# Application of Painlevé 3 equations to dynamical systems on 2-torus modeling Josephson junction

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# Plan

**Plan of the talk:**

- Rotation number, phase-lock areas.
- Josephson effect and dynamical system modeling it.
- Properties of phase-lock areas.
- Main results: theorems A and B.
- Ideas of the theorem A proof: application of isomonodromic deformations and Paineleve 3 equation.
- Some open problems.

Alexey Glutsyuk and YB
On families of constrictions in model of overdamped Josephson junction and Painlevé 3 equation
**https://arxiv.org/abs/2011.07839v4**

# Circle homeomorphisms and rotation number

$S^1 := \mathbb{R}/2\pi\mathbb{Z}$ oriented circle; $f : S^1 \to S^1$ **a positive homeomorphism.**

**Definition** of the **Poincaré rotation number** $\rho(f) \in \mathbb{R}/\mathbb{Z}$.
**H.Poincaré.** *Sur les courbes définies par les équations différentialles.* J. Math. Pures App. **I 167** (1885).

$\pi : \mathbb{R} \to S^1 :=$ the **universal covering projection:** $x \mapsto x(\operatorname{mod} 2\pi)$.
$F :=$ a **continuous lifting** of $f$ to $\mathbb{R}$.
Defined up to postcomposition
with translation by $2\pi m$, $m \in \mathbb{Z}$.

$\rho(F)(x) := \lim_{k\to+\infty} \frac{F^k(x)}{2\pi k}$.

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{F} & \mathbb{R} \\ \downarrow \pi & & \downarrow \pi \\ S^1 & \xrightarrow{f} & S^1 \end{array}$$

**The limit exists and is independent ! of** $x$.
$\rho(f) := \rho(F)(\operatorname{mod} \mathbb{Z})$.

**Henri Poincaré (1854–1912)**

# Rotation numbers: first examples

$\rho(F)(x) := \lim_{k\to+\infty} \frac{F^k(x)}{2\pi k}$.

$\rho(f) := \rho(F)(\text{mod}\,\mathbb{Z})$.

1) $f(x) = x + 2\pi a \quad => \quad \rho(f) = a(\text{mod}\,\mathbb{Z})$ for every $a \in \mathbb{R}$.
2) $f(x)$ has a **fixed point** $<=>$ $\rho(f) = 0(\text{mod}\,\mathbb{Z})$.
3) $f(x)$ has a ***q*-periodic point** $<=>$ $\rho(f) = \frac{p}{q}(\text{mod}\,\mathbb{Z})$ for some $p \in \mathbb{Z}$.

A **family** $f = f(x,u)$ of positive circle homeomorphisms $S^1 \to S^1$, $x \mapsto f(x,u)$;
$u :=$ the **parameter**; $u$ lies in a domain $U \subset \mathbb{R}^n$.

The **rotation number function:** $\rho(u) := \rho(f(.,u))$: $U \to S^1 = \mathbb{R}/\mathbb{Z}$.

**Main definition** (a version of **Arnold tongues**)
**Phase-lock areas:** subsets $\{u \in U \mid \rho(u) = r\} \subset U$ with **non-empty interiors.**

**Example.** Consider a family $f(x,u)$ of **circle diffeomorphisms.**
Let for some $u_0 \in U$ the diffeomorphism $f_{u_0}(x) = f(x,u_0)$ have a $q$-periodic point $x_0$ with $(f_{u_0}^q)'(x_0) \neq 1$. Then

$$\rho(u_0) = \frac{p}{q} \text{ for some } p = p(u_0) \in \mathbb{Z};$$

$$L_{\frac{p}{q}} := \{u \in U \mid \rho(u) = \frac{p}{q}\} \text{ is a } \textbf{phase-lock area.}$$

This example illustrates a **classical fact: stability** of the above $q$-periodic point.

## Rotation number and Poincaré map of flow on 2-torus

Differential equation on 2-torus $\mathbb{T}^2 := \mathbb{R}^2_{(\phi,\tau)}/2\pi\mathbb{Z}^2$

$$\begin{cases} \dot{\phi} = f(\phi,\tau) \\ \dot{\tau} = 1 \end{cases} \qquad f \in C^1, \quad f(\phi+2\pi,\tau) = f(\phi,\tau+2\pi) = f(\phi,\tau).$$

Solution $\phi = \phi(\tau) \in \mathbb{R}$. Uniquely defined by $\phi_0 = \phi(0)$.
If $\phi(\tau)$ is a solution, then $\phi(\tau + 2\pi)$ is also a solution.

