a}$.

Substituting the series into the equation and collecting the coefficients at powers of $z - z_0$, we obtain relations of the form

$$p^3 = p, \tag{7}$$

$$L_{\frac{j+1}{2}\kappa}(c_j) - \frac{j(j-1)}{2} c_j = f_j(z_0, p, c_0, \dots, c_{j-1}), \quad j \geqslant 0, \tag{8}$$

where

$$L_\sigma(c) \overset{def}{=} p^2 c + pcp + cp^2 + \sigma(pc - cp).$$

The first few functions f_i are easy to compute explicitly, for instance,

$$f_0 = 0, \qquad f_1 = -pc_0^2 - c_0 p c_0 - c_0^2 p - \frac{1}{2}(z_0 p + b_1 p + p b_2).$$

Since f_j in the right-hand sides of (8) do not contain c_j, the matrices c_j can be calculated from these equations recursively.

The number of arbitrary constants in the matrix p is equal to the dimension of the orbit of its Jordan form. It is easy to see that the Jordan form of any matrix p satisfying (7) is

$$p = \mathrm{diag}(E_k,\ -E_m,\ 0_{n-k-m}).$$

When a group acts on a manifold, the dimension of the orbit of a point is equal to the difference of the manifold dimension and the dimension of the stabilizer of this point, that is, the subgroup that leaves the point fixed.

In our case, the dimension of the manifold is n^2, the stabilizer consists of non-degenerate matrices which commute with p. It is easy to see that its dimension is $k^2 + m^2 + (n - k - m)^2$; whence it follows that the dimension of the orbit of p is equal to $2m(n - m) + 2k(n - k) - 2km$.

Lemma. The eigenvalues of the operator L_σ belong to the set

$$\lambda_0 = 0, \quad \lambda_{\pm 1} = 1 \pm \sigma, \quad \lambda_{\pm 2} = 1 \pm 2\sigma, \quad \lambda_3 = 3.$$

The space Mat_n decomposes into the direct sum of eigenspaces

$$\mathrm{Mat}_n = V_0 \oplus V_{-1} \oplus V_1 \oplus V_{-2} \oplus V_2 \oplus V_3, \qquad L_\sigma V_i = \lambda_i V_i,$$

with dimensions

$$\dim V_0 = (n-k-m)^2, \qquad \dim V_{\pm 1} = k(n-k) + m(n-m) - 2km,$$
$$\dim V_{\pm 2} = km, \qquad \dim V_3 = k^2 + m^2.$$

In the case when the eigenvalues coincide, the dimensions of the corresponding eigenspaces add up.

Proof.

We represent c as a 3×3 block matrix with the block sizes determined by the Jordan form of p:

$$c = \begin{array}{c} \\ k \\ m \\ l \end{array} \begin{array}{c} \begin{array}{ccc} k & m & l \end{array} \\ \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \end{array}, \qquad l = n - k - m.$$

Then it is easy to check that

$$L_\sigma(c) = \begin{pmatrix} 3c_{11} & (1+2\sigma)c_{12} & (1+\sigma)c_{13} \\ (1-2\sigma)c_{21} & 3c_{22} & (1-\sigma)c_{23} \\ (1-\sigma)c_{31} & (1+\sigma)c_{32} & 0 \end{pmatrix},$$

Using Lemma, we obtain that the maximal number of arbitrary constants is equal to $2n^2 - k^2 - m^2$. This is equal to $2n^2 - 1$ only for $k = 1$, $m = 0$ or for $k = 0$, $m = 1$. The second case is reduced to the first one by the change $y \to -y$. In the first case the Jordan form of p is $\mathrm{diag}(1, 0, \ldots, 0)$ and its orbit $\mathcal{O}$ consists of the matrices of the form $u\,v^T$, where u and v are column vector such that $u^T v = 1$.

Thus we require that:

- for any column vectors u and v such that $u^T v = 1$, there exists a formal solution of the form

$$y = \frac{u\,v^T}{z - z_0} + c_0 + c_1(z - z_0) + \cdots, \qquad p, c_j \in \mathrm{Mat}_n, \quad z_0 \in \mathbb{C},$$

  such that its coefficients c_i contain in total $2n^2 - 2n + 1$ arbitrary constants.

