ERGODIC THEOREMS FOR FLOWS IN IDEALS OF COMPACT OPERATORS

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Introduction

Advancing Lance's extension of the pointwise ergodic theorem for actions of the group of integers on von Neumann algebras, Conze and Dang-Ngoc[1] and Watanabe[2] studied continuous extensions of Lance's results. In particular, the noncommutative ergodic theorems were established for actions of the semigroups R_+^d and R_+ respectively. The corresponding ergodic theorem for actions of R_+ and with respect to bilaterally almost uniform convergence (in Egorov's sense) was initially considered by Junge and Xu[3]. In particular, they derived that these averages converge bilaterally almost uniformly in any noncommutative L^p-space for $1 \leq p < \infty$ and almost uniformly if $2 \leq p < \infty$.

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[1] Conze J.P., Dang-Ngoc N., Ergodic theorems for noncommutative dynamical systems, *Invent. Math.*, vol. 46, 1978, pp. 1–15.

[2] Watanabe S. Ergodic theorems for dynamical semi-groups on operator algebras, *Hokkaido Math. J.*, vol. 8, 1979, pp. 176–190.

[3] Junge M., Xu Q., Noncommutative maximal ergodic theorems, *J. Amer. Math. Soc.*, vol. 20, iss. 2, 2007, pp. 385–439.

Denote by $\{T_t\}_{t\geq 0}$ a strongly continuous semigroup of Dunford-Schwartz operators acting in fully symmetric ideal $\mathcal{C}_E$ of compact operators, and let

$$A_t(x) = \frac{1}{t}\int_0^t T_s(x)ds, \ x \in \mathcal{C}_E, t > 0,$$

be the corresponding ergodic averages. In the first part of main results of this note we establish that the net $A_t(x)$ converge to some $\widehat{x} \in \mathcal{C}_E$ with respect to the uniform norm $\|\cdot\|_\infty$ as $t \to \infty$ for all $x \in \mathcal{C}_E$. Besides, we show that if $\mathcal{C}_E \neq \mathcal{C}_1$ and $(E, \|\cdot\|_E) \subset c_0$ is separable space, then $\|A_t(x) - \widehat{x}\|_{\mathcal{C}_E} \to 0$ as $t \to \infty$. Note that, along with any Schatten ideals $\mathcal{C}_p, \ 1 \leq p < \infty,$ of compact operators, the family of such fully symmetric ideals $\mathcal{C}_E$ contains many noncommutative counterparts of classical symmetric sequence space, examples of which are given in the last section of this note.

In contrast to ergodic theorems for flows, there have been also established the noncommutative Wiener's local ergodic theorems for actions of the semigroups $\mathbb{R}_+^d$ and $\mathbb{R}_+$ respectively. The corresponding Wiener's local ergodic theorem for actions of $\mathbb{R}_+$ and with respect to bilaterally almost uniform convergence (in Egorov's sense) was initially considered in [4]. Later, Junge and Xu[5] derived that these averages converge bilaterally almost uniformly in any noncommutative L^p-space for $1 \leq p < \infty$ and almost uniformly if $2 \leq p < \infty$.

We consider actions of the semigroup $\mathbb{R}_+$ in Banach ideals E of compact operators (i.e. in noncommutative atomic symmetric spaces) and show that the corresponding ergodic averages $A_t(x)$ converge to x uniformly as $t \to 0^+$ for all $x \in E$. Besides, we show that if E has order continuous norm $\|\cdot\|_E$, then $\|A_t(x) - x\|_E \to 0$ as $t \to 0^+$.

[4] Castrandas E., A local ergodic theorem in semifinite von Neumann algebras, *Algebras Groups and Geometries*, vol. 13, 1966, pp. 71–80.

[5] Junge M., Xu Q., Noncommutative maximal ergodic theorems, *J. Amer. Math. Soc.*, vol. 20, iss. 2, 2007, pp. 385–439.

