Isomonodromic Laplace Transform with Coalescing Eigenvalues and Confluence of Fuchsian Singularities

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Problem. n-dimensional differential systems **isomonodromically depending on parameters** $u=(u_1,\dots,u_n)\in\mathbb{D}(u^c)$ polydisc.

$$\frac{dY}{dz}=\left(\Lambda(u)+\frac{A(u)}{z}\right)Y \quad \underset{\text{Laplace trsf.}}{\longleftrightarrow} \quad \frac{d\Psi}{d\lambda}=\sum_{k=1}^{n}\frac{B_k(u)}{\lambda-u_k}\Psi.$$

$$\Lambda(u)=\mathrm{diag}(u_1,\dots,u_n), \qquad B_k(u)=-E_k(A(u)+I).$$

$\mathbb{D}(u^c)$: center u^c is a coalescence point in the **coalescence locus**

$$\boldsymbol{\Delta}:=\mathbb{D}(\boldsymbol{u^c})\cap\left(\bigcup_{i\neq j}\{\boldsymbol{u_i-u_j=0}\}\right)\neq\emptyset$$

Coalescence of eigenvalues of $\Lambda(u)$ "resonant" irregular singularity $\longleftrightarrow$ **Confluence of Fuchsian singularities**.

Study fundamental matrix solutions (analyticity properties) and their monodromy data $\longrightarrow$ Extend the theory of isomonodromy deformations of Jimbo-Miwa-Ueno (1981) to a non-generic case with coalescences.

$$\frac{dY}{dz} = \left(\Lambda(u) + \frac{A(u)}{z}\right) Y$$

Problem has been solved in G.Cotti, B.Dubrovin, D.Guzzetti: Duke Math. J., 168, (2019). by direct analysis of the the system, its solutions and its Stokes phenomenon. (talks online: Banff 18w5025, or Isaac Newton Institute for Mathematical Sciences [CATW01]).

Goal of the talk. Start from

$$\frac{d\Psi}{d\lambda} = \sum_{k=1}^{n} \frac{B_k(u)}{\lambda - u_k} \Psi.$$

Combine isomonodromy deformations of Fuchsian systems, confluence of singularities and "isomonodromic" Laplace transform, re-obtain the results of G.Cotti, B.Dubrovin, D.Guzzetti..

Idea: Use results on integrable deformations of Fuchsian systems (Bolibrukh, Yoshida-Takano).

Reference for the talk: D.G.:arXiv:2101.03397 (2021).

References on similar issues

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• **Klimes M.**,
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– *Analytic Classification of Families of Linear Differential Systems Unfolding a Resonant Irregular Singularity*, (2020).
– *Confluence of Singularities in Hypergeometric Systems* (2020).

• **Loday-Richaud M., Remy P.**, Resurgence, *Stokes phenomenon and alien derivatives for level-one linear differential systems.* (2011)

• **P. Remy**. *Matrices de Stokes-Ramis et constantes de connexion pour les systèmes différentiels linéaires de niveau unique* (2012).

• **R. Schäfke**.
– *The connection problem for two neighboring regular singular points of general complex ordinary differential equations.* (1980).
– *A connection problem for a regular and an irregular singular point of complex ordinary differential equations.* (1984),
– *Über das globale analytische Verhalten der Normallosungen von* $(s-B)v'(s)=(B+t^{-1}A)v(s)$ *und zweier Arten von assoziierten Funktionen.* (1985)
– *Confluence of several regular singular points into an irregular singular one.* (1998).

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•**Galkin S., Golyshev V., Iritani H.**, *Gamma classes and quantum cohomology of Fano manifolds: gamma conjectures*, (2016).

INTRODUCTION – BACKGROUND 1 (G.Cotti, B.Dubrovin, D.Guzzetti (2019))

Deformations in $\mathbb{D}(u^0) = \left\{ u \in \mathbb{C}^n \;\middle|\; \max_{1\le j\le n} |u_j - u_j^0| \le \epsilon_0 \right\}$ polydisc at u^0.

No coalescence points, namely $u_i \neq u_j$ for $i \neq j$.

$$\frac{dY}{dz} = \left(\Lambda(u) + \frac{A(u)}{z}\right) Y,$$

$A(u)$ holomorphic

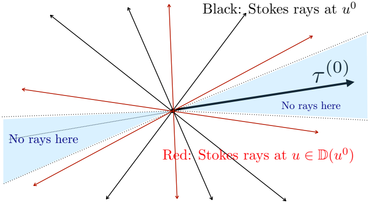
$$\Lambda(u) = \mathrm{diag}(u_1, ..., u_n).$$

- **Stokes rays of $\Lambda(u^0)$**

$$\Re((u_j^0 - u_k^0)z) = 0, \quad \Im((u_j^0 - u_k^0)z) < 0.$$

- **Admissible direction at u^0: $\arg z = \tau^{(0)}$**

not coinciding with any of the Stokes rays above.



- **Stokes rays of $\Lambda(u)$**

$$\Re((u_j - u_k)z) = 0, \quad \Im((u_j - u_k)z) < 0.$$

Stokes rays of $\Lambda(u)$ don't cross admissible direction (mod π), as u varies in $\mathbb{D}(u^0)$ small.

A classical result (Sibuya, Wasow, etc). $\mathbb{D}(u^0)$ small.

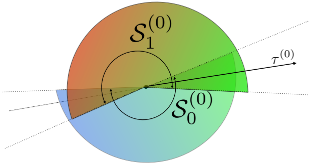
1) $\exists$ unique formal fundamental matrix solution

$$Y_F(z,u) = (I + \sum_{l=1}^{\infty} F_l(u) z^{-l}) z^{B(u)} e^{z\Lambda(u)}, \quad B(u) := \text{diag}(A_{11}, \dots, A_{nn});$$

$F_l(u)$ holomorphic in $\mathbb{D}(u^0)$ and recursively computable.

2) $\forall\ \nu \in \mathbb{Z}$, $\exists\ \delta > 0$ small

and sectors



$$\mathcal{S}_\nu^{(0)} : \quad (\tau^{(0)} + (\nu - 1)\pi) - \delta < \arg z < (\tau^{(0)} + \nu\pi) + \delta, \quad \nu \in \mathbb{Z},$$

and $\exists$ unique fundamental matrix solutions $Y_\nu(z,u)$ holomorphic in $\mathcal{R}(\mathbb{C}\backslash\{0\}) \times \mathbb{D}(u^0)$ having asymptotics

$$Y_\nu(z,u) \sim Y_F(z,u), \quad z \to \infty \text{ in } \mathcal{S}_\nu^{(0)}.$$

Here $\mathcal{R}$ is universal covering.

