On commuting difference operators

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[1]. G.S. Mauleshova, A.E. Mironov *Commuting difference operators of rank two*, Russian Math. Surveys, **70**:3 (2015), 557–559.

[2]. G.S. Mauleshova, A.E. Mironov *One–point commuting difference operators of rank 1*, Doklady Mathematics, **93**:1 (2016), 62–64.

[3]. G.S. Mauleshova, A.E. Mironov *One–point commuting difference operators of rank 1 and their relation with finite–gap Schrodinger operators*, Doklady Mathematics. **97**:1 (2018), 62–64.

[4]. G.S. Mauleshova, A.E. Mironov *Positive one–point commuting difference operators*, Integrable Systems and Algebraic Geometry, Part of London Mathematical Society Lecture Note Series, 1, Cambridge University Press, 2020, 395–412

$$L_n = \partial_x^n + \sum_{i=0}^{n-1} u_i(x)\partial_x^i, \quad L_m = \partial_x^m + \sum_{i=0}^{m-1} v_j(x)\partial_x^i.$$

Lemma (Schur, 1905)

If $L_n L_m = L_m L_n$ and $L_n L_s = L_s L_n$ $(L_n \neq const,)$ then

$$L_m L_s = L_s L_m.$$

Lemma (Burchnall, Chaundy, 1923)

If $L_nL_m = L_mL_n$, then there exist a non–trivial polynomial $Q(z,w)$ of two commuting variables such that $Q(L_n, L_m) = 0$.

Example

$$L_2 = \frac{d^2}{dx^2} - \frac{2}{x^2}, \qquad L_3 = \frac{d^3}{dx^3} - \frac{3}{x^2}\frac{d}{dx} + \frac{3}{x^3},$$

$$L_2^3 = L_3^2, \quad Q(z,w) = z^3 - w^2.$$

Spectral curve

$$\Gamma = \{(z,w) \in \mathbb{C}^2 : Q(z,w) = 0\}.$$

If $L_n\psi = z\psi, \quad L_m\psi = w\psi$, then $(z,w) \in \Gamma$.
rank of L_n and L_m is

$$\Gamma = \dim\{\psi : L_n\psi = z\psi, \quad L_m\psi = w\psi\}.$$

Baker – Akhiezer function $\psi(x, P)$

Spectral data of Krichever

$$\{\Gamma, q, k^{-1}, \gamma_1, \dots \gamma_g\}$$

Γ is algebraic curve, $q \in \Gamma,\ k^{-1}$ is a local parameter near q, $\gamma_1, \dots, \gamma_g \in \Gamma$. The Baker – Akhiezer function has the property:
1. $\psi = e^{kx}(1 + \frac{f(x)}{k} + \dots)$
2. on $\Gamma \backslash q$ the BA–function ψ is meromorphic with the poles at $\gamma_1, \dots \gamma_g$
For $\gamma_1, \dots \gamma_g$ in general position the BA – function ψ exists and unique.

Let $f(P)$ be meromorphic function on Γ with a unique pole at q of order n

$$f = k^n + c_{n-1}k^{n-1} + \cdots + c_0 + \frac{c_{-1}}{k} + \cdots$$

$$\partial_x^n \psi + u_{n-1}(x)\partial_x^{n-1} + \cdots + u_0(x)\psi = f\psi + e^{kx}(O(\frac{1}{k})).$$

From the uniqueness of BA–function it follows that

$$L_n \psi(x, P) = f(P)\psi(x, P).$$

Let $g(P)$ be a meromorphic function with unique pole at q of order m

$$L_m \psi(x, P) = g(P)\psi(x, P).$$

We have

$$(L_n L_m - L_m L_n)\psi(x, P) = 0 \Rightarrow L_n L_m = L_m L_n.$$

Example

$$\Gamma = \mathbb{C}/\{2w\mathbb{Z} + 2w'\mathbb{Z}\}, \quad q = 0,$$

$$\psi = e^{-x\zeta(z)} \frac{\sigma(z+x)}{\sigma(x)\sigma(z)},$$

$$(\partial_x^2 - 2\wp(x))\psi(x,z) = \wp(z)\psi(x,z),$$

$$(\partial_x^3 - 3\wp(x)\partial_x - \frac{3}{2}\wp'(x))\psi(x,z) = \frac{1}{2}\wp'(z)\psi(x,z).$$

$$L_2 = \partial_x^2 - u(x), \quad u(x+\tau) = u(x), \quad L_2\psi = \lambda\psi, \; \lambda \in \mathbb{R}.$$

