A refinement of the Grothendieck ring of varieties, and specialization of birational types

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joint work with John Christian Ottem

§ Motivation

N - Shinder 2019 : Stable rationality specializes in smooth & proper families in char 0

Tool "Nearby cycles functor" at the level of Grothendieck ring of varieties.

Kontsevich - Tschinkel 2019 : specialization of rationality

Tool Replace the Grothendieck ring by "Burnside ring" of birational types.

Aim present a common refinement.

§ The Grothendieck ring of varieties

F field

$K_0(\mathrm{Var}_F)$ abelian group with

- <u>generators</u> isomorphism classes $[X]$ of F-schemes X of finite type
- "scissor relation" if $Y \hookrightarrow X$ is a closed immersion, then $[X] = [Y] + [X \setminus Y]$.

Ring structure: $[X] \cdot [X'] = [X \times_F X']$

$$1 = [\mathrm{Spec}\, F]$$

$$\mathbb{L} = [\mathbb{A}^1_F].$$

<u>Note</u> $K_0(\mathrm{Var}_F)$ does not detect any non-reduced structure:

$$X_{red} \hookrightarrow X \rightsquigarrow [X] = [X_{red}]$$

<u>Problem</u> How much information about X is retained by $[X] \in K_0(\mathrm{Var}_F)$?

X, X' F-schemes of finite type. When

X, X' F-schemes of finite type. When do we have $[X] = [X']$?

<u>Obvious</u> sufficient condition:

if X, X' are piecewise isomorphic then $[X] = [X']$.

<u>not</u> necessary condition (Borisov 2018)

Z, Z' Calabi-Yau 3 folds not birational

$$\begin{tikzcd} Z \times \mathbb{A}^6_{\mathbb{C}} \arrow[dr, hook] & & Z' \times \mathbb{A}^6_{\mathbb{C}} \arrow[dl, hook'] \\ & W & \end{tikzcd}$$

(15-dimensional)

such that the complements C, C' are piecewise isomorphic

$$[Z \times \mathbb{A}^6_{\mathbb{C}}] = [W] - [C] = [W] - [C']$$

$$= [Z' \times \mathbb{A}^6_{\mathbb{C}}].$$

$= [Z' \times \mathbb{A}^6_{\mathbb{C}}]$.

But $Z \times \mathbb{A}^6_{\mathbb{C}}$ and $Z' \times \mathbb{A}^6_{\mathbb{C}}$ are not even birational.

Open problem: if X, X' are smooth and proper F-schemes such that $[X] = [X']$, does this imply that they are birational?

Larsen-Lunts (char $F = 0$): X, X' are stably birational. ($X \times \mathbb{P}^N_F$ is birational to $X' \times \mathbb{P}^N_F$ for $N \gg 0$)

§ A refinement of the Grothendieck ring

$d \in \mathbb{Z}_{\geq 0}$

$K_0(\mathrm{Var}_F^{\leq d})$ Grothendieck group of F-varieties of dimension $\leq d$.

abelian group with

- generators: isomorphism classes $[X]_d$ of F-schemes X of finite of F-schemes X of finite type and dimension $\leq d$

- Scissor relations: for $Y \hookrightarrow X$ closed,
$$[X]_d = [Y]_d + [X \setminus Y]_d .$$

$$K_0(\mathrm{Var}_F^{\dim}) = \bigoplus_{d \geq 0} K_0(\mathrm{Var}_F^{\leq d})$$

graded ring: $[X]_d \, [X']_e = [X \times_F X']_{d+e}$

$$1 = [\mathrm{Spec}\, F]_0$$
$$\mathbb{L} = [\mathbb{A}^1_F]_1$$

$\tau = [\mathrm{Spec}\, F]_1$ degree shift:
$$\tau^e \, [X]_d = [X]_{d+e}$$

PROP The ring morphism
$$K_0(\mathrm{Var}_F^{\dim}) \longrightarrow K_0(\mathrm{Var}_F) : [X]_d \mapsto [X]$$
forgetting degrees is surjective and has kernel $(\tau - 1)$.

PROP Comparison with the Burnside ring of Kontsevich – Tschinkel.

Bir_F^d = set of birational equivalence classes of integral F-schemes of finite type of dimension d.

$\{X\}$ = birational equivalence class of X

$$\mathrm{Bir}_F = \bigcup_{d \geq 0} \mathrm{Bir}_F^d$$

$$\mathbb{Z}[\mathrm{Bir}_F] = \bigoplus_{d \geq 0} \mathbb{Z}[\mathrm{Bir}_F^d]$$

X F-scheme of finite type

$$\{X\} = \{X_1\} + \dots + \{X_r\} \in \mathbb{Z}[\mathrm{Bir}_F]$$

$X_1, \dots, X_r$ irreducible components of X with induced reduced structure.

$\mathbb{Z}[\mathrm{Bir}_F]$ is a ring wrt

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$$\{X\}\{X'\} = \{X \times_F X'\}$$

Burnside ring of Kontsevich - Tschinkel

$\exists!$ ring morphism

$$K_0(\mathrm{Var}_F^{\dim}) \longrightarrow \mathbb{Z}[\mathrm{Bir}_F]$$

$$[X]_d \longmapsto \{X_1\} + \dots + \{X_r\}$$

$X_1, \dots, X_r$ irreducible components of X of dimension d.