- Stokes matrices $\mathbb{S}_\nu(u)$

$$Y_{\nu+1}(z,u) = Y_\nu(z,u)\mathbb{S}_\nu(u).$$

- At $z=0$

$$\exists \text{ "Levelt" form} \quad Y^{(0)}(z,u) = G^{(0)}(u)\Big(I + \sum_{j=1}^{\infty} a_j(u) z^j\Big) z^D z^L,$$

D diagonal of integers, L = Jordan + nilpotent.
$G^{(0)}(u)$ and $a_j(u)$ holomorphic in $\mathbb{D}(u^0)$; series uniformly convergent for $|z|$ bounded.

- Central connection matrix $C_\nu(u)$

$$Y_\nu(z,u) = Y^{(0)}(z,u)C_\nu(u).$$

- Monodromy for $z \longmapsto z e^{2\pi i}$.
  For $Y^{(0)}$ is $M := e^{2\pi i L}$.
  For Y_ν is $e^{2\pi i B}(\mathbb{S}_\nu \mathbb{S}_{\nu+1})^{-1} = C_\nu^{-1} M C_\nu$.

  $\mathbb{S}_{2\nu+1} = e^{-2\pi i \nu B}\mathbb{S}_1 e^{2\pi i \nu B}, \qquad \mathbb{S}_{2\nu} = e^{-2\pi i \nu B}\mathbb{S}_0 e^{2\pi i \nu B}.$

Definition. **Essential monodromy data** $\mathbb{S}_0$, $\mathbb{S}_1$, B, C_0, L, D.
System is (strongly) **isomonodromic on** $\mathbb{D}(u^0)$ if the above data are constant.

Theorem 1. *System strongly isomonodromic in $\mathbb{D}(u^0)$ $\Longleftrightarrow$ Y_ν, for every ν, and $Y^{(0)}$, satisfy the Frobenius integrable Pfaffian system*

$$dY = \omega(z,u)Y, \qquad \omega(z,u) := \left(\Lambda(u) + \frac{A(u)}{z}\right) dz + \sum_{k=1}^{n}\Big(zE_k + \omega_k(u)\Big)du_k,$$
$$\omega_k(u) := [F_1(u), E_k].$$

Equivalently, strongly isomonodromic $\Longleftrightarrow$ A satisfies

$$dA = \sum_{j=1}^{n}\Big[\omega_k(u), A\Big]du_k.$$

If the deformation is strongly isomonodromic, then $G^{(0)}(u)$ is a holomorphic fundamental solution of

$$dG = \Big(\sum_{j=1}^{n}\omega_k(u)du_k\Big)G,$$

and $J = G^{(0)}(u)^{-1}A(u)G^{(0)}(u)$, constant Jordan form.

Remark. The above theorem is analogous to the characterisation of isomonodromic deformations by Jimbo-Miwa-Ueno, including also possible resonances in A

Deformations in $\mathbb{D}(u^c)$, and u^c is a coalescence point

$$\frac{dY}{dz} = \left(\Lambda(u) + \frac{A(u)}{z}\right) Y,$$

$$\Delta = \mathbb{D}(u^c) \cap \left(\bigcup_{i \neq j} \{u_i - u_j = 0\}\right) \neq \emptyset \quad \text{there are coalescence points.}$$

Jimbo-Miwa-Ueno theory fails.

- A fundamental matrix solution $Y(z,u)$ is holomorphic on $\mathcal{R}\Big(\mathbb{C}\backslash\{0\} \times \mathbb{D}(u^c)\backslash\Delta\Big)$, but Δ is branching locus and $Y(z,u)$ may diverge along any direction approaching Δ.
- Monodromy data for fundamental solution $\mathring{Y}(z)$ of

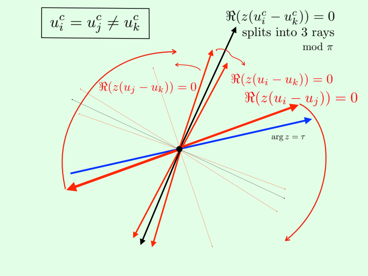
$$\frac{dY}{dz} = \left(\Lambda(u^c) + \frac{A(u^c)}{z}\right) Y, \quad \underline{\text{restricted at } u = u^c}$$

  differ from those of any fundamental solution $Y(z,u)$ at point $u \notin \Delta$.
- $F_l(u)$ in formal solution have poles at Δ.
- Serious problems with definition of asymptotics and Stokes sectors (see below)

- **Stokes rays of $\Lambda(u^c)$**

$$\Re((u_j^c - u_k^c)z) = 0, \quad \Im((u_j^c - u_k^c)z) < 0.$$

Admissible direction at u^c arg $z = \tau$ not containing the above Stokes rays.

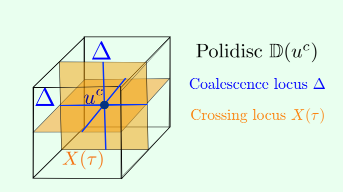


- Even if $\mathbb{D}(u^c)$ is small, as u varies **some Stokes rays of $\Lambda(u)$**

$$\Re((u_j - u_k)z) = 0, \quad \Im((u_j - u_k)z) < 0.$$

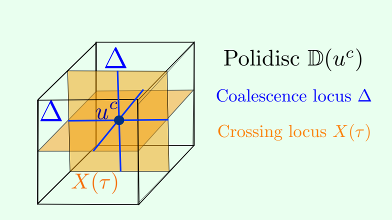
cross directions $\arg z = \tau \mod \pi$.

"Crossing locus" $X(\tau) = \{u \in \mathbb{D}(u^c)$ such that Stokes rays of $\Lambda(u)$ have directions $\arg z = \tau \mod \pi\}$.

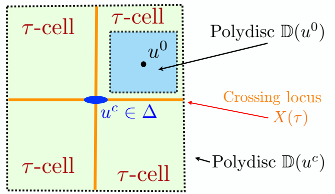


$\mathbb{D}(u^c)\backslash(\Delta \cup X(\tau))$ is not connected. Each (simply) connected component is a **topological cell (τ-cells)**.

Anyhow, isomonodromic deformations can be defined in polydisc $\mathbb{D}(u^0)$ contained in a τ-cell



$\Longrightarrow$



Extension of isomonodromic deformations to the whole $\mathbb{D}(u^c)$.