$\psi(x)$ — Bloch function, if

$$\psi(x+\tau) = e^{ip}\psi(x), \; p \in \mathbb{R}.$$

$\lambda \in$ spectrum, if there is Bloch function corresponding to λ. Operator L_2 is called **finite–gap** if the spectrum consist of finite number of intervals.

Lame operator

$$\Lambda = \{2\omega\mathbb{Z} + 2\omega'\mathbb{Z}\}$$

$$L_2 = \partial_x^2 - g(g+1)\wp(x+\omega).$$

L_2 is finite-gap (E.L. Ince).

Theorem (S.P. Novikov)
If $[L_2, L_{2g+1}] = 0$, then L_2 is finite–gap operator.

The inverse statement was proved by B.A. Dubrovin.
Spectral curve.

$$w^2 = z^{2g+1} + c_{2g} z^{2g} + \cdots + c_0.$$

Example

Treibich–Verdier operator

$$-\partial_x^2 + \sum_{i=0}^{3} a_i(a_i+1)\wp(x+\omega_i),$$

were ω_i are the half periods.

We denote by $\tilde{L}_k,\ \tilde{L}_s$ the operators of orders $k = N_- + N_+$ and $s = M_- + M_+$

$$\tilde{L}_k = \sum_{j=-N_-}^{N_+} u_j(n)T^j, \qquad \tilde{L}_s = \sum_{j=-M_-}^{M_+} v_j(n)T^j,$$

where $n \in \mathbb{Z},\ N_\pm, M_\pm \geq 0,\ u_{N_+}(n) = v_{M_+}(n) = 1$, T is the shift operator

$$Tf(n) = f(n+1), \qquad f : \mathbb{Z} \to \mathbb{C}.$$

If two difference operators $\tilde{L}_k$ and $\tilde{L}_s$ commute, then there is a nonzero polynomial $F(z,w)$ such that $F(\tilde{L}_k, \tilde{L}_s) = 0$. The polynomial F defines the *spectral curve* of the pair $\tilde{L}_k,\ \tilde{L}_s$

$$\Gamma = \{(z,w) \in \mathbb{C}^2 | F(z,w) = 0\}.$$

The common eigenvalues are parametrized by the spectral curve

$$\tilde{L}_k\psi = z\psi, \quad \tilde{L}_s\psi = w\psi, \quad (z, w) \in \Gamma.$$

The dimension of the space of common eigenfunctions of the pair $\tilde{L}_k, \ \tilde{L}_s$ for fixed eigenvalues is called the *rank* of $\tilde{L}_k, \ \tilde{L}_s$

$$l = \dim\{\psi : \tilde{L}_k\psi = z\psi, \quad \tilde{L}_s\psi = w\psi, \quad (z, w) \in \Gamma.\}$$

Any commutative ring of difference operators is isomorphic to a ring of meromorphic functions on a spectral curve with m poles. Such operators are said to be m–*point operators*.

Spectral data for two–point operators of rank 1 were found by I. M. Krichever and D. Mumford. Eigenfunctions of two–point operators of rank 1 (Baker–Akhiezer functions) can be found explicitly in terms of the theta function of the spectral curves. Spectral data for one–point operators of rank $l > 1$ were obtained by I. M. Krichever and S. P Novikov.

