Hyperbolic CR singularities

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Surfaces with CR singularity

Surface with CR singularity : real analytic surface $M \subset (\mathbb{C}^2, 0)$:

$$M : z_2 = z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2) + O^3(z_1, \bar{z}_1), \quad \gamma \geq 0.$$

r.a. perturbation of the *Bishop quadric* $Q_\gamma : z_2 = z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2)$
$\gamma \in \mathbb{R}^+$ — *Bishop invariant*

If $\gamma \neq \frac{1}{2}$, the origin is an isolated *Cauchy-Riemann singularity* :

- $\forall\, p \neq 0$, "$\mathbb{C} \not\subset T_pM$" (ie. totally real at $p \neq 0$)
- $T_0M = \{z_2 = 0\}$

M is said to be :

- *elliptic* si $0 \leq \gamma < \frac{1}{2}$
- *hyperbolic* if $\gamma > \frac{1}{2}$
- *parabolic* if $\gamma = \frac{1}{2}$

Geometry near an elliptic CR singularity

Questions

- *Holomorphic Flattening* : is $\phi(M) \subset \mathrm{Im}(z_2) = 0$?
- *What is the local hull of holomorphy ?*

Answers throught :

- Normal form of M with respect to holomorphic change of coordinates near the origin.

Normalization near an elliptic CR singularity

Theorem (Moser-Webster 1983)

If $0 < \gamma < \frac{1}{2}$, there exists a holomorphic change of variables near the origin such that M reads

$$x_2 = z_1 \bar{z}_1 + (\gamma + \delta x_2^s)(z_1^2 + \bar{z}_1^2), \quad y_2 = 0, \quad z_2 = x_2 + \mathrm{i} y_2$$

avec $\delta = \pm 1$ si $s \in \mathbb{N}^*$ ou $\delta = 0$ si $s = \infty$.

Complexification of M

Complexification of M: $(z_1, z_2, \bar{z}_1, \bar{z}_2) \leftarrow (z_1, z_2, w_1, w_2) =: (z, w) \in \mathbb{C}^4$

$$\mathcal{M} \subset \mathbb{C}^4 : \begin{cases} z_2 = z_1 w_1 + \gamma(z_1^2 + w_1^2) + H(z_1, w_1) \\ w_2 = z_1 w_1 + \gamma(z_1^2 + w_1^2) + \bar{H}(w_1, z_1) \end{cases}$$

Canonical projections : $\pi_1(z, w) = z$ et $\pi_2(z, w) = w$ for $(z, w) \in \mathcal{M}$.

According to Moser-Webster, π_1 et π_2 are 2-1 branched coverings:

- $\pi_2(z, w) = \pi_2(z', w')$, $(z, w), (z', w') \in \mathcal{M}$
  $\Longrightarrow$ unique solution $(z', w') =: \tau_1(z, w)$ with $z' \neq z$

$$\rightsquigarrow (z - z')\,(w + \gamma(z + z')) + H(z, w) - H(z', w) = 0$$

Moser-Webster involutions

$\rightsquigarrow$ pair of holomorphic involutions: pour $\gamma > 0$

$$\tau_1 : \begin{cases} z_1' = -z_1 - \frac{1}{\gamma} w_1 + \underbrace{h_1(z_1, w_1)}_{\mathrm{ord}_0 \geq 2} \\ w_1' = w_1 \end{cases} \quad ------- \quad \tau_1 \circ \tau_1 = Id$$

$$\tau_2 : \begin{cases} z_1' = z_1 \\ w_1' = -\frac{1}{\gamma} z_1 - w_1 + h_2(z_1, w_1) \end{cases} \quad ------- \quad \tau_2 \circ \tau_2 = Id$$

$$\tau_2 = \rho \tau_1 \rho, \quad \rho(z, w) := (\bar{w}, \bar{z})$$

Proposition (Moser-Webster 1983)

Holomorphic classification of surface $\mathcal{M} \in \mathbb{C}^4 \leftrightsquigarrow$ Holomorphic classification of (τ_1, τ_2)

Remark. Normal form of $M \subset \mathbb{C}^2 \leftrightsquigarrow$ Normal form of (τ_1, τ_2).