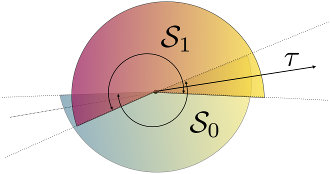
Theorem 2. [G.Cotti, B.Dubrovin, D.Guzzetti: Duke Math. J., 168, (2019).]
Assume: $A(u)$ holomorphic in $\mathbb{D}(u^c)$, strong isomonoromy in $\mathbb{D}(u^0)$, and

$$A_{ij}(u) = \mathcal{O}(u_i - u_j) \longmapsto 0 \qquad \text{whenever } u_i - u_j \to 0 \text{ aproaching } \Delta.$$

Then,

- *fundamental matrix solutions are holomorphic in $\mathcal{R}(\mathbb{C}\backslash\{0\}) \times \mathbb{D}(u^c)$.*
  Δ is not a branching locus.
- *Asymptotic relations still hold on the whole $\mathbb{D}(u^c)$ in wide sectors*

Background 1.



$Y_\nu(z,u) \sim Y_F(z,u), \quad z \to \infty, \quad u \in \mathbb{D}(u^c),$

in wide u-independent sectors $\mathcal{S}_\nu$

$(\tau + (\nu - 1)\pi) - \delta' < \arg z < (\tau + \nu\pi) + \delta';$

$\delta' > 0, \quad \nu \in \mathbb{Z}.$

- *The essential monodromy data* $\mathbb{S}_0$, $\mathbb{S}_1$, B, C_0, L, D *are well defined and constant on the whole* $\mathbb{D}(u^c)$.

It suffices to compute the data for fundamental matrix solutions $\mathring{Y}_\nu(z) \sim Y_F(z,u^c)$ *and* $\mathring{Y}^{(0)}(z)$ *of*

$$\frac{dY}{dz} = \left(\Lambda(u^c) + \frac{A(u^c)}{z}\right) Y, \quad \underline{\textit{restricted at } u = u^c}$$

- *Stokes matrices satisfy*

$$(\mathbb{S}_\nu)_{ij} = (\mathbb{S}_\nu)_{ji} = 0 \textit{ for every } i \neq j \textit{ such that } u_i^c = u_j^c.$$

□

Goal: deduce the results above for the $Y_\nu(z,u)$ and $\mathbb{S}_\nu$, using an isomonodromic Laplace transform and integrable deformations of Fuchsian systems.

INTRODUCTION- BACKGROUND 2

Balser-Jurkat-Lutz. SIAM J. Math Anal. 12 (1981) (generic case, diag(A) with no integers)

D.Guzzetti: Funkcial. Ekvac. 59 (2016) (general case, any A).

Systems not depending on parameters. $\Lambda = \Lambda(u^0)$, $A = A(u^0)$, u^0 fixed.

$$\frac{dY}{dz} = \left(\Lambda + \frac{A}{z}\right) Y \qquad \overset{\longleftrightarrow}{\text{Laplace}\ \vec{Y}(z) = \int_\gamma e^{\lambda z}\vec{\Psi}(\lambda)d\lambda,} \qquad \frac{d\Psi}{d\lambda} = \sum_{k=1}^{n} \frac{B_k}{\lambda - u_k^0}\Psi.$$

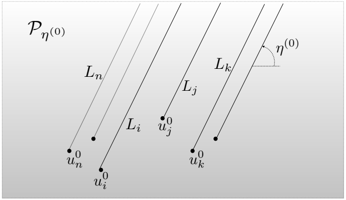
$$\text{if } \gamma \text{ is such that } e^{\lambda z}(\lambda - \Lambda)\vec{\Psi}(\lambda)\Big|_\gamma = 0.$$

Known results. There exist vector solutions

- $\vec{\Psi}_1(\lambda|\eta^{(0)}), \dots, \vec{\Psi}_n(\lambda|\eta^{(0)})$ selected.
- $\vec{\Psi}_1^{(sing)}(\lambda|\eta^{(0)}), \dots, \vec{\Psi}_n^{(sing)}(\lambda|\eta^{(0)})$ singular

$\eta^{(0)} \neq \arg(u_j^0 - u_k^0) \bmod \pi, \quad \forall\, 1 \leq j, k \leq n.$

admissible direction in $\lambda - plane$



$$\lambda \in \mathcal{P}_{\eta^{(0)}} := \left\{\lambda \in \mathcal{R}(\mathbb{C}\backslash\{u_1^0, \dots, u_n^0\}) \mid \eta^{(0)} - 2\pi < \arg(\lambda - u_k^0) < \eta^{(0)}, \quad 1 \leq k \leq n\right\}$$

- Connection coefficients c_{jk}: represent $\vec{\Psi}_k$ close to $\lambda = u_j^0$

$$\vec{\Psi}_k(\lambda|\eta^{(0)}) = \vec{\Psi}_j^{(sing)}(\lambda|\eta^{(0)})c_{jk} + \mathrm{reg}(\lambda - u_j^0)$$

Remark. They depend on direction $\eta^{(0)}$ of the branch-cut.

- We define

$$\vec{Y}_\nu(z|\nu) = \frac{1}{2\pi i}\int_{\gamma_k(\eta^{(0)}-\nu\pi)} e^{z\lambda}\vec{\Psi}_k^{(sing)}(\lambda|\eta^{(0)} - \nu\pi)\ d\lambda, \qquad \text{if } A_{kk} \notin \mathbb{Z}_-$$

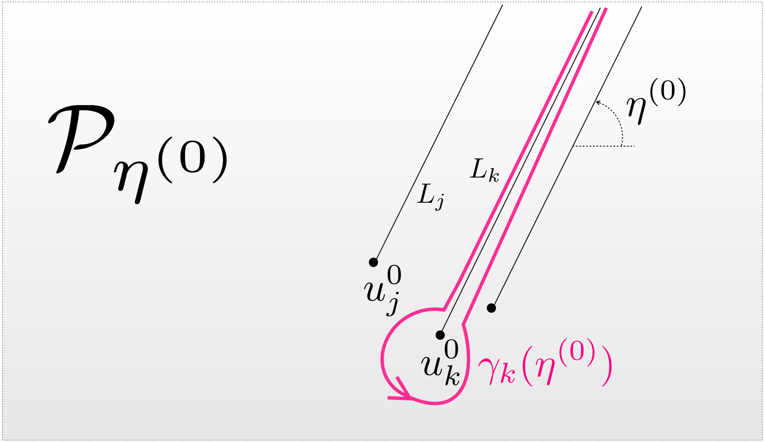
$$\vec{Y}_\nu(z|\nu) = \int_{L_k(\eta^{(0)}-\nu\pi)} e^{z\lambda}\vec{\Psi}_k(\lambda|\eta^{(0)} - \nu\pi)\ d\lambda, \qquad \text{if } A_{kk} \in \mathbb{Z}_-, \quad k = 1, \dots, n.$$