# Flow on torus

$$\begin{cases}\dot{\phi} = f(\phi,\tau) \\ \dot{\tau} = 1\end{cases} \qquad f \in C^1, \quad f(\phi+2\pi,\tau) = f(\phi,\tau+2\pi) = f(\phi,\tau).$$

**Rotation number of flow:**

$$\rho := \lim_{k\to+\infty} \frac{\phi(2\pi k)}{2\pi k} \in \mathbb{R}.$$

The **Poincaré map** of the circle $S^1 = \mathbb{R}_\phi/2\pi\mathbb{Z} = S^1_\phi \times \{0\} \subset \mathbb{T}^2_{(\phi,\tau)}$:

$$h : S^1_\phi \to S^1_\phi, \ (\phi(0),0) \mapsto (\phi(2\pi),0).$$

**Properties of the rotation number $\rho$:**

1) Independent on choice of the initial condition $\phi_0 = \phi(0)$.

2) the rotation number **of flow mod $\mathbb{Z}$,** equals
**the rotation number** of the **Poincaré map.**

# Phase-lock areas in family of dynamical systems on 2-torus

$$\begin{cases} \dot{\phi} = f(\phi, \tau; u) \\ \dot{\tau} = 1 \end{cases} ; \quad u \in U \subset \mathbb{R}^n \text{ is the parameter.} \qquad \text{(B)}$$

**The rotation number function** $\rho : U \to \mathbb{R}$**:**

$$\rho(u) := \lim_{k \to +\infty} \frac{\phi(2\pi k; u)}{2\pi k} \in \mathbb{R}.$$

**Phase-lock areas:** subsets $\{u \in U \mid \rho(u) = r\} \subset U$ with **non-empty interiors.**

**Example.** Let for $u = u_0$ (B) have an attracting (or repelling) **periodic orbit** (<=> $h$ have attracting (repelling) $q$-periodic point). Then

**Period of the orbit** $= 2\pi q,\ q \in \mathbb{Z}$; $\qquad \rho(u_0) = \dfrac{p}{q} \in \mathbb{Q}$ for some $p \in \mathbb{Z}$;

$L_{\frac{p}{q}} := \{u \in U \mid \rho(u) = \dfrac{p}{q}\}$ is a **phase-lock area.**

**Goal:** study phase-lock areas in a model of **Josephson junction (superconductivity).**

Phenomenon, when the **electric resistance** becomes **exactly zero.**

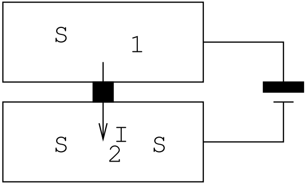
Occurs in **some** metals at temperature $\mathbb{T} < \mathbb{T}_{crit}$.

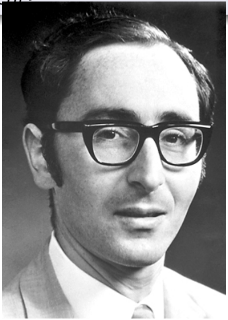
The resistance **jumps to zero,** once $\mathbb{T}$ becomes **less than** $\mathbb{T}_{crit}$.

## The Josephson effect

*Let two superconductors $S_1$, $S_2$ be separated by a very narrow dielectric or a very narrow metal layer, thickness $\leq 10^{-5}$cm ($<<$ distance in Cooper pair).*

*There exists a* **supercurrent $I_S$ through the dielectric.**



**Born in UK in 1940.**

**Nobel Prize in 1973**

*for "the discovery of tunnelling supercurrents".*

Supercurrent is carried by coherent **Cooper pairs** of electrons.

# Josephson effect

**Quantum mechanics.** State of $S_j$: wave function $\Psi_j = |\Psi_j| e^{i\chi_j}$;

$\chi_j$ are the **phases**, $\phi := \chi_1 - \chi_2$ is the **phase difference.**

**The first Josephson relation:** $I_S = I_c \sin\phi$, $I_c \equiv const$.

**Josephson voltage relation:** $V(t) = \frac{\hbar}{2e}\dot{\phi}$.

## General mathematical model

$$\varepsilon_1 \frac{d^2\phi}{dt^2} + \varepsilon_2 \frac{d\phi}{dt} + \sin\phi = f(t); \quad \varepsilon_1, \varepsilon_2 = const.$$