Appropriate coordinates

$$\tau_1 : \begin{cases} \xi' = \lambda\eta + \text{h.o.t.} \\ \eta' = \lambda^{-1}\xi + \text{h.o.t.} \end{cases}, \quad \tau_2 : \begin{cases} \xi' = \lambda^{-1}\eta + \text{h.o.t.} \\ \eta' = \lambda\xi + \text{h.o.t.} \end{cases},$$

$$\sigma := \tau_1 \circ \tau_2 : \begin{cases} \xi' = \lambda^2\xi + \text{h.o.t.} \\ \eta' = \lambda^{-2}\eta + \text{h.o.t.} \end{cases},$$

λ is a root of $\gamma\lambda^2 - \lambda + \gamma = 0$

Remark

- elliptic surface M, $0 < \gamma < \frac{1}{2} \Longrightarrow \lambda = \bar{\lambda}$ and $|\lambda| \neq 1$
  — origin is an hyperbolic fixed point of τ_1, τ_2 et $\tau_1 \circ \tau_2$
- hyperbolic surface M, $\gamma > \frac{1}{2} \Longrightarrow |\lambda| = 1$
  — origin is an elliptic fixed point of τ_1, τ_2 et $\sigma = \tau_1 \circ \tau_2$

Normal forms of involutions

Theorem (Moser-Webster 1983, formal normal form)

Assume: λ not a root of unity
Conclusion : exists a unique formal normalized transformation ψ s.t.

$$\psi^{-1}\circ\tau_1\circ\psi : \begin{cases} \xi' = \Lambda(\xi\eta)\eta \\ \eta' = \Lambda^{-1}(\xi\eta)\xi \end{cases}, \quad \psi^{-1}\circ\tau_2\circ\psi : \begin{cases} \xi' = \Lambda^{-1}(\xi\eta)\eta \\ \eta' = \Lambda(\xi\eta)\xi \end{cases},$$

where $\Lambda(t)\in\mathbb{C}[[t]]$. s.t. $\Lambda(t)=\bar{\Lambda}(t)$ (elliptic case) ou $\Lambda(t)\cdot\bar{\Lambda}(t)=1$ (hyperbolic case).

Theorem (Moser-Webster 1983, Convergence in elliptic case)

If $\lambda=\bar{\lambda}$ and $|\lambda|\neq 1$, then Λ and ψ are holomorphic on a neighborhood of the origin.

$\Longrightarrow$ Holomorphic equivalence of inital manifold M to NF manifold

Non exceptional hyperbolic CR singularity

$|\lambda| = 1$ not a root of unity (*non exceptionnal*).
Moser-Webster $\rightsquigarrow$ normalizing transformation ψ might not converge at the origin: no holomorphic equivalence to a normal form and even, no holomorphic flattening.

Theorem (Gong 1994: non exceptional degenerate case)

Assumptions:

1. $|\lambda| = 1$ *and* λ *satisfies* *diophantine condition:*

$$|\lambda^n - 1| > \frac{c}{n^\delta}$$

2. τ_1 *et* τ_2 *are* *formally linearizable* *(i.e.* $\Lambda(\xi\eta) = \lambda$*; i.e.* M *formally equivalent to the quadric),*

Then, ψ *is* *holomorphic* *in a neighborhood of the origin :* M *is holomorphically equivalent to the quadric.*