Then,

$$Y_\nu(z) = \left[\vec{Y}_1(z|\nu) \mid \dots \mid \vec{Y}_n(z|\nu)\right]$$

are fundamental matrix solutions of

$$\frac{dY}{dz} = \left(\Lambda + \frac{A}{z}\right)Y.$$



$\mathcal{P}_{\eta^{(0)}}$
$\eta^{(0)}$
L_k
L_j
u_j^0
u_k^0
$\gamma_k(\eta^{(0)})$

• **Stokes phenomenon** with $\tau^{(0)} = 3\pi/2 - \eta^{(0)}$, admissible in z-plane.

Studying how $\vec{\Psi}_k^{(sing)}$ and $\vec{\Psi}_k$ change when admissible directions are rotated $\eta^{(0)} \longmapsto \eta^{(0)} - \pi$ and $\eta^{(0)} \longmapsto \eta^{(0)} - 2\pi$, we obtain

$$(\mathbb{S}_1)_{jk} = \begin{cases} e^{-2\pi i A_{kk}}\alpha_k \, c_{jk} & \text{for } j \prec k, \\ 1 & \text{for } j = k, \\ 0 & \text{for } j \succ k, \end{cases} \quad (\mathbb{S}_2^{-1})_{jk} = \begin{cases} 0 & \text{for } j \prec k, \\ 1 & \text{for } j = k, \\ -e^{2\pi i (A_{jj} - A_{kk})}\alpha_k \, c_{jk} & \text{for } j \succ k. \end{cases}$$

$$\alpha_k := (e^{2\pi i A_{kk}} - 1) \text{ if } A_{kk} \notin \mathbb{Z}; \qquad \alpha_k := 2\pi i \text{ if } A_{kk} \in \mathbb{Z},$$

$$j \prec k \quad \text{means} \quad \Re((u_j - u_k)z)\Big|_{\arg z = \tau} < 0.$$

For the proof in generic case $A_{kk} \notin \mathbb{Z}$ see Balser-Jurkat-Lutz. SIAM J. Math Anal. 12 (1981) .
For the proof in general case, including $A_{kk} \in \mathbb{Z}$, see D.G.: Funkcial. Ekvac. 59 (2016).

For use of Laplace transform and higher Poincaré rank see also M. Loday-Richaud, P. Remy: J. Differential Equations 250, (2011).

Goal: unify background 1 + background 2, introducing dependence on deformation parameters u in the Laplace transform, and prove again Theorem 2.

Recall Theorem 2... already stated before...

$$\frac{dY}{dz} = \left(\Lambda(u) + \frac{A(u)}{z}\right) Y,$$

Assume: $A(u)$ holomorphic in $\mathbb{D}(u^c)$, strong isomonoromy in $\mathbb{D}(u^0)$, and

$$A_{ij}(u) = \mathcal{O}(u_i - u_j) \longmapsto 0 \quad \text{whenever } u_i - u_j \to 0 \text{ aproaching } \Delta.$$

Then,

- *fundamental matrix solutions are holomorphic in $\mathcal{R}(\mathbb{C}\backslash\{0\}) \times \mathbb{D}(u^c)$.*
- *Asymptotic relations still hold on the whole $\mathbb{D}(u^c)$*

$$Y_\nu(z,u) \sim Y_F(z,u), \quad z \to \infty \text{ in } \mathcal{S}_\nu.$$

$$\mathcal{S}_\nu := \quad (\tau + (\nu-1)\pi) - \delta' < \arg z < (\tau + \nu\pi) + \delta'.$$

- *The essential monodromy data $\mathbb{S}_0$, $\mathbb{S}_1$, B, C_0, L, D are well defined and constant on the whole $\mathbb{D}(u^c)$.*
- *Stokes matrices satisfy*

$$(\mathbb{S}_\nu)_{ij} = (\mathbb{S}_\nu)_{ji} = 0 \text{ for every } i \neq j \text{ such that } u_i^c = u_j^c.$$

Recall that

$$\frac{dY}{dz} = \left(\Lambda(u) + \frac{A(u)}{z} \right) Y$$

is (strongly) isomonodromic in $\mathbb{D}(u^0)$ contained in a τ-cell of $\mathbb{D}(u^c)$ if and only if

$$dA = \sum_{j=1}^{n} [\omega_j(u), A]\ du_j, \qquad \omega_j(u) = [F_1(u), E_j].$$

It is well known that a Fuchsian system

$$\frac{d\Psi}{d\lambda} = \sum_{k=1}^{n} \frac{B_k(u)}{\lambda - u_k} \Psi, \qquad B_k(u) = -E_k(A(u) + I),$$

is strongly isomonodromic in $\mathbb{D}(u^0)$ (constant Levelt exponents, constant connection matrices $\Rightarrow$ constant monodromy matrices) if and only if it is the λ-component of a Frobenius integrable Pfaffian system (integrable deformation)

$$d\Psi = P(\lambda, u)\Psi, \qquad P(z, u) = \sum_{k=1}^{n} \frac{B_k(u)}{\lambda - u_k} d(\lambda - u_k) + \sum_{k=1}^{n} \gamma_k(u) du_k.$$

L. Schlesinger...
A.A.Bolibrukh: Izv. Akad. Nauk SSSR Ser. Mat. 41 (1997),
A.A.Bolibrukh: J. of Dynamical Control Systems, 3, (1998).

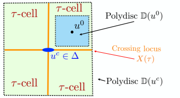
The integrability condition $dP = P \wedge P$ is the non-normalized Schlesinger system

$$\partial_i \gamma_k - \partial_k \gamma_i = \gamma_i \gamma_k - \gamma_k \gamma_i, \tag{1}$$

$$\partial_i B_k = \frac{[B_i, B_k]}{u_i - u_k} + [\gamma_i, B_k], \quad i \neq k \tag{2}$$

$$\partial_i B_i = -\sum_{k \neq i} \frac{[B_i, B_k]}{u_i - u_k} + [\gamma_i, B_i] \tag{3}$$

Lemma: Equivalence of deformation equations in $\mathbb{D}(u^0)$.