In physical works this equation is called the **Langevin equation.**

**Our main, special "overdamped" case:** $\varepsilon_1 = 0$, $\varepsilon_2 = 1$, $f(t) = B + A\cos\omega t$.

$$\frac{d\phi}{dt} = -\sin\phi + B + A\cos\omega t.$$

$$\frac{d\phi}{dt} = -\sin\phi + B + A\cos\omega t.$$

**V.M.Buchstaber, O.V.Karpov, S.I.Tertychnyi (2004):** the model is equivalent to a family of dynamical systems on two-torus $\mathbb{T}^2_{(\phi,\tau)} = \mathbb{R}^2/2\pi\mathbb{Z}^2$, $\tau = \omega t$:

$$\begin{cases} \dot{\phi} = \frac{1}{\omega}(-\sin\phi + B + A\cos\tau) \\ \dot{\tau} = 1 \end{cases}. \qquad (1)$$

**Our main problem:** Describe the **rotation number** $\rho$ of the flow (1) as a **function on the space** $(B, A; \omega)$ **with fixed** $\omega > 0$**.**

**Phase-lock area:**= a level set $\{\rho = r\}$ if it has a **non-empty interior.**

**Quantization effect (Buchstaber, Karpov, Tertychnyi, 2010):** **phase-lock areas** exist only for **integer rotation values.**

$$\begin{cases} \dot{\phi} = \frac{1}{\omega}(-\sin\phi + B + A\cos\tau) \\ \dot{\tau} = 1 \end{cases} . \quad (1)$$

Many works concern systems (1)

A subfamily of these systems occurred in the work by **Yu.S.Ilyashenko, J.Guckenheimer** (2001) from the slow-fast system point of view.

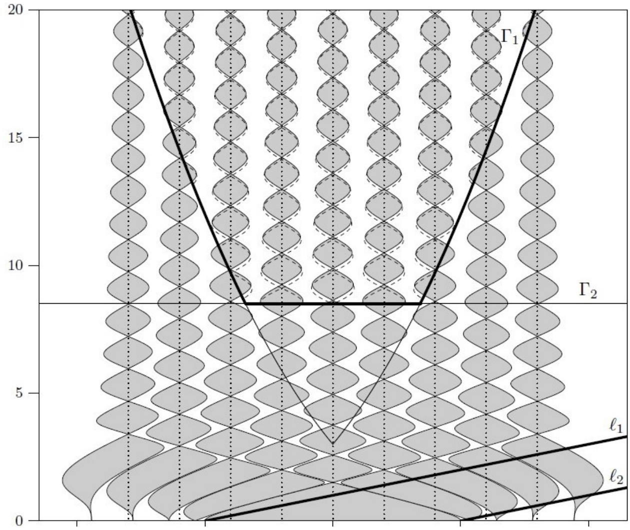
In a paper by **R. Foote, M. Levi, S. Tabachnikov** (2013) it was noticed that family (1) arises

in the investigation of some systems with non-holonomic connections.

In Prytz planimeter model and in cinematics of bicycle moving

Analogous equation describes the observed direction to a given point at infinity while moving along a geodesic in the hyperbolic plane.

# Boundaries of phase-lock areas



## Boundaries of phase-lock areas

**1).** There exist functions $\psi_{r,\pm}(A)$ **analytic in** $A \in \mathbb{R}$ such that the **boundary** $\partial L_r$ **is the union of their graphs:**

$$\partial L_r = \partial_+ L_r \cup \partial_- L_r, \ \partial_\pm L_r := \{B = \psi_{r,\pm}(A)\}.$$

$\psi_{r,\pm}(A)$ have asymptotics of **Bessel type** $r\omega \pm J_r(-\frac{A}{\omega}) + O(\frac{\ln |A|}{A})$, as $A \to \infty$.

**Observed** by **S.Shapiro, A.Janus, S.Holly** (1964);
Confirmed **numerically** by **V.M.Buchstaber. O.V.Karpov, S.I.Tertychnyi** (2005).

**Proved** by **A.V.Klimenko** and **O.L.Romaskevich** (2014).

**2)** Each $L_r$ is an infinite chain (garland) of domains going to infinity, separated by points of intersection $\partial_+ L_r \cap \partial_- L_r$.

Observed numerically by **Buchstaber, Karpov, Tertychnyi.**
**Proved** by **Klimenko** and **Romaskevich** (2014).

