$\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots, \varepsilon_n$ – i.i.d. $P\{\varepsilon_i = 1\} = P\{\varepsilon_i = -1\} = \frac{1}{2}$.

$a_1, a_2, a_3, \dots, a_n$

$$S_n = a_1\varepsilon_1 + a_2\varepsilon_2 + \dots + a_n\varepsilon_n \qquad S = a_1\varepsilon_1 + a_2\varepsilon_2 + \dots$$

1) $\underbrace{\sum a_i^2 \le 1}$ (Bentkus, Götze)

2) $|a_i| \le 1$ or $|a_i| \ge 1$ (T. Juškevičius)

1) $P\{S_n \ge x\}$

Hoeffding 63 $\quad P\{S_n \ge x\} \le \exp\{-\frac{x^2}{2}\}$, $x \ge 0$ (Th 2)

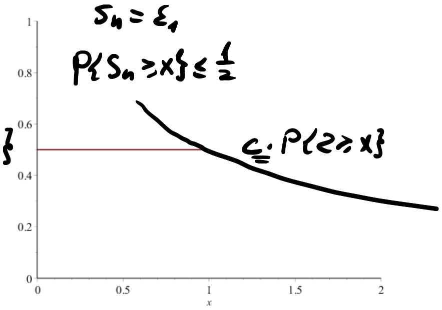
Eaton '70 $\quad Ef(S_n) \le \underbrace{Ef(Z)}_{x\to\infty} \le$

$$P\{S_n \ge x\} \underline{\le} \underbrace{E(x)} \sim \underset{4,46}{c_e} \cdot \underbrace{P\{Z \ge x\}} \to Z \sim N(0,1)$$

Pinelis '94 $\quad P\{S_n \ge x\} \le \underline{c_e}\, P\{Z \ge x\}$

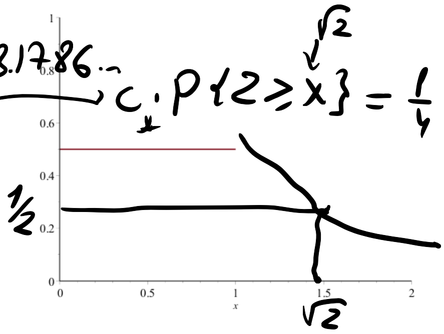
Bobkov, Houdré, Götze 2002 $\quad P\{S_n \ge x\} = \frac{1}{2} P\{\underbrace{S_{n-1} \ge x - a_1}\} + \frac{1}{2} P\{\underbrace{S_{n-1} \ge x + a_1}\}$

$c = 12\dots$



Bentkus $\quad P\{S_n \ge x\} \le \frac{1}{2x^2}$

$c \approx 4.00 \qquad c_* \approx \underbrace{\frac{1}{4P\{Z \ge \sqrt{2}\}}} \approx 3.1786\dots \quad c_* \cdot P\{Z \ge \overset{\sqrt{2}}{x}\} = \frac{1}{4}$