Equations (1)-(3) are equivalent to

$$dA = \sum_{j=1}^{n} [\omega_j(u), A]\, du_j, \qquad \omega_j(u) = [F_1(u), E_j]$$

if and only if $\gamma_j(u) = \omega_j(u), \quad j = 1, \dots, n.$

□

Namely,

$$\frac{dY}{dz} = \left(\Lambda(u) + \frac{A(u)}{z}\right) Y$$

is stronlgy isomonodromic if and only if so is

$$\frac{d\Psi}{d\lambda} = \sum_{k=1}^{n} \frac{B_k(u)}{\lambda - u_k} \Psi.$$

Remark.

$$B_j = -E_j(A+I) = \begin{pmatrix} 0 & & & 0 & & & 0 \\ \vdots & & & \vdots & & & \vdots \\ -A_{j1} & \cdots & -A_{j,j-1} & -A_{jj}-1 & -A_{j,j+1} & \cdots & -A_{jn} \\ \vdots & & & \vdots & & & \vdots \\ 0 & & & 0 & & & 0 \end{pmatrix}.$$

$$\partial_i\gamma_k - \partial_k\gamma_i = \gamma_i\gamma_k - \gamma_k\gamma_i,$$

$$\partial_i B_k = \frac{[B_i, B_k]}{u_i - u_k} + [\gamma_i, B_k], \quad i \neq k, \qquad B_k(u) = -E_k(A(u) + I)$$

$$\partial_i B_i = -\sum_{k\neq i} \frac{[B_i, B_k]}{u_i - u_k} + [\gamma_i, B_i]$$

Lemma [Integrability in $\mathbb{D}(u^c)$]. *Assume that $A(u)$ is holomorphic on the whole $\mathbb{D}(u^c)$.*

Then, the Pfaffian system

$$d\Psi = P(\lambda, u)\Psi, \qquad P(z, u) = \sum_{k=1}^{n} \frac{B_k(u)}{\lambda - u_k} d(\lambda - u_k) + \sum_{k=1}^{n} \gamma_k(u) du_k.$$

is Frobenius integrable on the whole $\mathbb{D}(u^c)$ with holomorphic matrix coefficients

i.e. non-normalized Schlesinger system has holomorphic solution on the whole $\mathbb{D}(u^c)$

if and only if

$$\left(A(u)\right)_{ij} \longrightarrow 0, \quad \Longleftrightarrow \quad [B_i(u), B_j(u)] \longrightarrow 0, \quad \textit{whenever } u_i - u_j \to 0 \textit{ in } \mathbb{D}(u^c).$$

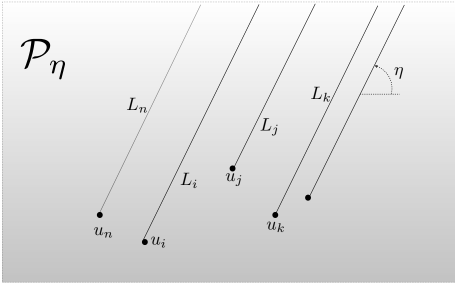
Preliminary Main Result

Preparation.

We start by defining and admissible direction in λ-plane for the Fuchsian system, starting from **admissible direction for $\Lambda(u^c)$ in z-plane**, namely
$\arg z = \tau$ not a Stokes ray of $\Lambda(u^c)$.

For each $u \in \mathbb{D}(u^c)$, consider in λ-plane
branch-cuts $L_1 = L_1(\eta)$, ..., $L_n = L_n(\eta)$
issuing from $u_1, \ldots, u_n$ with direction

$$\boldsymbol{\eta := 3\pi/2 - \eta},$$



Sheet

$$\mathcal{P}_\eta(u) := \Big\{\lambda \in \mathcal{R}(\mathbb{C}\backslash\{u_1, \ldots, u_n\}) \mid \eta - 2\pi < \arg(\lambda - u_k) < \eta, \quad 1 \le k \le n\Big\}.$$

We define the domain

$$\mathcal{D} := \bigcup_{u \in \mathbb{D}(u^c)} \Big\{(\lambda, u) \mid \lambda \in \mathcal{P}_\eta(u)\Big\}$$

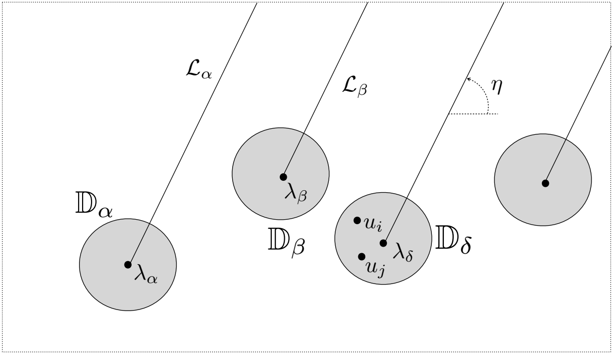
Note. $\mathbb{D}(u^c)$ is "sufficiently" small...

Central coalescence point

$$(u_1^c,\ldots,u_n^c) = (\underbrace{\lambda_1,\ldots,\lambda_1}_{p_1 \text{ times}}, \underbrace{\lambda_2,\ldots,\lambda_2}_{p_2 \text{ times}},\ldots,\underbrace{\lambda_s,\ldots,\lambda_s}_{p_s \text{ times}}),$$

$$p_1+p_2+\cdots+p_s=n.$$

$\mathbb{D}_\alpha$ = the disc at $\boldsymbol{\lambda_\alpha = u_k^c}$, $\alpha = 1,2,\ldots,s$.



Theorem 3.[D.G. arXiv:2101.03397]
Assume that

$$d\Psi = P(\lambda,u)\Psi, \qquad P(z,u)=\sum_{k=1}^{n}\frac{B_k(u)}{\lambda-u_k}d(\lambda-u_k)+\sum_{k=1}^{n}\gamma_k(u)du_k,$$

$$B_j(u)=-E_j(A(u)+I),$$

is Frobenius integrable in $\mathbb{D}(u^0)$.
Moreover, assume that

$$\big(A(u)\big)_{ij}\longrightarrow 0,\ \text{for } u_i-u_j\to 0 \text{ in } \mathbb{D}(u^c).$$

Then there are selected vector solutions of $d\Psi = P(\lambda,u)\Psi$

$$\vec{\Psi}_1(\lambda,u\,|\eta),\ \dots\ ,\ \vec{\Psi}_n(\lambda,u\,|\eta) \quad \text{holomorphic on } \mathcal{D},$$

and singular solutions with regular singularity at $u_1,\dots,u_n$,

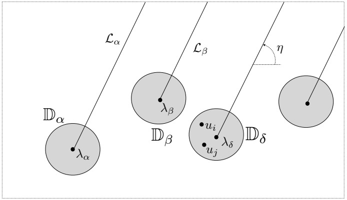
$$\vec{\Psi}_1^{(sing)}(\lambda,u\,|\eta),\ \dots\ ,\ \vec{\Psi}_n^{(sing)}(\lambda,u\,|\eta) \quad \text{holomorphic on } \mathcal{D}.$$

They are characterized as follows.