The separation points with $A \neq 0$ are called **constriction points (constrictions).**

The separation points of $L_r$ with $A = 0$ exist for $r \neq 0$, are called **growth points** and their abscissas $B_r$ satisfy the equation $B_r^2 - r^2\omega^2 = 1$.

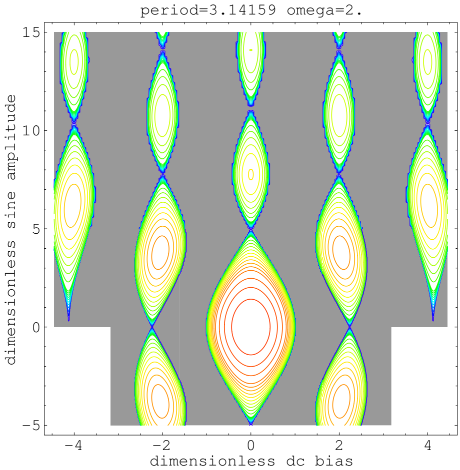
The phase-lock area $L_0$ has no growth points; it intersects the $B$-axis by $[-1, 1]$.

We present pictures of phase-lock areas for different values of $\omega$. They are **symmetric** with respect to coord. axes: $(B, A) \mapsto (-B, A)$; $(B, A) \mapsto (B, -A)$.

Taking into account these symmetries, we present only upper parts of the pictures.

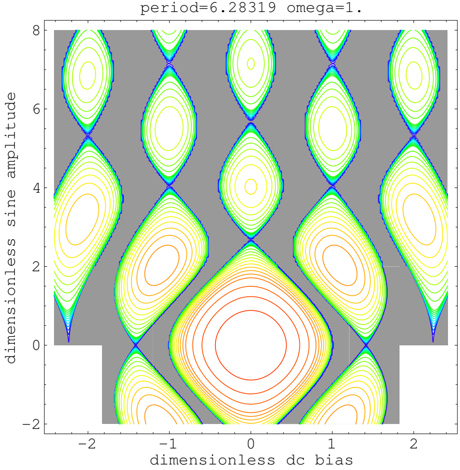
# Phase-lock areas for $f(t) = B + A\cos\omega t$, $\omega = 2$

Each phase-lock area $L_r$ is an infinite chain (garland) of domains going to infinity, separated by points of intersection $\partial_+ L_r \cap \partial_- L_r$.
The separation points with $A \neq 0$ are called **constriction points (constrictions).**



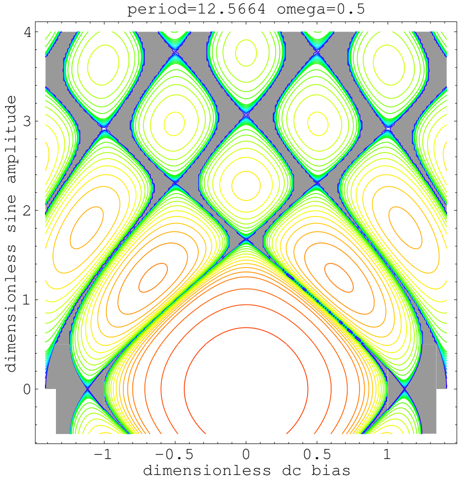
# Phase-lock areas for $f(t) = B + A\cos\omega t$, $\omega = 1$

- infinitely many **constrictions** in every phase-lock area.



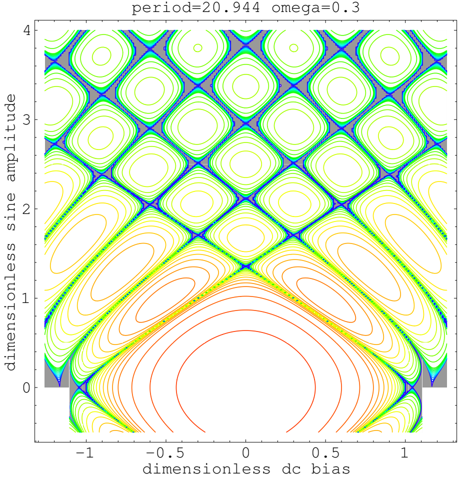
# Phase-lock areas for $f(t) = B + A\cos\omega t, \omega = 0.5$

- infinitely many **constrictions** in every phase-lock areas.



# Phase-lock areas for $f(t) = B + A\cos\omega t$, $\omega = 0.3$

- infinitely many **constrictions** in every phase-lock areas.