It is surjective and its kernel is $(\mathcal{I})$.

COR X, X' reduced F-schemes of finite type of pure dimension d

Then X, X' are birational iff

$$[X]_d \equiv [X']_d \mod (\mathcal{I})$$

in $K_0(\mathrm{Var}_F^{\dim})$

| in $K_0(\mathrm{Var}_F^{\dim})$.

§ Motivic nearby fiber

C smooth complex curve

$0 \in C(\mathbb{C})$, t local coordinate on C at 0

$X \to C \setminus \{0\}$ smooth and proper family of relative dimension $\leq d$

Question: Can we give a meaningful interpretation to "$\lim_{t \to 0} [X_t]_d$" ?

$\uparrow$

$K_0(\mathrm{Var}_{\mathbb{C}}^{\dim})$

Idea Assume that $X \to C \setminus \{0\}$ extends to a strictly semistable family $\mathfrak{X} \to C$:

* $\mathfrak{X} \to C$ is proper & flat

* $\mathfrak{X} \to \Delta$ is proper & flat
* $\mathfrak{X}$ is smooth over $\mathbb{C}$
* $\mathfrak{X}_0$ is a reduced divisor with strict normal crossings

$$\sum_{i \in I} E_i$$

$\forall \; \emptyset \neq J \subset I$ we set $E_J = \bigcap_{j \in J} E_j$.

Then "$\lim_{t \to 0} [X_t]_d$"

$$= \sum_{\emptyset \neq J \subset I} (-1)^{|J|-1} [E_J]_{d-|J|+1} \cdot [\mathbb{P}_{\mathbb{C}}^{|J|-1}]_{|J|-1}$$

$$\in K_0(\mathrm{Var}_{\mathbb{C}}^{\leq d})$$

Motivation: this is the expression that computes the class of the limit mixed Hodge structure of

limit mixed Hodge structure of $X \to \mathbb{C}\setminus\{0\}$ at 0 in the Grothendieck ring of Hodge structures.
(Steenbrink weight spectral sequence).

For applications, it will be more convenient to work with a more general class of models.

k field of characteristic 0

$$R = \bigcup_{n>0} k[[t^{1/n}]] , \quad K = \bigcup_{n>0} k((t^{1/n})).$$

X smooth proper K-scheme

$\mathfrak{X}$ R-model of X
(proper & flat R-scheme endowed with $\mathfrak{X}_K \to X$)

we say that $\mathfrak{X}$ is <u>strictly toroidal</u> if, Zariski-locally, there exists a smooth morphism

$$\mathfrak{X} \longrightarrow \operatorname{Spec} \frac{R[M]}{(t^q - \chi^m)}$$

where

- M is a toric monoid (ie, monoid of lattice points in a strictly convex rational polyhedral cone)
- $q \in \mathbb{Q}_{>0}$
- $m \in M$ such that $\dfrac{k[M]}{(\chi^m)}$ is reduced.

<u>Eg</u> X defined over $k((t))$

$\mathfrak{X}$ strict normal crossings model over $k[[t]]$

($\mathfrak{X}$ proper & flat over $k[[t]]$, $\mathfrak{X}$ regular, $(\mathfrak{X}_k)_{red}$ strict normal crossings)

normalization of

normalization of

$$\mathfrak{X} \times_{k[[t]]} R \text{ is a strictly toroidal model.}$$

<u>Eg</u> $f_0, \dots, f_r \in k[Z_0, \dots, Z_n]$

general homogeneous polynomials of degrees $d_0, \dots, d_r > 0$ such that

$d_0 = d_1 + \dots + d_r$. Then

$$\mathfrak{X} = \operatorname{Proj} \frac{R[Z_0, \dots, Z_n]}{(t f_0 - f_1 \cdots f_r)}$$

is strictly toroidal.

$\mathfrak{X}$ strictly toroidal

<u>Stratum</u> E of $\mathfrak{X}_k$ = connected component of an intersection of irreducible components of $\mathfrak{X}_k$

$\rightsquigarrow$ Can attach $P(E)$ = projective toric k-variety … $\mathrm{codim}(E)$

$P(E)$ = Toric ... variety of dim = codim(E) reflecting the toroidal structure of $\mathfrak{X}$ at generic point of E.

<u>Thm</u> (Denef - Loeser, Hrushovski - Kazhdan, N - Shinder, N - Ottem)

$\exists!$ ring morphism

$$\mathrm{Vol} : K_0(\mathrm{Var}_K^{\dim}) \longrightarrow K_0(\mathrm{Var}_k^{\dim})$$

such that $\forall X$ smooth and proper K-scheme of $\dim \leq d$,

$\forall \mathfrak{X}$ strictly toroidal R-model,

$$\mathrm{Vol}([X]_d) = \sum_{\substack{E \text{ stratum} \\ \text{in } \mathfrak{X}_k}} (-1)^{\mathrm{codim}(E)} [E]_{d-\mathrm{codim}(E)} \cdot [P(E)]_{\mathrm{codim}(E)}.$$

We have $\mathrm{Vol}(\tau) = \tau$, $\mathrm{Vol}(\amalg) = \amalg$.