Let l^∞ (respectively, c_0) be the Banach lattice of bounded (respectively, converging to zero) sequences $\{\xi_n\}_{n=1}^\infty$ of complex numbers equipped with the uniform norm $\|\{\xi_n\}\|_\infty = \sup\limits_{n\in\mathbb{N}} |\xi_n|$, where $\mathbb{N}$ is the set of natural numbers.

If $\xi = \{\xi_n\}_{n=1}^\infty \in l^\infty$, then the non-increasing rearrangement $\xi^* = \{\xi_n^*\}_{n=1}^\infty$ of ξ is defined by

$$\xi_n^* = \inf \left\{ \sup_{n\notin F} |\xi_n| : F \subset \mathbb{N},\ |F| < n \right\}.$$

The Hardy-Littlewood-Polya partial order in the space l^∞ is defined as follows:

$$\xi = \{\xi_n\} \prec\prec \eta = \{\eta_n\} \iff \sum_{n=1}^m \xi_n^* \leq \sum_{n=1}^m \eta_n^* \quad \text{for all} \quad m \in \mathbb{N}.$$

A non-zero linear subspace $E \subset l^\infty$ with a Banach norm $\|\cdot\|_E$ is called a *symmetric* (*fully symmetric*) sequence space if

$$\eta \in E,\ \xi \in l^\infty,\ \xi^* \le \eta^* \text{ (respectivelly, } \xi^* \prec\prec \eta^*) \implies$$

$$\xi \in E \ \text{ and } \ \|\xi\|_E \le \|\eta\|_E.$$

Let $(\mathcal{H}, (\cdot,\cdot))$ be an infinite-dimensional separable Hilbert space over the field $\mathbb{C}$ of complex numbers, and let $(\mathcal{B}(\mathcal{H}), \|\cdot\|_\infty)$ be the C^*-algebra of bounded linear operators in $\mathcal{H}$. Denote by $\mathcal{K}(\mathcal{H})$ (respectively, $\mathcal{F}(\mathcal{H})$) the two-sided ideal of compact (respectively, finite rank) linear operators in $\mathcal{B}(\mathcal{H})$.

Let $\mathcal{B}_h(\mathcal{H}) = \{x \in \mathcal{B}(\mathcal{H}) : x = x^*\}$, $\mathcal{B}_+(\mathcal{H}) = \{x \in \mathcal{B}_h(\mathcal{H}) : x \geq 0\}$, and let $\tau : \mathcal{B}_+(\mathcal{H}) \to [0, \infty]$ be the canonical trace on $\mathcal{B}(\mathcal{H})$, that is, $\tau(x) = \sum\limits_{i=1}^{\infty}(x\varphi_i, \varphi_i)$, $x \in \mathcal{B}(\mathcal{H})$, where $\{\varphi_i\}_{i=1}^{\infty}$ is an orthonormal basis in $\mathcal{H}$. Let $\mathcal{P}(\mathcal{H})$ be the lattice of projections in $\mathcal{H}$. If $\mathbf{1}$ is the identity of $\mathcal{B}(\mathcal{H})$ and $e \in \mathcal{P}(\mathcal{H})$, we will write $e^{\perp} = \mathbf{1} - e$.

Let $x \in \mathcal{B}(\mathcal{H})$, and let $\{e_\lambda\}_{\lambda \geq 0}$ be the spectral family of projections for the absolute value $|x| = (x^*x)^{1/2}$ of x, that is, $e_\lambda = \{|x| > \lambda\}$. If $t > 0$, then the t-th generalized singular number of x, or the non-increasing rearrangement of x, is defined as (see [6])

$$\mu_t(x) = \inf\{\lambda > 0 : \ \tau(e_\lambda^{\perp}) \leq t\}.$$

[6]Fack T., Kosaki H, Generalized s-numbers of τ-measurable operators, *Pacific. J. Math.*, vol. 123, 1986, pp. 269–300.