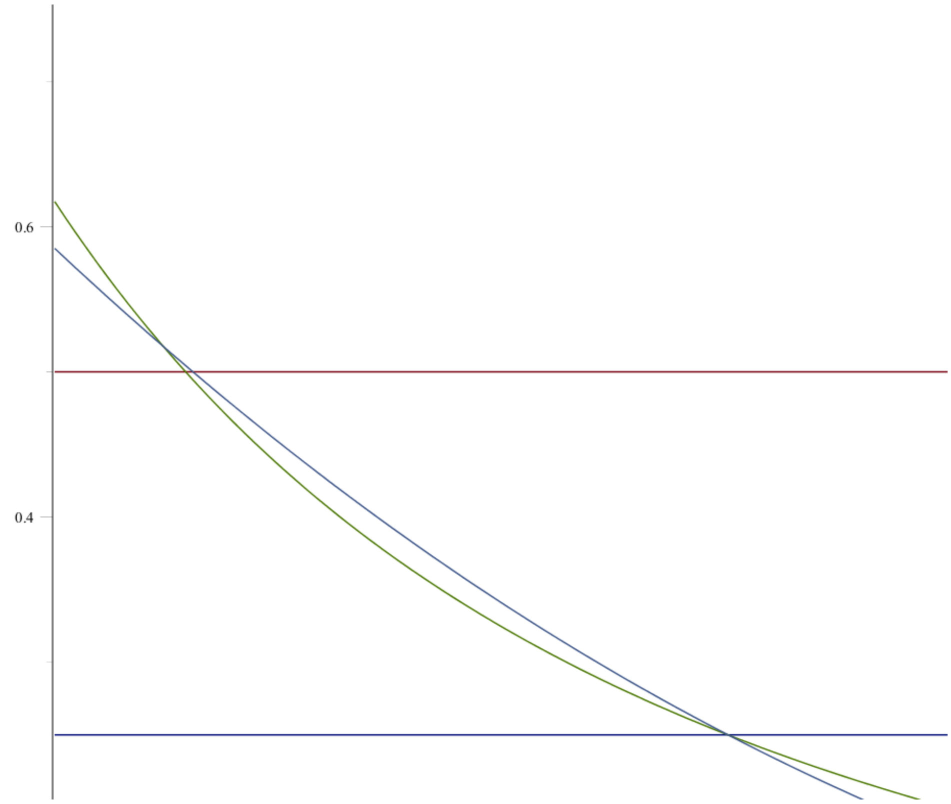


$c \approx 3.22$

Bentkus $c \approx 3.18$

1) a_i – "small" (localized case)

2) at least one a_i – "large" (delocalized case)

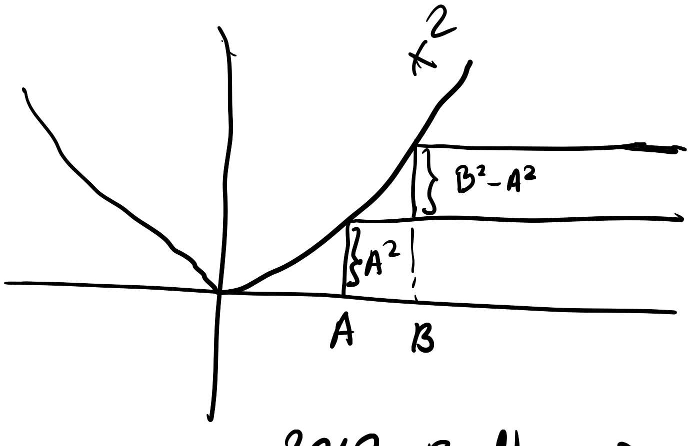


$a_1 \geq a_2 \geq \dots \geq a_n \geq 0 \quad \sum a_i^2 = 1$

$$\left| P\{S_n \geq x\} - P\{N \geq x\} \right| \leq C_B \cdot (a_1^3 + \dots + a_n^3) \leq C_B \cdot \max a_i = C_B a_1$$

$\underbrace{a_1 > 0{,}16}$, $a_1' < 0.16$ Shevtsova (2010) $C_B \leq 0{,}56$

1



$$A^2 \cdot P\{S \geq A\} + (B^2 - A^2) \cdot P\{S \geq B\} \leq$$
$$\leq \frac{1}{2} ES^2 = \frac{1}{2}.$$

2010 Bentkus, D $\quad c = (4 \cdot P\{Z \geq \sqrt{2}\})^{-1} \approx 3.1786\dots$

Conjecture (Tomaszewski 86)

$a_1, \dots, a_n \in \mathbb{R} : \quad \sum a_i^2 = 1.$

$S_n = \varepsilon_1 a_1 + \varepsilon_2 a_2 + \dots + \varepsilon_n a_n \qquad |S_n| \le 1 \text{ or } |S_n| > 1$

2^n

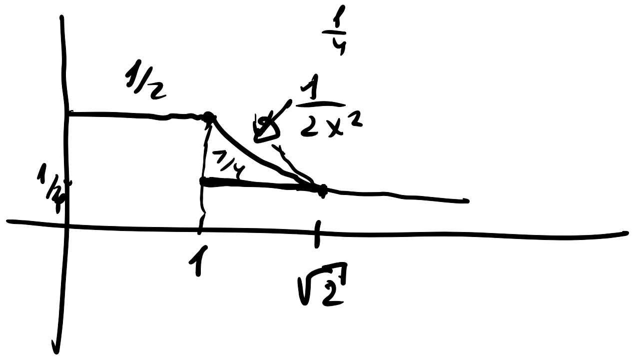
$P\{|S_n| > 1\} \le \frac{1}{2} \qquad P\{S > 1\} \le \frac{1}{4}.$

$P\{S_n \ge x\} \le \frac{1}{2x^2}, \quad x > 1$

$\frac{1}{4}$

$S_2 = \frac{\varepsilon_1 + \varepsilon_2}{\sqrt{2}}$

$P\{S_n \ge \sqrt{2}\} \le \frac{1}{4}.$



Conjecture (Bobkov? 90's)

$$P\{S_n \ge x\} \le \max_{k \ge 1} \mathbb{P}\left\{\frac{\varepsilon_1 + \dots + \varepsilon_k}{\sqrt{k}} \ge x\right\}$$