The conditions that the eigenvalues $\lambda_{\pm 1}$ are resonant for some index j, are given by equations

$$1 \pm \frac{\kappa(j+1)}{2} = \frac{j(j-1)}{2}$$

Cancelling this by $j+1$ (recall that $j \geqslant 0$), we obtain the resonance conditions

$$\lambda_{\pm 1} : \;\; j = 2 \pm \kappa.$$

This implies that the parameter κ must be such that both numbers $2 \pm \kappa$ are non-negative integers. This leaves the admissible values

$$\kappa = 0, \, \pm 1, \, \pm 2.$$

Part 2. Case $\kappa = 0$

Let $p = \mathrm{diag}(1, 0, \ldots, 0)$. In this case the operator L_σ is of the form

$$L_\sigma : \quad \begin{matrix} & 1 & n-1 \\ 1 & \multicolumn{2}{c}{\multirow{2}{*}{}} \\ \end{matrix}$$

$$L_\sigma : \quad \begin{array}{c} \\ 1 \\ n-1 \end{array} \begin{array}{c} \begin{array}{cc} 1 & n-1 \end{array} \\ \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \end{array} \mapsto \begin{pmatrix} 3c_{11} & (1+\sigma)c_{12} \\ (1-\sigma)c_{21} & 0 \end{pmatrix}.$$

In order to analyze equations (8), we write them block-wise by dividing the involved matrices into blocks of appropriate sizes:

$$c_j = \begin{pmatrix} c_{j,11} & c_{j,12} \\ c_{j,21} & c_{j,22} \end{pmatrix}, \quad b_1 = \begin{pmatrix} b_{1,11} & b_{1,12} \\ b_{1,21} & b_{1,22} \end{pmatrix},$$

$$b_2 = \begin{pmatrix} b_{2,11} & b_{2,12} \\ b_{2,21} & b_{2,22} \end{pmatrix}, \quad a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

If the linear operator in the left-hand side in the equation (8) for c_j is invertible, then c_j is uniquely determined. If, for some j (resonance), the corresponding operator is degenerate then the answer contains arbitrary constants in the amount equal to the dimension of the kernel;

Conditions on the matrices b_1, b_2 and a arise from the requirement of existence of solutions for inhomogeneous linear systems (8) for the resonance values of j.

In the case $\kappa = 0$ the resonance values are $j = 0, 1, 2$ and 3. Let us consider equations (8) for $j = 0, 1, 2, 3$.

For $j = 0$ we find $c_{0,11} = c_{0,12} = c_{0,21} = 0$ and the block $c_{0,22}$ is arbitrary.

Next, for $j = 1$ we obtain

$$c_{1,11} = -\frac{1}{6}(z_0 + b_{1,11} + b_{2,11}), \quad c_{1,12} = -\frac{1}{2} b_{2,12}, \quad c_{1,21} = -\frac{1}{2} b_{1,21}$$

and $c_{1,22}$ is arbitrary. So far, no conditions for the coefficients b_1, b_2 and a appear.

For $j = 2$ we find, by substituting the obtained values, that

$$c_{2,11} = -\frac{1}{4}(1 + a_{11}),$$

$$c_{2,22} = \frac{1}{2}(a_{22} + b_{1,22}c_{0,22} + c_{0,22}b_{2,22} + z_0 c_{0,22}) + c_{0,22}^3,$$

and the blocks $c_{2,12}$ and $c_{2,21}$ are arbitrary. In addition, we obtain the restrictions on the coefficients in the form of the relations

$$(b_{2,12} - b_{1,12})c_{0,22} = a_{12}, \qquad c_{0,22}(b_{1,21} - b_{2,21}) = a_{21}.$$

Since the block $c_{0,22}$ is arbitrary, this implies that

$$b_{1,12} = b_{2,12}, \qquad b_{1,21} = b_{2,21}, \qquad a_{12} = a_{21} = 0. \tag{9}$$

Finally, for $j = 3$ it is sufficient to write down the condition for the 1×1 block

$$b_{1,12}b_{1,21} - 2b_{2,12}b_{1,21} + b_{2,12}b_{2,21} = 0, \tag{10}$$

because all other equations are solved uniquely with respect to $c_{3,12}$, $c_{3,21}$ and $c_{3,22}$. The condition (10) follows from (9). For $j > 3$, there are no resonances and therefore all equations for c_j are solved uniquely.