A non-zero linear subspace $X \subset \mathcal{B}(\mathcal{H})$ with a Banach norm $\|\cdot\|_X$ is called noncommutative symmetric (fully symmetric) if the conditions

$$x \in X,\ y \in \mathcal{B}(\mathcal{H}),\ \mu_t(y) \leq \mu_t(x)\ \forall\ t > 0$$

$$\text{(respectively, } \int_0^s \mu_t(y)dt \leq \int_0^s \mu_t(x)dt\ \forall\ s > 0 \text{ (writing } y \prec\prec x\text{))}$$

imply that $y \in X$ and $\|y\|_X \leq \|x\|_X$.

The spaces $(\mathcal{B}(\mathcal{H}), \|\cdot\|_\infty)$ and $(\mathcal{K}(\mathcal{H}), \|\cdot\|_\infty)$, as well as the classical Banach two-sided ideals

$$\mathcal{C}_p = \{x \in \mathcal{K}(\mathcal{H}) :\ \|x\|_p = \tau(|x|^p)^{1/p} < \infty\},\ 1 \leq p < \infty,$$

are examples of noncommutative fully symmetric spaces.

If $x \in \mathcal{K}(\mathcal{H})$, then $|x| = \sum\limits_{n=1}^{m(x)} s_n(x)p_n$ (if $m(x) = \infty$, the series converges uniformly, i.e. with respect to the uniform norm $\|x\|_\infty = \sup\limits_{\xi\in\mathcal{H}, \|\xi\|_{\mathcal{H}}=1} \|x(\xi)\|_{\mathcal{H}}$), where $\{s_n(x)\}_{n=1}^{m(x)}$ is the set of eigenvalues of the compact operator $|x|$ in the decreasing order, and p_n is the projection onto the eigenspace corresponding to $s_n(x)$. Consequently, the non-increasing rearrangement $\mu_t(x)$ of $x \in \mathcal{K}(\mathcal{H})$ can be identified with the sequence $\{s_n(x)\}_{n=1}^{\infty}$, $s_n(x) \downarrow 0$ (if $m(x) < \infty$, we set $s_n(x) = 0$ for all $n > m(x)$).

Fix an orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ in $\mathcal{H}$. Let p_n be the one-dimensional projection on the subspace $\mathbb{C}\cdot\varphi_n \subset \mathcal{H}$. If $(X, \|\cdot\|_X) \subset \mathcal{K}(\mathcal{H})$ is a symmetric space then the set

$$E(X) = \left\{ \xi = \{\xi_n\}_{n=1}^{\infty} \in c_0 : \ x_\xi = \sum_{n=1}^{\infty} \xi_n p_n \in X \right\}$$

(the series converges uniformly) is a symmetric sequence space with respect to the norm $\|\xi\|_{E(X)} = \|x_\xi\|_X$. Consequently, each symmetric space $(X, \|\cdot\|_X) \subset \mathcal{K}(\mathcal{H})$ generates a symmetric sequence space $(E(X), \|\cdot\|_{E(X)}) \subset c_0$. The converse is also true: every symmetric sequence space $(E, \|\cdot\|_E) \subset c_0$ generates a symmetric space $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E}) \subset \mathcal{K}(\mathcal{H})$ by the following rule (see, for example, [[7], Chapter 3, Section 3.5]):

$$\mathcal{C}_E = \{x \in \mathcal{K}(\mathcal{H}) : \ \{s_n(x)\} \in E\}, \quad \|x\|_{\mathcal{C}_E} = \|\{s_n(x)\}\|_E.$$

[7] Lord S., Sukochev F. and Zanin D., Singular Traces. *Walter de Gruyter GmbH*, Berlin/Boston, 2013.

The pair $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is called a Banach ideal of compact operators (cf. [8], Chapter III]). It is known that $(\mathcal{C}_p, \|\cdot\|_p) = (\mathcal{C}_{l^p}, \|\cdot\|_{\mathcal{C}_{l^p}})$ for all $1 \leq p < \infty$ and $(\mathcal{K}(\mathcal{H}), \|\cdot\|_\infty) = (\mathcal{C}_{c_0}, \|\cdot\|_{\mathcal{C}_{c_0}})$.