$a_1 = a_2 = \dots = a_k = \frac{1}{\sqrt{k}}, \quad a_i = 0, \; i \ge k+1$

Aleksey Zhubr. (90's preprint, 2012 published) $\leftarrow$ | 2012 Pinelis)

$n \ge 9, \quad x = \sqrt{\frac{37}{5}}$

$P\{|S_n| < 1\} \ge \frac{3}{8} = 0.375 \qquad P\overset{H:}{\{}|S_n| \le 1\} \ge \frac{1}{2}.$

D. Kleitman, R. Holzman 92

$S_4 = \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4}{\sqrt{4}} \qquad P\{|S_4| < 1\} = \frac{3}{8}.$

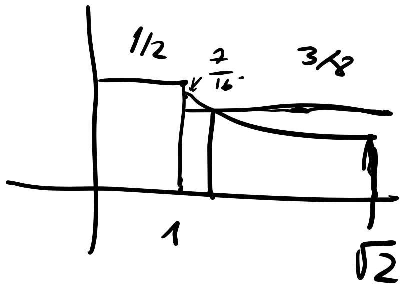
$a_1 = a_2 = a_3 = a_4 = \frac{1}{2}.$

Ben-Tal et al '02 $\ge \frac{1}{3}$. (

Shevnitsov '12 $\geq 0{,}36$ $\qquad$ $P\{|S| \leq x\} = 1 - P\{|S| > x\}$.

Dzindzalieta '14 $P\{|S| \leq x\} \geq \underset{=}{\frac{1}{2}} - \underbrace{\frac{1}{4}\left(1 - \sqrt{2 - \frac{2}{x^2}}\right)}$ $\qquad 1 \leq x \leq \sqrt{2}$

Boppana et al 2020
Boppana, Holzman 2020 $\Big\}\geq 0.428$



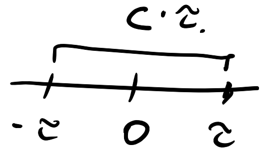
Dvorak 2020 ≥ 0.46

Nathan Keller and Ohad Klein 2020 $x = 1$

$\geq \frac{1}{2}$

D. Sëtre: $\quad \sum a_i^2 \leq 1$, $|a_i| \leq \tau \leq 1 \Rightarrow \exists\, c > 0$

then $\quad P\{|S_n| \leq \tau\} \geq c\tau$.



Keller, Klein $\quad a_i \leq 0{,}22$

$$P\{S_n \leq x\} \geq P\{Z \geq x\} - \max\left(0.084,\ P\{|Z| \leq \tfrac{a_1}{2}\}\right)$$

Conjecture:

1) $|P\{S_n \leq x\} - P\{Z \leq x\}| \leq P\{Z \in (0, a_1)\} < \frac{1}{\sqrt{2\pi}} a_1$

2) $\underbrace{P\{S_n > 1\} \geq \frac{7}{64}}$ $\qquad$ Oleszkiewicz $\geq \frac{1}{20}$.

$\geq \frac{1}{16}$.

$P\{S_n > 1\} \leq \frac{1}{4}$ $\qquad$ $a_i \neq 1$

S_5

$S_n = a_1\varepsilon_1 + \dots + a_n\varepsilon_n \qquad 0 \le |a_i| \le 1$

$P\{S_n = x\}. \qquad a_i = 1$

Erdös' 45 $\qquad R_n = \overset{\pm 1}{\varepsilon_1} + \dots + \overset{\pm 1}{\varepsilon_n}$

$$P\{S_n = x\} \le P\{R_n \in \{0,1\}\} \le \binom{n}{\lfloor \frac{n}{2} \rfloor} \cdot 2^{-n}$$

$S_{n,x}$: $\{1,2,\dots,n\} = [n]$ $\quad S_n = a_1\varepsilon_1 + \dots + a_n\varepsilon_n$

$$\mathcal{F}_{x} = \{A \subset \{1,\dots,n\} : \sum_{i\in A} a_i - \sum_{j \in A^c} a_j \overset{\ge}{=} x\}$$

$$P\{S_n = x\} = \underbrace{|\mathcal{F}_x|}\cdot 2^{-n} \qquad \text{Erdös 45}$$

$\mathcal{F}$ family of subsets of $[n]$ is called

1) an antichain if $A \notin B$ for all $A, B \in \mathcal{F}$

2) a k-intersecting if $|A \cap B| \ge k$ for all $A, B \in \mathcal{F}$

1) Erdös' 45. $\mathcal{F}_x$ – antichain.