Non degenerate hyperbolic CR singularity surface

$$\tau_1 : \left\{ \begin{array}{l} \xi' = \lambda\eta + \text{h.o.t.} \\ \eta' = \lambda^{-1}\xi + \text{h.o.t.} \end{array} \right. , \quad \tau_2 : \left\{ \begin{array}{l} \xi' = \lambda^{-1}\eta + \text{h.o.t.} \\ \eta' = \lambda\xi + \text{h.o.t.} \end{array} \right.$$

$$\lambda := e^{\frac{i}{2}\alpha}, \frac{\alpha}{\pi} \in \mathbb{R} \setminus \mathbb{Q}, \quad \Lambda(\xi\eta) = \lambda + \sum_{n\geq 1} \tilde{c}_n(\xi\eta)^n.$$

Theorem (S.-Zhao 2020)

Assume $\Lambda(\xi\eta) \neq \lambda$. If $r > 0$ is small enough, there exists a "asymptotic full measure" parameters set $\mathcal{O}_r \subset]-r^2, r^2[$ s.t. $\forall\ \omega \in \mathcal{O}_r$, $\exists\ \mu_\omega \in \mathbb{R}$ and an holomorphic transformation Ψ_ω, Withney smooth in ω, on $\mathcal{C}_\omega^r := \{\xi\eta = \omega,\ |\xi|, |\eta| < r\}$ with $\Psi_\omega \circ \rho = \rho \circ \Psi_\omega$ and s.t. , on $\mathcal{C}_\omega^r$,

$$\Psi_\omega^{-1} \circ \tau_1 \circ \Psi_\omega : \left\{ \begin{array}{l} \xi' = e^{\frac{i}{2}\mu_\omega}\eta \\ \eta' = e^{-\frac{i}{2}\mu_\omega}\xi \end{array} \right. , \quad \Psi_\omega^{-1} \circ \tau_2 \circ \Psi_\omega : \left\{ \begin{array}{l} \xi' = e^{-\frac{i}{2}\mu_\omega}\eta \\ \eta' = e^{\frac{i}{2}\mu_\omega}\xi \end{array} \right. ,$$

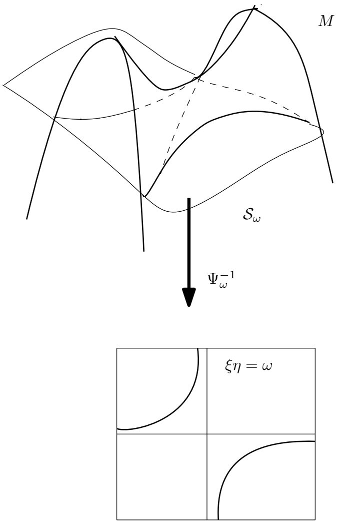
Remark $\Psi_\omega(\mathcal{C}_\omega^r)$ is a holomorphic invariant set of τ_i's and their restriction is conjugated to a linear map . "Asymptotic full measure"$= \frac{|\mathcal{O}_r|}{2r^2} \xrightarrow{r\to 0} 1$.

Geometric consequences

Theorem (S.-Zhao 2020)

Let M be a surface with an hyberbolic CR singularity at the origin which is non exceptionnal and not formally equivalent to a quadric. Then: there exist a neighborhood of the origin and a Whitney smooth family of holomorphic curves $\{\mathcal{S}_\omega\}_{\omega\in\mathcal{O}}$ which intersects M along holomorphic hyperbolas : 2 real curves which are simultaneously holomorphically mapped to the two branches of the hyperbolas $\xi\eta = \omega$, $\omega \neq 0$.

Intersection of M by holomorphic curves



Intersection of M by holomorphic curves (elliptic case)

Intersection of M along 2 real lines at the origin

Theorem (Klingenberg 1985)

Let M be a surface with an hyberbolic CR singularity at the origin with $\lambda = e^{\frac{i}{2}\alpha}$ satisfiying the diophantine condition above. Then , there exists a unique holomorphic curve intersecting M along 2 totally real curves interesecting transversally at the origin.

Remarque. These are the "traces" of 2 lines $\xi\eta = 0$.