Hardy-Littlewood-Polya partial order in the Banach ideal $\mathcal{K}(\mathcal{H})$ is defined by

$$x \prec\prec y,\ x, y \in \mathcal{K}(\mathcal{H}) \iff \{s_n(x)\} \prec\prec \{s_n(y)\}.$$

We say that a Banach ideal $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is fully symmetric, if conditions $y \in \mathcal{C}_E$, $x \in \mathcal{K}(\mathcal{H})$, $x \prec\prec y$ entail that $x \in \mathcal{C}_E$ and $\|x\|_{\mathcal{C}_E} \leq \|y\|_{\mathcal{C}_E}$. It is clear that $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is a fully symmetric ideal if and only if $(E, \|\cdot\|_E)$ is a fully symmetric sequence space.

[8] Gohberg I.C., Krein M.G., Introduction to the theory of linear nonselfadjoint operators, *Translations of Mathematical Monographs*, vol. 18, 1969, Amer. Math. Soc., Providence, RI 02904.

A linear contraction $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is called a Dunford-Schwartz operator (writing $T \in DS$), if $T(\mathcal{C}_1) \subset \mathcal{C}_1$ and $\|T(x)\|_1 \leq \|x\|_1$ for all $x \in \mathcal{C}_1$. We will write $T \in DS^+$ if T is a positive Dunford-Schwartz operator, that is, $T \in DS$ and $T(\mathcal{B}_+(\mathcal{H})) \subset \mathcal{B}_+(\mathcal{H})$.

Any fully symmetric ideal $\mathcal{C}_E$ is an exact interpolation space in the Banach pair $(\mathcal{C}_1, \mathcal{B}(\mathcal{H}))$ (see [[9], Theorem 2.4]). It then follows that $T(\mathcal{C}_E) \subset \mathcal{C}_E$ and $\|T\|_{\mathcal{C}_E \to \mathcal{C}_E} \leq 1$ for all $T \in DS$. In particular, $T(\mathcal{K}(\mathcal{H})) \subset \mathcal{K}(\mathcal{H})$ and the restriction of T on $\mathcal{K}(\mathcal{H})$ is a linear contraction (also denoted by T).

[9] P.G. Dodds P.G., T.K. Dodds T.K. and Pagter B., Fully symmetric operator spaces. *J. Integr. Equat. Oper. Theory*, vol. 15, 1992, pp. 942–972.

The following theorem establishes an extension of any linear contraction $T : \mathcal{C}_1 \to \mathcal{C}_1$ with the property $\|T(x)\|_\infty \leq \|x\|_\infty$ for all $x \in \mathcal{C}_1$ up to the Dunford-Schwarz operator $\widetilde{T} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$.

Theorem 1

Let $T : \mathcal{C}_1 \to \mathcal{C}_1$ be a linear contraction such that $\|T(x)\|_\infty \leq \|x\|_\infty$ for all $x \in \mathcal{C}_1$. Then there exists a unique operator $\widetilde{T} \in DS$ such that $\widetilde{T}(x) = T(x)$ for all $x \in \mathcal{C}_1$, and $\widetilde{T}$ is $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{C}_1)$-continuous.

Let $\mathbb{R}$ be the set of real numbers and let $\mathbb{R}_+ = \{t \in \mathbb{R},\ t \geq 0\}$. In what follows, $\{T_t\}_{t\in\mathbb{R}_+} \subset DS^+$ is a semigroup such that $T_0(x) = x$ for all $x \in \mathcal{B}(\mathcal{H})$.

A semigroup $\{T_t\}_{t\in\mathbb{R}_+}$ is said to be strongly continuous on fully symmetric ideal $\mathcal{C}_1$, if

$$\lim_{t\to s} \|T_t(x) - T_s(x)\|_{\mathcal{C}_1} = 0.$$

for each $x \in \mathcal{C}_1$.