Take the following spectral data

$$S = \{\Gamma, \gamma_1, \dots, \gamma_g, q, k^{-1}, P_n\},$$

where Γ is the Riemannian surface of genus g, $\gamma = \gamma_1 + \cdots + \gamma_g$ is the non–special divisor on Γ, $q \in \Gamma$ is the marked point, k^{-1} is the local parameter nearby q, $P_n \in \Gamma$ is the set of points, $n \in \mathbb{Z}$.

Theorem 1

There is a unique function of the Baker–Akhiezer $\psi(n,P),\ n\in\mathbb{Z},\ P\in\Gamma$, which has the following properties.

1. The divisor of zeros and poles ψ has the form

$$\gamma_1(n)+\ldots+\gamma_g(n)+P_1+\ldots+P_n-\gamma_1-\ldots-\gamma_g-nq,$$

if $n\geq 0$ and has the form

$$\gamma_1(n)+\ldots+\gamma_g(n)-P_{-1}-\ldots-P_n-\gamma_1-\ldots-\gamma_g-nq,$$

if $n<0$.

2. In the neighborhood q function ψ has the expansion

$$\psi=k^n+O(k^{n-1}).$$

For any meromorphic functions $f(P)$ and $g(P)$ on Γ with a single pole order m and s in q with expansions

$$f(P) = k^m + O(k^{m-1}), \quad g(P) = k^s + O(k^{s-1})$$

there are only difference operators

$$\tilde{L}_m = T^m + u_{m-1}(n)T^{m-1} + \ldots + u_0(n),$$

$$\tilde{L}_s = T^s + v_{s-1}(n)T^{s-1} + \ldots + v_0(n)$$

such that

$$\tilde{L}_m\psi = f(P)\psi, \quad \tilde{L}_s\psi = g(P)\psi.$$

The operators $\tilde{L}_m, \ \tilde{L}_s$ commute.

Consider the hyperelliptic spectral curve Γ defined by the equation

$$w^2 = F_g(z) = z^{2g+1} + c_{2g}z^{2g} + \ldots + c_0,$$

for the base point we take $q = \infty$. Let $\psi(n, P)$ be the corresponding to the Baker–Akhiezer function. Then there exist commuting operators $\tilde{L}_2, \ \tilde{L}_{2g+1}$ such that

$$\tilde{L}_2\psi = \big((T + U_n)^2 + W_n\big)\psi = z\psi, \quad \tilde{L}_{2g+1}\psi = w\psi.$$

Theorem 2

The relation

$$\tilde{L}_2 - z = (T + U_n + U_{n+1} + \chi(n,P))(T - \chi(n,P)),$$

holds, where

$$\chi = \frac{\psi(n+1,P)}{\psi(n,P)} = \frac{S_n}{Q_n} + \frac{w}{Q_n},$$

$$S_n(z) = -U_n z^g + \delta_{g-1}(n) z^{g-1} + \ldots + \delta_0(n), \quad Q_n = -\frac{S_{n-1} + S_n}{U_{n-1} + U_n}.$$

The functions U_n, W_n, S_n satisfy the equation

$$F_g(z) = S_n^2 + (z - U_n^2 - W_n) Q_n Q_{n+1}.$$

Corallary 1

The functions $S_n(z), U_n, W_n$ satisfy the equation

$$(U_n + U_{n+1})(S_n - S_{n+1}) - (z - U_n^2 - W_n)Q_n +$$

$$(z - U_{n+1}^2 - W_{n+1})Q_{n+2} = 0.$$

Theorem 3

In the case of an elliptic spectral curve Γ, given by the equation

$$w^2 = F_1(z) = z^3 + c_2 z^2 + c_1 z + c_0,$$

operator $\tilde{L}_2$ type

$$\tilde{L}_2 = (T + U_n)^2 + W_n,$$

where

$$U_n = -\frac{\sqrt{F_1(\gamma_n)} + \sqrt{F_1(\gamma_{n+1})}}{\gamma_n - \gamma_{n+1}}, \quad W_n = -c_2 - \gamma_n - \gamma_{n+1},$$

γ_n — arbitrary function parameter, commute with some operator

$$\tilde{L}_3 = L_2(T + U_n) - \gamma_{n+2} T - (\sqrt{F_1(\gamma_n)} + U_n \gamma_n).$$

Theorem 4

The operator

$$L_2^\sharp = (T + r_1\cos(n))^2 + \frac{r_1^2\sin(g)\sin(g+1)}{2\cos^2(g+\frac{1}{2})}\cos(2n),$$

$r_1 \neq 0$ commutes with a operator $L_{2g+1}^\sharp$.