**Constrictions:**$=$ the separation points in $L_r$, $r \in \mathbb{Z}$, with $A \neq 0$.

The constrictions $\mathcal{A}_{r,1}, \mathcal{A}_{r,2}, \mathcal{A}_{r,3}, \dots$ with $A > 0$ are ordered by their $A$-coordinates.

.

**Theorem 1 (quantization of constrictions) (Y.B., A.Glutsyuk):** *All the constrictions* $\mathcal{A}_{r,k}$ *lie in the line* $\Lambda_r := \{B = r\omega\} :=$ **the axis** *of the area* $L_r$.

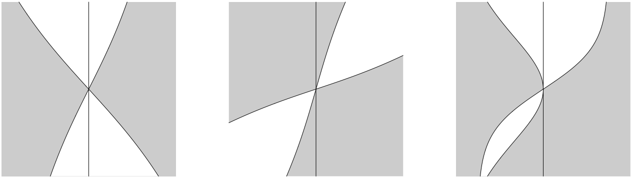
This is a confirmation of an experimental fact that was earlier discovered in numerical simulations (**Tertychnyi, Filimonov, Kleptsyn, Schurov, 2011**).

**Previous results (Filimonov, Glutsyuk, Kleptsyn, Schurov, 2014)** Each constriction $\mathcal{A}_{r,k}$ lies in a line $\{B = \ell\omega\}$, where $\ell \in [0, r]$ and $\ell \equiv r(mod 2\mathbb{Z})$.

Theorem 1 holds for $\omega \geq 1$.

**Definition.** **A priori possible types of constrictions:**

**Positive** **Negative** **Neutral**



**Theorem 2 (Y.B., A.Glutsyuk).** Each constriction is **positive.**

**Known:** For every constriction $\ell := \frac{B}{\omega} \in \mathbb{Z}$
**(Filimonov, Glutsyuk, Kleptsyn, Schurov, 2014).**

**Definition.** **Ghost constriction:**= a constriction satisfying one of two conditions:
- either the **rotation number** $\rho \neq \ell$,
- or the constriction is **non-positive.**

**Theorems 1 and 2** state that there are **no ghost constrictions.**

# Plan of proof of main results: no ghost constrictions

**Known properties** of constrictions: $\ell := \frac{B}{\omega} \in \mathbb{Z}$, $\ell \in [0, \rho]$, $\ell \equiv \rho (\mathrm{mod}\, 2)$.

**Ghost constriction**:= either non-positive, or $|\ell| < |\rho|$

**Theorem A.** For every given $\ell \in \mathbb{Z}$ each constriction $(\ell\omega, A; \omega)$ can be deformed to another constriction of the same type, $\ell$, $\rho$, with **arbitrarily small $\omega$.**

$=>$ **Ghost constrictions** are deformed to **ghost constrictions** with small $\omega$.

Proof is based on equivalent description of the model by **complex linear equations on $\overline{\mathbb{C}}$** and studying their **isomonodromic deformations.**

**Theorem B.** For every given $\ell \in \mathbb{Z}$ and every $\omega > 0$ small enough there are **no ghost constrictions** with $B = \ell\omega$.

Theorem B is proved by methods of theory of **slow-fast** families of dynam. systems.

## Equivalent description of model by linear systems on $\overline{\mathbb{C}}$.

$$\frac{d\phi}{d\tau} = \frac{1}{\omega}(\sin\phi + B + A\cos\tau) = \frac{\sin\phi}{\omega} + \ell + 2\mu\cos\tau. \qquad (2)$$

**V.M.Buchstaber, O.V.Karpov, S.I.Tertychnyi, 2004.** The variable changes

$$z := e^{i\tau}, \quad \Phi := ie^{i\phi}$$

transforms (2) to Riccati equation

$$\frac{d\Phi}{dz} = z^{-2}((\ell z + \mu(z^2+1))\Phi + \frac{z}{2\omega}(\Phi^2+1)). \qquad (3)$$

$\Phi(z)$ is a solution of (3) $<=>$ $\Phi(z) = \frac{v}{u}(z)$, where $Y = (u,v)(z)$ is a solution of

$$Y' = \left(\frac{\mathrm{diag}(-\mu,0)}{z^2} + \frac{R}{z} + \mathrm{diag}(-\mu,0)\right)Y, \quad R = \begin{pmatrix} -\ell & -\frac{1}{2\omega} \\ \frac{1}{2\omega} & 0 \end{pmatrix}, \qquad (4)$$