Idea of the proof : KAM (Kolmogorv-Arnold-Moser) scheme

KAM (Kolmogorv-Arnold-Moser) scheme — formulation

Pair of holomorphic involutions

$$\tau_1 : \begin{cases} \xi' = e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\eta + p(\xi,\eta) \\ \eta' = e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\xi + q(\xi,\eta) \end{cases}, \quad \tau_2 = \rho \circ \tau_1 \circ \rho,\ \rho : (\xi,\eta) \mapsto (\bar{\xi},\bar{\eta})$$

restricted to a "crown" $\mathcal{C}^r_{\omega,\beta} := \{|\xi\eta - \omega| < \beta,\ |\xi|, |\eta| < r\}$, $\omega \in \mathcal{O} \subset]-r^2, r^2[$.

- Non degeneracy: $\exists\ s \in \mathbb{N}^*, \forall\ \omega \in \mathcal{O},\ |\alpha^{(s)}(\omega)| > \frac{1}{2}$,

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- Non degeneracy: $\exists\ s \in \mathbb{N}^*, \forall\ \omega \in \mathcal{O},\ |\alpha^{(s)}(\omega)| > \frac{1}{2}$,
- Smallness: unique decomposition :

$$p(\xi,\eta) = p^{0,0}(\xi\eta) + \sum_{l\geq 1} p^{l,0}(\xi\eta)\xi^l + \sum_{j\geq 1} p^{0,j}(\xi\eta)\eta^j,$$

$$\|p\|_{\omega,\beta,r} := \sum_{l\cdot j=0} \sup_{|\xi\eta-\omega|<\beta} |p^{l,j}(\xi\eta)| r^{l+j} < \varepsilon, \quad \|q\|_{\omega,\beta,r} < \varepsilon$$

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- Non degeneracy: $\exists\ s\in\mathbb{N}^*, \forall\ \omega\in\mathcal{O}$, $|\alpha^{(s)}(\omega)|>\frac{1}{2}$,
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$$\|p\|_{\omega,\beta,r} := \sum_{l\cdot j=0}\sup_{|\xi\eta-\omega|<\beta}|p^{l,j}(\xi\eta)|r^{l+j}<\varepsilon, \quad \|q\|_{\omega,\beta,r}<\varepsilon$$

- Skew condition: $\|e^{\frac{i}{2}\alpha(\xi\eta)}\eta q + e^{-\frac{i}{2}\alpha(\xi\eta)}\xi p\|_{\omega,\beta,r} < \varepsilon^{\frac{3}{2}}$

KAM scheme — transformation

$$r \rightsquigarrow r_+ < r, \quad \mathcal{O} \rightsquigarrow \mathcal{O}_+ \subset \mathcal{O}, \quad \tau_i \rightsquigarrow \tau_{i,+} = \psi^{-1} \circ \tau_i \circ \psi$$

defined on a smaller "crown".
For $0 < r_+ < r$, $\beta_+ = \beta^{\frac{5}{4}}$, $\varepsilon_+ = \varepsilon^{\frac{5}{4}}$ $(\beta \sim \varepsilon^{\frac{1}{40s}})$

$$\mathcal{O}_+ := \left\{ \omega \in \mathcal{O} \cap] - r_+^2, r_+^2[: |e^{\mathrm{i}k\alpha(\omega)} - 1| > \varepsilon^{\frac{1}{64s}}, \ \ 1 \leq |k| \lesssim |\ln \varepsilon| \right\}$$

Using "approximate solutions of cohomological equations", one builds a transformation

$$\psi(\xi,\eta) = (\mathrm{Id} + \mathcal{U})(\xi,\eta) = \begin{pmatrix} \xi + u(\xi,\eta) \\ \eta + v(\xi,\eta) \end{pmatrix}$$

s.t. $\psi \circ \rho = \rho \circ \psi$, $\|u\|_{\omega,\beta_+,r_+}, \|v\|_{\omega,\beta_+,r_+} < \varepsilon^{\frac{49}{50}}$ and

$$\|\eta u + \xi v\|_{\omega,\beta_+,r_+} < \varepsilon^{\frac{61}{32}} + \varepsilon^{-\frac{1}{16}} \|e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\eta q + e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\xi p\|_{\omega,\beta,r} < \varepsilon^{\frac{5}{4}}.$$