Theorem 5
The operator

$$L_2^{\checkmark} = (T + \alpha_2 n^2 + \alpha_0)^2 - g(g+1)\alpha_2^2 n^2, \quad \alpha_2 \neq 0$$

commutes with a operator $L_{2g+1}^{\checkmark}$.

We consider one–point ε–difference operators of rank 1 having the form

$$L_k = \frac{T_\varepsilon^k}{\varepsilon^k} + u_{k-1}(x,\varepsilon)\frac{T_\varepsilon^{k-1}}{\varepsilon^{k-1}} + \ldots + u_0(x,\varepsilon),$$

where T_ε is the operator of shift by ε, i.e., $T_\varepsilon f(x) = f(x+\varepsilon)$, $f : \mathbb{C} \to \mathbb{C}$. Let Γ be the hyperelliptic spectral curve determined by the equation

$$w^2 = F_g(z) = z^{2g+1} + c_{2g}z^{2g} + \ldots + c_0,$$

and let $q = \infty$. Suppose that the operator

$$L_2 = \frac{T_\varepsilon^2}{\varepsilon^2} + A(x,\varepsilon)\frac{T_\varepsilon}{\varepsilon} + B(x,\varepsilon)$$

commutes with L_{2g+1}.

Theorem 6

The relation

$$L_2 - z = \left(\frac{T_\varepsilon}{\varepsilon} + A(x,\varepsilon) + \chi(x+\varepsilon,\varepsilon,z)\right)\left(\frac{T_\varepsilon}{\varepsilon} - \chi(x,\varepsilon,z)\right),$$

holds, where

$$\chi = \frac{S(x,\varepsilon,z)}{Q(x,\varepsilon,z)} + \frac{w}{Q(x,\varepsilon,z)},$$

$$S(x,\varepsilon,z) = -\delta_g(x,\varepsilon)z^g + \delta_{g-1}(x,\varepsilon)z^{g-1} + \ldots + \delta_0(x,\varepsilon),$$

$$A(x,\varepsilon) = \delta_g(x,\varepsilon) + \delta_g(x+\varepsilon,\varepsilon),$$

$$Q(x,\varepsilon,z) = -\frac{S(x-\varepsilon,\varepsilon,z) + S(x,\varepsilon,z)}{A(x-\varepsilon,\varepsilon)}.$$

The functions A, B, S, Q satisfy the equation

$$F_g(z) = S^2(x,\varepsilon,z) + (z - B(x,\varepsilon))Q(x,\varepsilon,z)Q(x+\varepsilon,\varepsilon,z).$$

The functions A, B, S, Q satisfy the equation

$$A(x,\varepsilon)(S(x,\varepsilon,z) - S(x+\varepsilon,\varepsilon,z)) - (z - B(x,\varepsilon))Q(x,\varepsilon,z) +$$
$$(z - B(x+\varepsilon,\varepsilon))Q(x+2\varepsilon,\varepsilon,z) = 0.$$