Let $\{T_t\}_{\in R_+} \subset DS^+$ be a strongly continuous semigroup on $\mathcal{C}_1$. Fix $x \in \mathcal{C}_1$. Then for any given $y \in \mathcal{B}(\mathcal{H})$ the function $\varphi_{x,y}(t) = \tau(T_t(x)y)$ is continuous on R_+. Therefore, if μ is the Lebesgue measure on R_+, then the map $U_x : R_+ \to \mathcal{C}_1$ defined as $U_x(t) = T_t(x)$ is weakly μ-measurable [[10], Ch.V, § 4], that is, the complex function $\tau(U_x(t)y)$ is a measurable function on (R_+, μ) for all $y \in \mathcal{B}(\mathcal{H})$ (recall that $(\mathcal{C}_1)^* = \mathcal{B}(\mathcal{H})$ and every $f \in (\mathcal{C}_1)^*$ has the following form $f(x) = \tau(xy)$ for some $y \in \mathcal{B}(\mathcal{H})$). Since, in addition, $U_x(R_+)$ is a separable subset in $\mathcal{C}_1$, Pettis theorem [[10], Ch.V, § 4] entails that the real function $\|U_x(t)\|_1 = \|T_t(x)\|_1$ is μ-measurable on R_+. Since $\|T_t(x)\|_1 \leq \|x\|_1$, it follows that $\|T_s(x)\|_1$ is an integrable function on $[0, t]$ for any $t > 0$. By [[10], Ch.V, § 5, Theorem 1], the function $T_s(x)$ is Bochner μ-integrable on $[0, t]$, $t > 0$.

[10] Yosida K. Functional Analysis, Springer Verlag, Berlin-Göttingen-Heidelberg, 1965.

This means that there is a sequence of measurable step maps

$$\varphi_n : (R_+, \mu) \to \mathcal{C}_1,\ \varphi_n(s) = \sum_{i=1}^{k_n} x_i^{(n)} \chi_{A_i^{(n)}}(s),\ x_i^{(n)} \in \mathcal{C}_1,\ \mu(A_i^{(n)}) < \infty,$$

such that $\| T_s(x) - \varphi_n(s)\|_1 \to 0$ as $n \to \infty$ for μ-almost everywhere $s \in R_+$. In this case, the Bochner integral is defined by the following equality

$$\int_0^t T_s(x) ds = \| \cdot \|_1 - \lim_{n\to\infty} \sum_{i=1}^{k_n} x_i^{(n)} \mu(A_i^{(n)}) \in \mathcal{C}_1.$$

Therefore, for any $x \in \mathcal{C}_1$ and $t > 0$ there exists the Bochner integral $A_t(x) = \frac{1}{t}\int_0^t T_s(x) ds \in \mathcal{C}_1$. It is clear that $\|A_t(x)\|_1 \leq \|x\|_1$ and $\|A_t(x)\|_\infty \leq \|x\|_\infty$ for all $x \in \mathcal{C}_1$.

Consequently, by Theorem 1, there exists a unique operator $\widetilde{A_t} \in DS$ such that $\widetilde{A_t}(x) = A_t(x)$ for all $x \in \mathcal{C}_1$, and $\widetilde{A_t}$ is $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{C}_1)$ - continuous. Below, the operator $\widetilde{A_t}$ is denoted by A_t.

Theorem 2

Let $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ be a fully symmetric Banach ideal, and let $\{T_t\}_{t \in R_+} \subset DS^+$ be a strongly continuous semigroup on $\mathcal{C}_1$. Then
(i). (IET). Given $x \in \mathcal{C}_E$, the averages $A_t(x)$ converge to some $\widehat{x} \in \mathcal{C}_E$ with respect to the uniform norm $\|\cdot\|_\infty$ as $t \to \infty$.
(ii). (MET). If $\mathcal{C}_E \neq \mathcal{C}_1$ and $(E, \|\cdot\|_E) \subset c_0$ is separable space, then $\|A_t(x) - \widehat{x}\|_{\mathcal{C}_E} \to 0$ as $t \to \infty$.

Applications to Orlicz and Lorentz Banach ideals

In this section we give applications of Theorem 2 to Orlicz and Lorentz ideals of compact operators.