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If $\mathfrak{X}$ is <u>smooth</u> over R then

$$\mathrm{Vol}([X]_d) = [\mathfrak{X}_k]_d .$$

<u>Proof</u> weak factorization + log geometry

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§ <u>Applications</u>

(1) <u>Kontsevich - Tschinkel</u> Specialization of birational types

S Noetherian $\mathbb{Q}$- scheme

$\mathfrak{X} \to S$, $\mathfrak{X}' \to S$ smooth and proper.

Then $\{ s \in S \mid \underbrace{\mathfrak{X}_{\bar{s}}}$ is birational to $\mathfrak{X}'_{\bar{s}} \}$

geometric point centered at s

is a countable union of closed subsets of S' (ie, it is closed under specialization).

Pf Reduce to $S' = k[[t]]$.

$$\begin{tikzcd}
K_0(\mathrm{Var}_K^{\dim}) \arrow[r, "\mathrm{Vol}"] \arrow[d, "/(\tau)"'] & K_0(\mathrm{Var}_k^{\dim}) \arrow[d, "/(\tau)"] \\
\mathbb{Z}[\mathrm{Bir}_K] \arrow[r] & \mathbb{Z}[\mathrm{Bir}_k]
\end{tikzcd}$$

$$\{\mathcal{X}_K\} \mapsto \{\mathcal{X}_k\}$$
$$\{\mathcal{X}_K\} = \{\mathcal{X}'_K\} \qquad \{\mathcal{X}_k\} = \{\mathcal{X}'_k\}$$

(2) Obstructions to (stable) rationality

k alg closed

C smooth k-curve

$0 \in C(k)$, t local coordinate on C at 0

$\hat{\mathcal{O}} \simeq k[[t]]$

on Ciato

$$\hat{\mathcal{O}}_{C,0} \cong k[[t]]$$

$X \longrightarrow C \setminus \{0\}$ smooth and proper.

$\mathcal{X} \longrightarrow C$ strictly toroidal extension.

<u>If</u>

$$\sum_{\substack{E \text{ stratum} \\ \text{in } \mathcal{X}_0}} (-1)^{\dim(E)} \{ E \times \mathbb{P}_k^{\operatorname{codim}(E)} \}$$

$$\neq \{ \mathbb{P}_k^{\dim(\mathcal{X}/C)} \} \text{ in } \mathbb{Z}[\mathrm{Bir}_k],$$

then a very general fiber of $X \longrightarrow C \setminus \{0\}$ is <u>irrational</u>.

You get a similar obstruction for stable rationality.

<u>Example</u> (N - Ottem)

Very general quartic fivefold is not stably rational.

Degeneration for the quartic fivefold:

$F \in k[Z_0, \dots, Z_6]$

very general quartic form

wrt the condition that it is symmetric in Z_5, Z_6.

$$\mathfrak{X} = \operatorname{Proj} R[Z_0, \dots, Z_6, \underset{\substack{\uparrow \\ \text{degree } 2}}{y}] \Big/ \begin{pmatrix} yt - Z_5 Z_6, \\ y^2 - F \end{pmatrix}.$$

→ strictly toroidal

→ $\mathfrak{X}_K$ generic fiber: smooth quartic fivefold in $\mathbb{P}^6_K$

→ $\mathfrak{X}_k = E_1 \cup E_2$

$$E_1 \Updownarrow \{Z_5 = y^2 - F = 0\} \cong \{Z_6 = y^2 - F = 0\} \Updownarrow E_2$$

very general

$E_1 \cap E_2$ quartic double

$E_1 \cap E_2$: very general quartic double fourfold

non-stably rational by Hassett-Pirutka-Tschinkel (2019)

$$\mathrm{Vol}(\{\mathcal{X}_K\}) = \{E_1\} + \{E_2\}$$

$$- \{(E_1 \cap E_2) \times \mathbb{P}^1_k\}$$

in $\mathbb{Z}[\mathrm{Bir}_k]$

$$= 2\{E_1\} - \{\underbrace{(E_1 \cap E_2) \times \mathbb{P}^1_k}_{\neq \{\mathbb{P}^5_k\}}\}$$

$$\neq \{\mathbb{P}^5_k\}.$$

$$\rightsquigarrow \{\mathcal{X}_K\} \neq \{\mathbb{P}^5_K\}$$

so $\mathcal{X}_K$ is not rational.

Repeating the argument for

$\mathcal{X} \times_R \mathbb{P}^N_R$, we see that

$\mathcal{X}_K$ is even non-stably rational.

Monodromy $\hat{\mu} = \varprojlim_k \mu_{n/k}$

$$K_0(\mathrm{Var}^{\dim}_{k((t))}) \to K_0^{\hat{\mu}}(\mathrm{Var}^{\dim}_k)$$