Selected vector solutions

- **If** $A_{kk} \notin \mathbb{N}$,

$$\vec{\Psi}_k(\lambda, u \,|\eta) = \vec{\psi}_k(\lambda, u \,|\eta)(\lambda - u_k)^{-A_{kk}-1}.$$



$\vec{\psi}_k(\lambda, u \,|\eta)$ *is a vector valued function ,*

holomorphic of $(\lambda, u) \in \mathbb{D}_\alpha \times \mathbb{D}(u^c)$.

Behaviour at u_k

$$\vec{\psi}_k(\lambda, u \,|\eta) = f_k \vec{e}_k + \sum_{\ell=1}^{\infty} \vec{b}_\ell^{\,(k)}(u)(\lambda - u_k)^\ell, \qquad \text{for } \lambda \to u_k,$$

uniformly convergent, coefficients $\vec{b}_\ell^{\,(k)}(u)$ *holomorphic on* $\mathbb{D}(u^c)$.
Unique with choice

$$f_k = \begin{cases} \Gamma(A_{kk}+1), & A_{kk} \in \mathbb{C}\backslash\mathbb{Z}, \\ \dfrac{(-1)^{A_{kk}}}{(-A_{kkk}-1)!}, & A_{kk} \in \mathbb{Z}_- := \{-1, -2, \dots\}, \end{cases}$$

- **If** $A_{kk} \in \mathbb{N}$,

$$\vec{\Psi}_k(\lambda, u \,|\eta) = \sum_{\ell=0}^{\infty} \vec{d}_\ell^{\,(k)}(u)(\lambda - u_k)^\ell, \qquad \textit{for } \lambda \to u_k,$$

holomorphic of $(\lambda, u) \in \mathbb{D}_\alpha \times \mathbb{D}(u^c)$, *with* $u_k^c = \lambda_\alpha$.

$\vec{d}_\ell^{\,(k)}(u)$ *holomorphic in* $\mathbb{D}(u^c)$, *expansion is uniformly convergent.*

$\vec{\Psi}_k$ *uniquely identified by the existence of the singular solution* $\vec{\Psi}_k^{(sing)}$ *given below.*

- *For* j, k *such that* $u_j^c = u_k^c$, $\vec{\Psi}_j(\lambda, u \,|\eta)$ *and* $\vec{\Psi}_k(\lambda, u \,|\nu)$ *are*
    - *either linearly independent,*
    - *or at least one of them is zero (possibly for* A_{jj} *or* A_{kk} *in* $\mathbb{N}$*)*

Singular vector solutions

- *For $A_{kk} \in \mathbb{C}\backslash\mathbb{Z}$.* [algebraic or logarithmic branch-point]

$$\vec{\Psi}_k^{(sing)}(\lambda, u\ |\eta) = \underbrace{\vec{\Psi}_k(\lambda, u\ |\eta)}_{\text{selected sol. is singular!}} = \vec{\psi}_k(\lambda, u\ |\eta)(\lambda - u_k)^{-A_{kk}-1}.$$

- *For $A_{kk} \in \mathbb{Z}_- = \{-1, -2, ...\}$.* [logarithmic branch-point]

$$\vec{\Psi}_k^{(sing)}(\lambda, u\ |\eta) = \vec{\Psi}_k(\lambda, u\ |\eta)\ln(\lambda - u_k) + \sum_{m\neq k}^{*} r_m \vec{\Psi}_m(\lambda, u\ |\eta)\ln(\lambda - u_m) + \vec{\phi}_k(\lambda, u\ |\eta),$$

$$\underset{\lambda\to u_k}{=} \vec{\Psi}_k(\lambda, u\ |\eta)\ln(\lambda - u_k) + \mathrm{reg}(\lambda - u_k), \qquad r_m \in \mathbb{C},$$

$\sum_{m\neq k}^{*} =$ *sum over all m such that $u_m^c = u_k^c$ and $A_{mm} \in \mathbb{Z}_-$.*

$\vec{\Psi}_k$, $\vec{\Psi}_m$, $\vec{\phi}_k$ *holomorphic in $\mathbb{D}_\alpha \times \mathbb{D}(u^c)$, where $\lambda_\alpha = u_k^c$.*

• *For $A_{kk} \in \mathbb{N}$.* [logarithmic branch-point and pole]

$$\vec{\Psi}_k^{(sing)}(\lambda, u \,|\eta) = \vec{\Psi}_k(\lambda, u \,|\eta) \ln(\lambda - u_k) + \frac{\vec{\psi}_k(\lambda, u \,|\eta)}{(\lambda - u_k)^{A_{kk}+1}},$$

$\vec{\Psi}_k$, $\vec{\psi}_k$ *holomorphic in* $\mathbb{D}_\alpha \times \mathbb{D}(u^c)$, *with* $\lambda_\alpha = u_k^c$.

where

$$\vec{\psi}_k(\lambda, u \,|\eta) = \Gamma(A_{kk} + 1)\vec{e}_k + \sum_{\ell=1}^{\infty} \vec{b}_\ell^{(i)}(u)(\lambda - u_k)^\ell, \qquad \text{for } \lambda \to u_k,$$

uniformly convergent, coefficients $\vec{b}_\ell^{(k)}(u)$ *are holomorphic on* $\mathbb{D}(u^c)$.

□

Definition of connection coefficient.

$$\vec{\Psi}_k(\lambda, u\,|\eta) \underset{\lambda\to u_j}{=} \vec{\Psi}_j^{(sing)}(\lambda, u\,|\eta)\, \boldsymbol{c_{jk}} + \mathrm{reg}(\lambda - u_j), \qquad \lambda \in \mathcal{P}_\eta,$$

$c_{jk}^{(\nu)} := 0$, $\forall k = 1, \ldots, n$, when $\vec{\Psi}_j^{(sing)} \equiv 0$, possibly for $A_{jj} \in -\mathbb{N} - 2$.