1. Let Φ be an *Orlicz function*, that is, $\Phi : [0, \infty) \to [0, \infty)$ is left-continuous, convex, increasing and such that $\Phi(0) = 0$ and $\Phi(u) > 0$ for some $u \neq 0$. Let

$$l_\Phi(N) = \left\{ \xi = \{\xi_n\}_{n=1}^{\infty} \in l^\infty : \sum_{n=1}^{\infty} \Phi\left(\frac{|\xi_n|}{a}\right) < \infty \text{ for some } a > 0 \right\}$$

be the corresponding *Orlicz sequence space*, and let $\|\xi\|_\Phi = \inf\left\{ a > 0 : \sum_{n=1}^{\infty} \Phi \right.$ (
be the *Luxemburg norm* in $l_\Phi(N)$. It is well-known that $(l_\Phi(N), \|\cdot\|_\Phi)$ is a fully symmetric sequence space.

If $\Phi(u) > 0$ for all $u \neq 0$, then $\sum_{n=1}^{\infty} \Phi\left(\frac{1}{a}\right) = \infty$ for each $a > 0$. Hence $\mathbf{1} = \{1, 1, ...\} \notin l_\Phi(N)$ and $l_\Phi(N) \subset c_0$. If $\Phi(u) = 0$ for all $0 \leq u < u_0$, then $\mathbf{1} \in l_\Phi$ and $l_\Phi(N) = l^\infty$.

It is said that an Orlicz function Φ satisfies (Δ_2)-*condition at* 0 if there exist $u_0 \in (0, \infty)$ and $k > 0$ such that $\Phi(2u) < k \cdot \Phi(u)$ for all $0 < u < u_0$. It is well known that an Orlicz function Φ satisfies (Δ_2)-condition at 0 if and only if $(l_\Phi(N), \|\cdot\|_\Phi)$ is a separable space.

We also note that $l_\Phi(N) = l^1$ as sets if and only if $\limsup\limits_{u \to 0} \frac{\Phi(u)}{u} > 0$ [[11], Chapter 16, §16.2].

[11] Rubshtein B.A., Grabarnik G.Ya., Muratov M.A., and Pashkova Yu.S. Foundations of Symmetric Spaces of Measurable Functions. Lorentz, Marcinkiewicz and Orlicz Spaces. Springer International Publishing, Switzerland, 2016.

Set $\mathcal{C}_\Phi = \mathcal{C}_{l_\Phi(N)}$ and $\|x\|_\Phi = \|x\|_{\mathcal{C}_{l_\Phi(N)}}$, $x \in \mathcal{C}_\Phi$. Theorem 2 yield the following.

Theorem 3

Let Φ be an Orlicz function, let $\{T_t\}_{t\in R_+} \subset DS^+$ be a strongly continuous semigroup on $\mathcal{C}_1$. Then
(i). If $\Phi(u) > 0$ for all $u > 0$ and $x \in \mathcal{C}_\Phi$, then the averages $A_t(x)$ converge to some $\widehat{x} \in \mathcal{C}_\Phi$ with respect to the uniform norm as $t \to \infty$;
(ii). If $\lim\limits_{u\to 0} \frac{\Phi(u)}{u} = 0$ and the Orlicz function Φ satisfy (Δ_2)-condition at 0, then $\|A_t(x) - \widehat{x}\|_\Phi \to 0$ as $t \to \infty$.

2. Let ψ be a concave function on $[0, \infty)$ with $\psi(0) = 0$ and $\psi(t) > 0$ for all $t > 0$, and let

$$\Lambda_\psi(N) = \left\{ \xi = \{\xi_n\}_{n=1}^{\infty} \in l^\infty : \ \|\xi\|_{\Lambda_\psi} = \sum_{n=1}^{\infty} \xi_n^*(t)(\psi(n) - \psi(n-1)) < \infty \right\}$$

be the corresponding *Lorentz sequence space*. It is well-known that $(\Lambda_\psi(N), \|\cdot\|_{\Lambda_\psi})$ is a fully symmetric sequence space (see, for example, [[12] , Part III, Ch.9, § 9.1]); in addition, if $\psi(\infty) = \infty$, then $\mathbf{1} \notin \Lambda_\psi(N)$ and $\Lambda_\psi(N) \subset c_0$. If $\psi(\infty) < \infty$, then $\mathbf{1} \in \Lambda_\psi(N)$ and $\Lambda_\psi(N) = l^\infty$. In addition, the space $(\Lambda_\psi(N), \| \cdot \|_{\Lambda_\psi})$ is separable if and only if $\psi(+0) = 0$ and $\psi(\infty) = \infty$ [[12], Ch.9, §9.3, Theorem 9.3.1].