In other words, (3) is the **projectivization** of (4).

$$\frac{d\phi}{d\tau} = \frac{1}{\omega}(\sin\phi + B + A\cos\tau) = \frac{\sin\phi}{\omega} + \ell + 2\mu\cos\tau. \tag{2}$$

$$Y' = \left(\frac{\mathrm{diag}(-\mu,0)}{z^2} + \frac{R}{z} + \mathrm{diag}(-\mu,0)\right)Y, \quad R = \begin{pmatrix} -\ell & -\frac{1}{2\omega} \\ \frac{1}{2\omega} & 0 \end{pmatrix}, \tag{4}$$

**The monodromy operator** $M$ of system (4).
Acts on its local solution space $\mathbb{C}^2 = \mathbb{C}^2 \times \{z_0\}$ at $z_0 \in \mathbb{C}^*$
by analytic extension along counterclockwise circuit around 0.

**Fact.** $(B, A; \omega)$ is a **constriction** $<=>$ (4) has **trivial monodromy:** $M = Id$.
**D.A.Filimonov, A.G., V.A.Kleptsyn, I.V.Schurov, 2014.**

# Linear systems with irregular nonresonant singularities.
## Classical theory (Birkhoff, Jurkat, Lutz, Peyerimhoff, Balser, Sibuya).

$$Y' = \left(\frac{K}{z^2} + \frac{R}{z} + N + O(z)\right) Y, \quad Y = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{C}^2, \tag{5}$$

Germ at 0; $K, R, N \in \mathrm{Mat}_2(\mathbb{C})$, $K$ has **distinct eigenvalues** $\lambda_1 \neq \lambda_2$.

Then we say that 0 is **irregular non-resonant singularity of Poincaré rank 1.**

Two germs (5), (5)' are **analytically equivalent,** if there exists a germ of holomorphic $\mathrm{GL}_2(\mathbb{C})$-valued function $H(z)$ such that $Y = H(z)\widetilde{Y}$ sends (5) to (5)'.

(5)$\simeq$(5)' **formally**, if this holds for a **formal** invertible matrix power **series** $\widehat{H}(z)$.

**Theorem.** System (5) is **formally equivalent** to a unique **formal normal form**

$$\widetilde{Y}' = \left(\frac{\widetilde{K}}{z^2} + \frac{\widetilde{R}}{z}\right) \widetilde{Y}, \ \widetilde{K} = \mathrm{diag}(\lambda_1, \lambda_2), \ \widetilde{R} = \mathrm{diag}(b_1, b_2). \tag{6}$$

Here $\widetilde{K} = \mathbf{H}^{-1} K \mathbf{H}$ for some $\mathbf{H} \in \mathrm{GL}_2(\mathbb{C})$; $\widetilde{R} =$ **the diagonal part** of $\mathbf{H}^{-1} R \mathbf{H}$.

# Linear systems with irregular nonresonant singularities.
## Classical theory (Birkhoff, Jurkat, Lutz, Peyerimhoff, Balser, Sibuya).

$$Y' = \left(\frac{K}{z^2} + \frac{R}{z} + N + O(z)\right) Y \quad (5)$$

is **generically not analytically equivalent** to its formal normal form

$$\widetilde{Y}' = \left(\frac{\widetilde{K}}{z^2} + \frac{\widetilde{R}}{z}\right)\widetilde{Y}, \ \ \widetilde{K} = \text{diag}(\lambda_1, \lambda_2), \ \ \widetilde{R} = \text{diag}(b_1, b_2): \quad (6)$$

for generic (5) the **normalizing series** $\widehat{H}(z)$ **diverges.**

There exists a covering $\mathbb{C}^* = S_0 \cup S_1$ by two sectors with vertex 0 and **analytic** functions $H_j : S_j \cap D_r \to \text{GL}_2(\mathbb{C})$, $H_j \in C^\infty(\overline{S}_j \cap D_r)$, s.t. $Y = H_j(z)\widetilde{Y}$: (5)$\mapsto$(6).

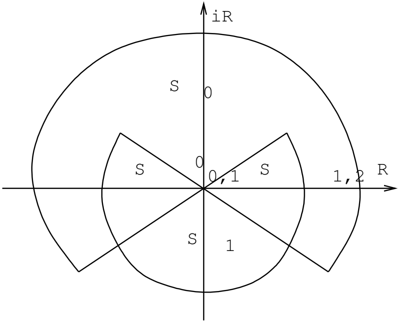
The matrix functions $H_j$ are **unique** up to left multipl. by const diagonal matrices.