Note. c_{jk} are uniquely defined, for uniqueness of the singular behaviour of $\vec{\Psi}_j^{(sing)}$ (but $\vec{\Psi}_j^{(sing)}$ is not uniquely def. if $A_{jj} \in \mathbb{Z}_-$).

Corollary of Theorem 3. *They are* **isomonodromic connection coefficients**, *independent of* $u \in \mathbb{D}(u^c)$. *They satisfy the vanishing relations*

$$c_{jk} = 0 \qquad \text{for } j \neq k \text{ such that } u_j^c = u_k^c.$$

Our goal:

Use selected and singular solutions in a suitable Laplace transform to re-obtain results for irregular system.
In particular $c_{jk} = 0 \Rightarrow (\mathbb{S}_\nu)_{jk} = (\mathbb{S}_\nu)_{kj} = 0$ **for** $i \neq j$ **s.t.** $u_j^c = u_k^c$.

Main ingredient of the proof. Use results on integrable deformations of Fuchsian systems (Yoshida-Takano (1976), Bolibrukh (1977)).

$$\text{Recall that } (u_1^c, \dots, u_n^c) = (\underbrace{\lambda_1, \dots, \lambda_1}_{p_1 \text{ times}}, \underbrace{\lambda_2, \dots, \lambda_2}_{p_2 \text{ times}}, \dots, \underbrace{\lambda_s, \dots, \lambda_s}_{p_2 \text{ times}})$$

Consider for example $u_1, \dots, u_{p_1} \to \lambda_1$. We can always label so that

$$\underbrace{A_{11}, \dots, A_{q_1 q_1},}_{\notin \mathbb{Z}} \quad \underbrace{A_{q_1+1, q_1+1}, \dots, A_{p_1 p_1},}_{\in \mathbb{Z}} \quad 0 \le q_1 \le p_1$$

Theorem. *The Paffian system (with assumptions of Theorem 3)*

$$d\Psi = \Big(\sum_{k=1}^{n} \frac{B_k(u)}{\lambda - u_k} d(\lambda - u_k) + \sum_{k=1}^{n} \gamma_k(u) du_k\Big)\Psi.$$

admits the fundamental matrix solution

$$\Psi^{(p_1)}(\lambda, u) = G^{(p_1)} U^{(p_1)}(\lambda, u) \cdot \prod_{j=1}^{p_1} (\lambda - u_j)^{T^{(j)}} \cdot \prod_{j=q_1+1}^{p_1} (\lambda - u_j)^{R^{(j)}},$$

- $G^{(p_1)} \in GL(n, \mathbb{C})$ *reducing* $B_1(u^c), \dots, B_{p_1}(u^c)$ *to Jordan forms* $T^{(1)}, \dots, T^{(p_1)}$.
- $U^{(p_1)}(\lambda, u) =$ *matrix function holomorphic in* $\mathbb{D}_1 \times \mathbb{D}(u^c)$.
- *The exponents* $R^{(j)}$ *are nilpotent matrices.*

$$T^{(j)}=\mathrm{diag}(0,\dots,0,\underbrace{-1-A_{jj}}_{\text{position } j},0,\dots,0),\ \text{for } A_{jj}\neq -1;\quad T^{(j)}=0,\ \text{for } A_{jj}=-1.$$

$$R^{(j)}=\begin{pmatrix}0 & \cdots & & & & \cdots & 0\\ \vdots & & & & & & \vdots\\ 0 & \cdots & 0 & 0 & r^{(j)}_{p_1+1} & \cdots & r^{(j)}_n\\ \vdots & & & & & & \vdots\\ 0 & \cdots & & & & \cdots & 0\end{pmatrix}\quad \longleftarrow \text{ row } j,\qquad A_{jj}\in\mathbb{Z}_-;$$

$$R^{(j)}=\left[\vec{0}\,\Big|\,\cdots\,\Big|\,\vec{0}\,\Big|\,\sum_{m=p_1+1}^{n} r^{(j)}_m\vec{e}_m\,\Big|\,\vec{0}\,\Big|\,\cdots\,\Big|\,\vec{0}\right],\qquad A_{jj}\in\mathbb{N}.$$

$$[T^{(i)},T^{(j)}]=0,\quad i,j=1,\dots,p_1;$$
$$[R^{(j)},R^{(k)}]=0,\quad [T^{(i)},R^{(j)}]=0,\quad i=1,\dots,p_1,\quad i\neq j,\quad j,k=q_1+1,\dots,p_1,$$

We take certain vector solutions given by linear combinations of columns of $\Psi^{(p_1)}(\lambda, u)$.

$$\vec{\Psi}_k(\lambda, u) := \begin{cases} \Psi^{(p_1)}(\lambda, u) \cdot \vec{e}_k, & A_{kk} \in \mathbb{C}\backslash\mathbb{N}; \\ \Psi^{(p_1)}(\lambda, u) \cdot \sum_{\ell=p_1+1}^{n} r_\ell^{(k)} \vec{e}_\ell, & A_{kk} \in \mathbb{N}. \end{cases}$$

$$\vec{\Psi}_k^{(sing)}(\lambda, u) := \begin{cases} \Psi^{(p_1)}(\lambda, u) \cdot \vec{e}_k, & A_{kk} \in \mathbb{C}\backslash\mathbb{Z}_-, \\ \Psi^{(p_1)}(\lambda, u) \cdot \dfrac{\vec{e}_\ell}{r_\ell^{(k)}}, & A_{kk} \in \mathbb{Z}_-, \quad \text{for } \ell \in \{p_1+1, \dots, n\} \text{ with } r_\ell^{(k)} \neq 0 \\ 0, & A_{kk} \in \mathbb{Z}_-, \quad \text{if } r_\ell^{(k)} = 0 \text{ for all } \ell \in \{p_1+1, \dots, n\}. \end{cases}$$

Then, from the analytic properties of $\Psi^{(p_1)}(\lambda, u)$, with several technical steps we prove that the above definitions exactly provide the selected and singular solutions in the statement of Theorem 3.
□

Remark. $M \cdot \vec{e}_k$ is the k-th column of a matrix M, where $\vec{e}_k = k$-th standard unit vector in $\mathbb{C}^n$.