KAM scheme — new perturbation

$$\rightsquigarrow \psi^{-1} \circ \tau_1 \circ \psi : \begin{cases} \xi' = e^{\frac{i}{2}\alpha_+(\xi\eta)}\eta + p_+(\xi,\eta) \\ \eta' = e^{-\frac{i}{2}\alpha_+(\xi\eta)}\xi + q_+(\xi,\eta) \end{cases} \text{on } \mathcal{C}^{r_+}_{\omega,\beta_+}$$

$$\sup_{|\xi\eta-\omega|<\beta} |\alpha_+(\xi\eta) - \alpha(\xi\eta)| < \varepsilon.$$

New size: for $\varepsilon_+ = \varepsilon^{\frac{5}{4}}$,

$$\|p_+\|_{\omega,\beta_+,r_+}, \|q_+\|_{\omega,\beta_+,r_+} < \varepsilon^{\frac{61}{32}} + \varepsilon^{-\frac{1}{16}} \|e^{\frac{i}{2}\alpha(\xi\eta)}\eta q + e^{-\frac{i}{2}\alpha(\xi\eta)}\xi p\|_{\omega,\beta,r} < \varepsilon_+$$

New skew condition:

$$\|e^{\frac{i}{2}\alpha_+(\xi\eta)}\eta q_+ + e^{-\frac{i}{2}\alpha_+(\xi\eta)}\xi p_+\|_{\omega,\beta_+,r_+} < \varepsilon^{\frac{61}{32}} < \varepsilon^{\frac{15}{8}} = \varepsilon_+^{\frac{3}{2}}.$$

KAM scheme — cancelation

Remarque. In the computation of $e^{\frac{\mathrm{i}}{2}\alpha_+(\xi\eta)}\eta q_+ + e^{-\frac{\mathrm{i}}{2}\alpha_+(\xi\eta)}\xi p_+$, one needs to consider the part

$$\begin{pmatrix} (e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta+\eta u+\xi v+uv)} - e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)})\eta \\ (e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta+\eta u+\xi v+uv)} - e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta)})\xi \end{pmatrix} \text{of} \begin{pmatrix} p_+ \\ q_+ \end{pmatrix}:$$

$$\begin{aligned}
& e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\eta \cdot (e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta+\eta u+\xi v+uv)} - e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta)})\xi \\
& + e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\xi \cdot (e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta+\eta u+\xi v+uv)} - e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)})\eta \\
= \; & (\xi\eta) \cdot \left(-\frac{\mathrm{i}}{2}\alpha'(\xi\eta)(\eta u+\xi v+uv)\right) + (\xi\eta) \cdot \left(\frac{\mathrm{i}}{2}\alpha'(\xi\eta)(\eta u+\xi v+uv)\right) \\
& + O^2(\eta u+\xi v+uv) \\
= \; & O^2(\eta u+\xi v+uv)
\end{aligned}$$

Main CR singularity results

Moser-Webster: $M_n \hookrightarrow \mathbb{C}^n$ with smallest dim. of complex tangent at 0 : $p = 1$.

- Smaller dimension : $M_m \hookrightarrow \mathbb{C}^n$, $m < n \rightsquigarrow$ Coffman [Houston '04, Pacific '06, Memoirs AMS'10]
- Higher degeneracy $p \geq 1 \rightsquigarrow$ Gong-S. [Invent. '16+ JDG '19]
- $\gamma = 0$: No involution $\rightsquigarrow$ Holomorphic classification Huang-Yin [Invent. '09]
- Flattening in higher dimension $\rightsquigarrow$ Huang-Yin [Math. Ann.'16+Adv. Math.'17], Huang-Fan [GAFA '18]
- Hyperbolic exceptional case (λ root of unity)$\rightsquigarrow$ on-going work with Martin Klimes.