Theorem 7

The operator

$$L_2 = (\frac{T_\varepsilon}{\varepsilon} + \delta_1(x,\varepsilon))^2 + W(x,\varepsilon),$$

where

$$\delta_1(x,\varepsilon) = -\frac{\sqrt{F_1(\gamma(x,\varepsilon))} + \sqrt{F_1(\gamma(x+\varepsilon,\varepsilon))}}{\gamma(x,\varepsilon) - \gamma(x+\varepsilon,\varepsilon)},$$

$$W(x,\varepsilon) = -c_2 - \gamma(x,\varepsilon) - \gamma(x+\varepsilon,\varepsilon),$$

and $\gamma(x,\varepsilon)$ is an arbitrary functional parameter, commutes with the operator L_3. The spectral curve of the pair $L_2,\ L_3$ is determined by the equation

$$w^2 = F_1(z) = z^3 + c_2 z^2 + c_1 z + c_0.$$

Example

If $\gamma(x,\varepsilon) = \wp(x-\varepsilon)$ in Theorem 7, then

$$L_2 = \frac{T_\varepsilon^2}{\varepsilon^2} + (-2\zeta(\varepsilon) - \zeta(x-\varepsilon) + \zeta(x+\varepsilon))\frac{T_\varepsilon}{\varepsilon} + \wp(\varepsilon),$$

Moreover,

$$L_2 = \partial_x^2 - 2\wp(x) + O(\varepsilon).$$

Consider the function $A_g(x, \varepsilon)$ defined as follows. We put

$$A_1 = -2\zeta(\varepsilon) - \zeta(x - \varepsilon) + \zeta(x + \varepsilon)$$

and

$$A_2 = -\frac{3}{2}\big(\zeta(\varepsilon) + \zeta(3\varepsilon) + \zeta(x - 2\varepsilon) - \zeta(x + 2\varepsilon)\big),$$

where $\zeta(x)$ is the Weierstrass function. Next, for odd $g = 2g_1 + 1$, we put

$$A_g = A_1 \prod_{k=1}^{g_1} \left(1 + \frac{\zeta(x - (2k+1)\varepsilon) - \zeta(x + (2k+1)\varepsilon)}{\zeta(\varepsilon) + \zeta((4k+1)\varepsilon)}\right),$$

and for even $g = 2g_1$, we put

$$A_g = A_2 \prod_{k=2}^{g_1} \left(1 + \frac{\zeta(x - 2k\varepsilon) - \zeta(x + 2k\varepsilon)}{\zeta(\varepsilon) + \zeta((4k-1)\varepsilon)}\right).$$

Theorem 8
The operator

$$L_2 = \frac{T_\varepsilon^2}{\varepsilon^2} + A_g(x,\varepsilon)\frac{T_\varepsilon}{\varepsilon} + \wp(\varepsilon)$$

commutes with L_{2g+1}. Moreover,

$$L_2 = \partial_x^2 - g(g+1)\wp(x) + O(\varepsilon).$$

Treibich–Verdier operator

$$-\partial_x^2 + \sum_{i=0}^{3} a_i(a_i+1)\wp(x+\omega_i),$$

were ω_i are the half periods and

$$(\wp'(x))^2 = 4\wp^3(x) - g_1\wp(x) - g_2.$$

Let

$$\left(\left(\frac{1}{\sin^2(x)}\right)'\right)^2 = 4\left(\frac{1}{\sin^2(x)}\right)^2\left(\frac{1}{\sin^2(x)} - 1\right).$$

Special case

$$\partial_x^2 - \left(\frac{6}{\sin^2(x)} + \frac{2}{\cos^2(x)}\right).$$

Consider the function

$$A_2 = (-\frac{3}{4}\varepsilon + \frac{9}{8})\big(2\cot(\varepsilon) + \tan(\varepsilon - x) + \tan(\varepsilon + x)\big)\times$$

$$\big(\cot(\varepsilon) + \cot(3\varepsilon) - \cot(2\varepsilon - x) - \cot(2\varepsilon + x)\big).$$

Theorem 9

The operator

$$L_2 = \frac{T_\varepsilon^2}{\varepsilon^2} + A_2(x,\varepsilon)\frac{T_\varepsilon}{\varepsilon} + \wp(\varepsilon)$$

commutes with L_5. Moreover,

$$L_2 = \partial_x^2 - \left(\frac{6}{\sin^2(x)} + \frac{2}{\cos^2(x)}\right) + O(\varepsilon).$$