Sperner Lemma' 1928

$$\mathcal{F} = \{A \subset [n] : |A| = \lfloor \tfrac{n}{2} \rfloor\}, \quad |\mathcal{F}| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

middle layer

$$P\{S_n = x\} \le \binom{n}{\lfloor \frac{n}{2} \rfloor} \cdot 2^{-n}.$$

D. Juškevičius, Šileikis '12 $k = \lceil x \rceil$ ceiling

$\mathcal{F}_{x}$ — antichain, k-intersecting.

$$|\mathcal{F}| \le \binom{n}{t} \qquad t = \left\lceil \frac{n+k}{2} \right\rceil \qquad (A \cap B)$$

$0 < a_i \le 1$ $k = \lceil x \rceil$ $R_n = \varepsilon_1 + \dots + \varepsilon_n$

$$P\{S_n = x\} \le \begin{cases} P\{R_{n \wedge k^2} = k\} & n-k \in 2\mathbb{Z} \\ P\{R_{n-1 \wedge k^2} = k\} & n-k \in 2\mathbb{Z}+1 \end{cases}$$

If $0 \le a_i \le 1$ $k = \lceil x \rceil$

$$P\{S_n = x\} \le P\{R_{k^2} = k\}.$$

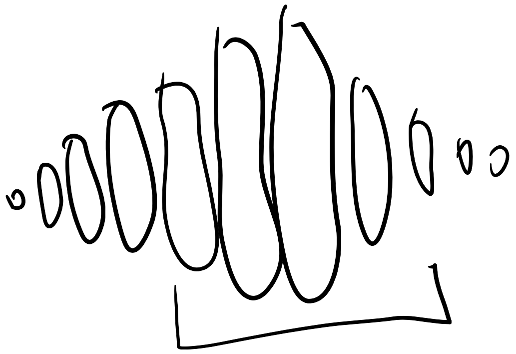
this is optimale

$P\{R_j = k\}$ $j = k^2$.

$P\{S_n \ge x\}$ $\mathcal{F}_{\ge x}$

k-intersecting

Katona '64



$\uparrow t = \frac{k+n}{2}$

$$P\{S_n \geq x\} \leq \begin{cases} P\{R_{\underline{n}} \geq k\} & n-k \in 2\mathbb{Z} \\ P\{R_{\underline{n-1}} \geq k\} & n-k \in 2\mathbb{Z}+1 \end{cases}$$

Erdős 45 $\quad a_1, \dots, a_n \in \mathbb{R}^1$

Kleitman '66 $\quad \underbrace{a_1, \dots, a_n}$ $\in \mathbb{R}^d \qquad \varepsilon_1, \dots, \varepsilon_n \;\pm 1$

$\|a_i\| \leq 1$

$$P\{S_n = x\} \leq P\{R_n = \{0,1\}\} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

D. Juškevičius 2020

$\|a_i\|_2 \leq 1 \quad , a_i \in \mathbb{R}^d$

$$P\{S_n = x\} \leq P\{R_n = k + \{0,1\}\} = \binom{n}{\lceil \frac{n+k}{2} \rceil} \cdot 2^{-n}.$$

$x = (\|x\|_2, 0, 0, \dots, 0) \qquad d = 1$

$S_n = x \Rightarrow S_n^{(1)} = \|x\|_2 \,,\; S_n^{(2)} = 0, \dots, S_n^{(d)} = 0.$

Kyle Luh, D. Xiang 2020

$\|a_i\| \leq 1$ $\boxed{d=1}$

$-1;1$

Conjecture 1. Let $U_1 \dots, U_n$ be independent uniform random variables on the arithmetic progression $\{-m+1, -m+3, \dots, m-3, m-1\}^n$ with $m \geq 3$. Then for all non-zero $x \in \mathbb{R}^d$ and non-zero $v_i \in \mathbb{R}^d$ with $||v_i||_2 \leq 1$ we have that for $m \in 2\mathbb{Z}+1$

$$\underbrace{\mathbb{P}(v_1U_1 + \cdots + v_nU_n = x) \leq \mathbb{P}(U_1 + \cdots + U_n = k)}$$ $k = \lceil x \rceil$

and for $m \in 2\mathbb{Z}$

$$\mathbb{P}(v_1U_1 + \cdots + v_nU_n = x) \leq \mathbb{P}(U_1 + \cdots + U_n = k + \delta_{n,k})$$

$\{0,1\}$

where k is the lower integer part of $||x||_2$.

$a_i = v_i$ | Milner multisets.

$\{-1;1\}^n$