Thus, we proved that equations (8) with $\kappa = 0$ are solved with arbitrary blocks $c_{0,22}$, $c_{1,22}$, $c_{2,12}$, $c_{2,21}$ and $c_{3,11}$. Moreover, we obtained the solvability conditions, which mean that the matrices a and $b_1 - b_2$ commute with $p = \mathrm{diag}(1, 0, \ldots, 0)$.

By choosing as p the matrices of the form

$$\mathrm{diag}(0, 0, \ldots, 1, \ldots, 0),$$

we obtain that a and $b_1 - b_2$ must commute with any such matrix, which means that they are diagonal.

According to Remark 2, the matrices TaT^{-1} and $T(b_1 - b_2)T^{-1}$ must be diagonal for any nondegenerate T. This means that a and $b_1 - b_2$ are scalar matrices. Using Remark 1, we arrive at the equation P_2^1.

Part 3. Reductions of partial differential equations

Consider the equation:

$$u_t = u_{xxx} + 3[u, u_{xx}] - 6uu_xu - 3(u_x + u^2)c - 3c(u_x - u^2), \qquad c \in \mathrm{Mat}_n\,.$$

The zero curvature representation this mKdV equation is given by the matrices

$$U = \begin{pmatrix} 0 & 1 \\ c - \lambda & -2u \end{pmatrix},$$

$$V = 2\begin{pmatrix} 2u(c-\lambda) & -u_x - u^2 - c - 2\lambda \\ (u_x - u^2 - c - 2\lambda)(c-\lambda) & 2\lambda u - u_{xx} - [u, u_x] + 2u^3 + 2cu + 2uc \end{pmatrix}.$$

Consider the following group-invariant substitution:

$$u = \varepsilon\tau e^{\log(\tau)d} y(z) e^{-\log(\tau)d}, \quad \tau = t^{-1/3}, \quad z = \varepsilon\tau x, \quad 3\varepsilon^3 = -1,$$

$$c = (\varepsilon\tau)^2 e^{\log(\tau)d} c_0 e^{-\log(\tau)d}, \quad d \in \mathrm{Mat}_n,$$

where c_0 is an arbitrary constant matrix such that $2c_0 + [d, c_0] = 0$. Indeed, the differentiation with respect to τ shows that this relation implies that the matrix c is constant as well. As the result, we get

$$y''' = 3[y'', y] + 6yy'y + y + zy' + 3(y' + y^2)c_0 + 3c_0(y' - y^2) + [d, y].$$

By eliminating y''' and y'' in virtue of equation of the form (2) we find that if $\kappa = -2$, $3c_0 = b$ and $d = -a$ then this third order equation is a consequence of equation P_2^3.

The standard manipulations with the matrices U and V lead to the Lax representation (4) with

$$B = \begin{pmatrix} 0 & 1 \\ \frac{1}{3}b - \zeta & -2y \end{pmatrix},$$

$$A = \frac{1}{\zeta} \begin{pmatrix} 2\zeta y - \frac{2}{3}yb - \frac{1}{2}(a+1) & 2\zeta + y' + y^2 + \frac{1}{3}b + \frac{z}{2} \\ (2\zeta - y' + y^2 + \frac{1}{3}b + \frac{z}{2})(\frac{1}{3}b - \zeta) & -2\zeta y - [y, y'] + \frac{1}{3}by + \frac{1}{3}yb + \frac{1}{2}a \end{pmatrix}$$

for equation P_2^3 with $\kappa = -2$.

A similar diversity of integrable matrix generalizations should be expected for other Painlevé equations, too. As one of examples, we present in [6] the following matrix version of the P_4 equation

$$y'' = \frac{y'^2}{2\,y} + \frac{3}{2}y^3 + 4zy^2 + 2(z^2 - \alpha)y + \frac{\beta}{y},$$

which contains the matrix constant $c \in \mathrm{Mat}_n$ and the scalar constant $\alpha \in \mathbb{C}$:

$$y'' = \frac{1}{2}(y'+2c)y^{-1}(y'-2c)+\frac{1}{2}[y,y']+\frac{3}{2}y^3+4zy^2+2(z^2-\alpha)y+cy+yc.$$

We have found an isomonodromic Lax pair for this equation.