NF (6) has **canonical solution basis,** fund. matr. $W(z) = \text{diag}(\widetilde{y}_1(z), \widetilde{y}_2(z))$.

$X^j(z) := H_j(z)W(z)$ are **canonical sectorial solution bases** for system (5) in $S_j$.

## Classical theory (Birkhoff, Jurkat, Lutz, Peyerimhoff, Balser, Sibuya).

**Example.** Let $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 - \lambda_2 < 0$.




On intersect. component $S_{j,j+1}$ two canonical solution bases $X^0(z)$, $X^1(z)$ of (5).

$$X^1(z) = X^0(z) C_0 \text{ on } S_{0,1}; \quad X^0(z) \exp(2\pi i \widetilde{R}) = X^1(z) C_1 \text{ on } S_{1,2} \qquad (7)$$

$C_0$, $C_1$ **Stokes matrices**. **Unipotent;** $C_0$ is **upper-triangular,** $C_1$ is **lower-triang.**

**Theorem.** (5) is **analytically** $\simeq$ to form. norm. form (6) <=> $C_0 = C_1 = Id$; (5)$\simeq$(5)' anal. <=> (6)=(6)', $(C_0', C_1') = D(C_0, C_1)D^{-1}$ for some diag. matr. $D$.

# Linear systems on $\overline{\mathbb{C}}$. Monodromy–Stokes data.

$$Y' = \left(\frac{K}{z^2} + \frac{R}{z} + N\right) Y, \qquad K, R, N \in \mathrm{End}(\mathbb{C}^2), \tag{8}$$

where each one of the matrices $K$ and $N$ at $0$, $\infty$ has **distinct real eigenvalues.**

Fix a $z_0 \in S_0$, e.g., $z_0 = 1$. Let $M : \mathbb{C}^2 \times \{z_0\} \to \mathbb{C}^2 \times \{z_0\}$ - **monodromy.**

In the sector $S_0$ two fund. solution matrices $X^{0,0}(z)$, $X^{0,\infty}(z)$: from $0$ and $\infty$. Their 4 columns $f_{10}(z)$, $f_{20}(z)$, $f_{1\infty}(z)$, $f_{2\infty}(z)$ are solutions of (8).

$\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1 = \overline{\mathbb{C}}$ tautological projection.

$$q_{kp} := \pi(f_{kp}(z_0)) \in \mathbb{CP}^1 = \overline{\mathbb{C}}, \;\; q := (q_{10}, q_{20}, q_{1\infty}, q_{2\infty}) \in \overline{\mathbb{C}}^4.$$

$(q, M) \simeq (q', M') \;:=\;$ if there exists an $H \in \mathrm{GL}_2(\mathbb{C})$, $H(q) = q'$, $H^{-1}M'H = M$.

$[q, M] = (q, M)/\simeq \;:=\;$ **the monodromy–Stokes data.**

**Definition.** A family of systems (8) is **isomonodromic,** if $[q, M] = const$.

# Method of proof of Theorem A (i.e., reduction to small $\omega$).

$$Y' = \left(\frac{\mathrm{diag}(-\mu, 0)}{z^2} + \frac{R}{z} + \mathrm{diag}(-\mu, 0)\right) Y, \qquad R = \begin{pmatrix} -\ell & -\frac{1}{2\omega} \\ \frac{1}{2\omega} & 0 \end{pmatrix} \qquad \textbf{(Jos)}$$

lie in the **4-dimensional** family of **normalized** $\mathbb{R}_+$**-Jimbo type linear systems:**

$$Y' = \left(\frac{\tau}{z^2} G \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} G^{-1} + \frac{1}{z} \begin{pmatrix} -\ell & -R_{21} \\ R_{21} & 0 \end{pmatrix} + \tau \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}\right) Y, \ \tau \in \mathbb{R}_+, \qquad (\mathbf{J}(\mathbb{R}_+))$$

$K, R, N$ are real 2x2-matrices, $\quad R_{21} > 0, \ \ell \in \mathbb{R},$

where $G \in \mathrm{SL}_2(\mathbb{R}), \ G^{-1}RG = \begin{pmatrix} -\ell & * \\ * & 0 \end{pmatrix}$; the elements $*$ are arbitrary.