[12] Rubshtein B.A., Grabarnik G.Ya., Muratov M.A., and Pashkova Yu.S. Foundations of Symmetric Spaces of Measurable Functions. Lorentz, Marcinkiewicz and Orlicz Spaces. Springer International Publishing, Switzerland, 2016.

It is clear that $\lim\limits_{t\to\infty}\frac{\psi(t)}{t}>0$ if and only if the norms $\|\cdot\|_{\Lambda_\psi}$ and $\|\cdot\|_1$ are equivalent on $\Lambda_\psi(N)$, i.e. the equality $\Lambda_\psi(N)=l^1$ (as sets) is true.

Set $\mathcal{C}_{\Lambda_\psi}=\mathcal{C}_{\Lambda_\psi(N)}$ and $\|x\|_{\Lambda_\psi}=\|x\|_{\mathcal{C}_{\Lambda_\psi(N)}},\ x\in\mathcal{C}_{\Lambda_\psi}$. Theorem 2 imply the following.

Theorem 4

Let ψ be a concave function on $[0,\infty)$ with $\psi(0)=0,\ \psi(t)>0$ for all $t>0$, and let $\{T_t\}_{t\in R_+}\subset DS^+$ be a strongly continuous semigroup on $\mathcal{C}_1$. Then
(i). If $\psi(\infty)=\infty$, then the averages $A_t(x)$ converge to some $\widehat{x}\in\mathcal{C}_{\Lambda_\psi}$ with respect to the uniform norm;
(ii). If $\psi(+0)=0,\ \psi(\infty)=\infty$ and $\lim\limits_{t\to\infty}\frac{\psi(t)}{t}=0$, then $\|A_t(x)-\widehat{x}\|_{\Lambda_\psi}\to 0$ as $t\to\infty$.

The following Theorem is a version of local individual ergodic theorem for fully symmetric ideals.

Theorem 5

Let $\mathcal{C}_E$ be a fully symmetric ideal, and let $\{T_t\}_{t \in R_+} \subset DS^+$ be a strongly continuous semigroup on $\mathcal{C}_1$. If $x \in \mathcal{C}_E$, then $\|A_t(x) - x\|_\infty \to 0$ as $t \to 0$.

A fully symmetric ideals $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is said to have order continuous norm if

$$\|x_n\|_{\mathcal{C}_E} \downarrow 0 \quad \text{whenever} \quad 0 \leq x_n \in \mathcal{C}_E \quad \text{and} \quad x_n \downarrow 0.$$

It is well known that the fully symmetric ideals $(\mathcal{C}_p, \|\cdot\|_p)$, $1 \leq p < \infty$, $(\mathcal{C}_\Phi, \|\cdot\|_\Phi)$ (the Orlicz function Φ satisfy (Δ_2)-condition at 0), $\mathcal{C}_{\Lambda_\psi}, \|\cdot\|_{\Lambda_\psi}$ $(\psi(+0) = 0,\ \psi(\infty) = \infty)$ have order continuous norms.

The following Theorem is a version of local mean ergodic theorem for fully symmetric ideals.

Theorem 6

Let $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ be a fully symmetric ideal with order continuous norm, and let $\{T_t\}_{t\in R_+} \subset DS^+$ be a strongly continuous semigroup on $\mathcal{C}_1$. If $x \in \mathcal{C}_E$, then $\|A_t(x) - x\|_{\mathcal{C}_E} \to 0$ as $t \to 0$.

Thank you for attention!