Family $\mathbf{J}(\mathbb{R}_+)$ is foliated by **isomonodromic families** parametrized by $\tau$, obtained from well-known **Jimbo isomonodromic families** via variable changes.
In isom. families $\ell \equiv const$, and $w(\tau) := -\frac{R_{12}(\tau)}{\tau K_{12}(\tau)}$ satisfies **Painlevé 3 equation:**

$$w'' = \frac{(w')^2}{w} - \frac{w'}{\tau} + w^3 - 2\ell\frac{w^2}{\tau} - \frac{1}{w} + (2\ell - 2)\frac{1}{\tau} \qquad \text{(P3)}$$

# Method of proof of Theorem A (i.e., reduction to small $\omega$).

**Key argument 1.** **Systems Jos** correspond to **1st order poles with res**$=1$ of solutions $w(\tau) := -\frac{R_{12}(\tau)}{\tau K_{12}(\tau)}$ of P3: $K_{12} = 0$ and $R_{12} \neq 0$ on **Jos**.

$=>$ **Jos** is a **local cross-section** to the **isomonodromic foliation** of $\mathbf{J}(\mathbb{R}_+)$.

$$\Sigma_\ell := \{\text{systems in } \mathbf{J}(\mathbb{R}_+) \text{ with trivial monodromy and given } \ell \in \mathbb{Z}\}.$$

**Key argument 2.** $\Sigma_\ell$ is a 2-dim. submanifold foliated by isomonodromic leaves.

With **submersive** projection $\mathcal{R} : \Sigma_\ell \to \mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$ constant along the leaves.

$\mathcal{R}$ is the **cross-ratio** of $(q_{10}, q_{20}, q_{1\infty}, q_{2\infty})$; an **analytic invariant** of lin. system.

$Constr_\ell := \Sigma_\ell \cap$ **Jos** $=$ **constrictions** with $B = \ell\omega$;
**Non-trivial fact:** $\mathcal{R}(Constr_\ell)$ lies in $\mathbb{R}$: does not contain $\infty$.

**Corollary of 1 and 2.** $Constr_\ell$ is **local cross-section** to isomon. foliation of $\Sigma_\ell$

$=>$ each component $\mathcal{C}$ in $Constr_\ell$ is **1-to-1 parametr.** by interval $(a, b) = \mathcal{R}(\mathcal{C})$.

$$\mathcal{R} := \frac{(q_{10} - q_{1\infty})(q_{20} - q_{2\infty})}{(q_{10} - q_{2\infty})(q_{20} - q_{1\infty})}.$$

For each component $\mathcal{C}$ in $Constr_\ell$ one has

$$\mathcal{R} : \mathcal{C} \to \mathcal{R}(\mathcal{C}) = (a, b) \subset \mathbb{R} \quad \textbf{is an analytic diffeomorphism.}$$

The **inverse:** $(\mathcal{R}|_{\mathcal{C}})^{-1} : x \in (a, b) \mapsto C(x) \in \mathcal{C}$.

**Theorem.** For every $c \in \{a, b\} \setminus \{0\}$ there exists a sequence $x_k \in (a, b)$, $x_k \to c$, such that $\omega_k := \omega(C(x_k)) \to 0$.

This allows to **deform** analytically a **ghost constriction** (if it exists) to another one, with **arbitrarily small** $\omega$.

# Open problems

- 'Key argument 1' means that if one can obtain constriction points via poles of a solution $w(\tau)$ of Painlevé 3

$$w'' = \frac{(w')^2}{w} - \frac{w'}{\tau} + w^3 - 2\ell\frac{w^2}{\tau} - \frac{1}{w} + (2\ell - 2)\frac{1}{\tau}.$$

  What real solutions have an infinite lattice of simple poles with residue 1 converging to $+\infty$?
  (It is known that Painlevé 3 of the above form has a one-dimentional family of Bessel solutions and recent numerical experience has shown that their small deformations have an infinite lattice of poles too.)

- For what $\ell$ Painlevé 3 has solutions with no real poles, (smth like the tronquée solutions)?

## Open problems

- **Theorem 2(Buchstaber-Glutsyuk)** *A point in the parameter space of the Josephson system lies in the boundary of a phase-lock area $\Longleftrightarrow$ the corresponding Josepson system monodromy either has Jordan cell type, or is the identity.*
  So 'Key argument 1' works for the boundaries as well. Calculate improved asymptotics for boundaries.
- Study geometry of the three-dimensional phase-lock area portrait of family.
- Study geometry of phase-lock areas for generalized Josephson systems.