To prove that connection coefficients are constant, we can show from the definitions that for a small loop

$$(\lambda - u_j) \mapsto (\lambda - u_j)e^{2\pi i}, \quad j \neq k,$$

we have at each $u \in \mathbb{D}(u^c)$:

$$\vec{\Psi}_k \longmapsto \vec{\Psi}_k + (e^{-2\pi i A_{jj}} - 1)c_{jk}\vec{\Psi}_j \qquad \text{for } A_{jj} \notin \mathbb{Z}$$

$$\vec{\Psi}_k \longmapsto \vec{\Psi}_k + 2\pi i c_{jk}\vec{\Psi}_j, \qquad \text{for } A_{jj} \in \mathbb{Z}$$

Then, essentially, we use the isomonodromic properties of matrix solutions of the Fuchsian Pfaffian system, and some subtle technical issues (arising form the fact that $\det[\vec{\Psi}_1 \mid \dots \mid \vec{\Psi}_n] = 0$ may be $= 0$ if A has some integer eigenvalues...).

To see that

$$c_{jk} = 0 \quad \text{for } j \neq k \text{ such that } u_j^c = u_k^c$$

just look at the definition $\boldsymbol{\vec{\Psi}_k = \vec{\Psi}_j^{(sing)}\, c_{jk} \; + \mathrm{reg}(\lambda - u_j)}$ and analytic properties of $\vec{\Psi}_k$, $\vec{\Psi}_j$ at u_k and u_j.

□

Main Result

We are ready to re-obtain a main result of Cotti, Dubrovin, D.G. in Duke Math. J., 168, (2019)., as far as Stokes phenomenon is concerned.

$$\vec{\Psi}_k(\lambda,u)\equiv\vec{\Psi}_k(\lambda,u\,|\eta),\quad \vec{\Psi}_k^{(sing)}(\lambda,u)\equiv\vec{\Psi}_k^{(sing)}(\lambda,u\,|\eta)\quad \text{branch-cuts with direction } \eta$$

- Suppose that u is fixed in a τ-cell of $\mathbb{D}(u^c)$, with $\tau=3\pi/2-\eta$.

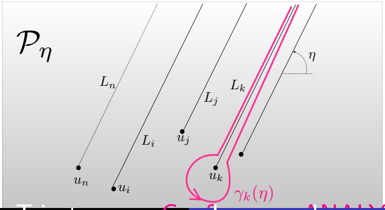
  For $\nu\in\mathbb{Z}$ we define the Laplace transforms:

$$\vec{Y}_k(z,u\,|\nu):=\frac{1}{2\pi i}\int_{\gamma_k(\eta-\nu\pi)} e^{z\lambda}\vec{\Psi}_k^{(sing)}(\lambda,u\,|\eta-\nu\pi)d\lambda,\qquad \text{for } A_{kk}\notin\mathbb{Z}_-,\qquad (4)$$

$$\vec{Y}_k(z,u\,|\nu):=\int_{L_k(\eta-\nu\pi)} e^{z\lambda}\vec{\Psi}_k(\lambda,u\,|\eta-\nu\pi)d\lambda,\qquad \text{for } A_{kk}\in\mathbb{Z}_-.\qquad (5)$$

and the matrix

$$Y_\nu(z,u):=\left[\vec{Y}_1(z,u\,|\nu)\,\Big|\,\dots\,\Big|\,\vec{Y}_n(z,u\,|\nu)\right],\qquad \text{fixed } u\in\tau\text{-cell.},$$



Theorem 4.[D.G. arXiv:2101.03397]

1) *The $Y_\nu(z,u)$, define by Laplace transf. above, are holomorphic in* $(\lambda, u) \in \mathcal{R}(\mathbb{C}\backslash\{0\}) \times \mathbb{D}(u^c)$.

2) *They are fundamental matrix solutions of* $\frac{dY}{dz} = \left(\Lambda(u) + \frac{A(u)}{z}\right) Y$.

3) *They have asymptotic behaviour uniform in* $u \in \mathbb{D}(u^c)$

$$Y_\nu(z,u) \sim (I + \sum_{l=1}^{\infty} F_l(u) z^{-l}) z^B e^{z\Lambda}, \qquad B = \mathrm{diag}(A),$$

$$z \to \infty \text{ in } \mathcal{S}_\nu := \left\{(\tau + (\nu-1)\pi) - \delta' < \arg z < (\tau + \nu\pi) + \delta'\right\}.$$

The coefficients $F_l(u)$ *are holomorphic in* $\mathbb{D}(u^c)$.

4) *Stokes matrices defined by*

$$\boldsymbol{Y_{\nu+1}(z,u) = Y_\nu(z,u)}\mathbb{S}_\nu,$$

- *are constant in the whole* $\mathbb{D}(u^c)$,
- *are expressed in terms of isomonodr. connection coefficients:*

$$(\mathbb{S}_0)_{jk} = \begin{cases} e^{2\pi i A_{kk}}\alpha_k\ c_{jk}, & j \prec k,\ u_j^c \neq u_k^c, \\ 1 & j = k, \\ 0 & j \succ k,\ u_j^c \neq u_k^c, \\ 0 & j \neq k\ ,\ u_j^c = u_k^c, \end{cases}$$

$$(\mathbb{S}_1^{-1})_{jk} = \begin{cases} 0 & j \neq k,\ u_j^c = u_k^c, \\ 0 & j \prec k,\ u_j^c \neq u_k^c, \\ 1 & j = k, \\ -e^{2\pi i(A_{kk}-A_{jj})}\alpha_k\ c_{jk} & j \succ k,\ u_j^c \neq u_k^c, \end{cases}$$

$$\mathbb{S}_{2\nu+1} = e^{-2\pi i \nu B}\mathbb{S}_1 e^{2\pi i\nu B}, \quad \mathbb{S}_{2\nu} = e^{-2\pi i\nu B}\mathbb{S}_0 e^{2\pi i\nu B}$$

Therefore

$$(\mathbb{S}_\nu)_{jk} = (\mathbb{S}_\nu)_{kj} = 0 \quad \textit{for } j \neq k \textit{ such that } u_j^c = u_k^c.$$

Relation $j \prec k$, *for* $u_j^c \neq u_k^c$, *means* $\Re(z(u_j^c - u_k^c))\Big|_{\arg z = \tau} < 0$.

$$\alpha_k := (e^{2\pi i A_{kk}} - 1),\ \textit{if } A_{kk} \notin \mathbb{Z}; \qquad \alpha_k := 2\pi i,\ \textit{if } A_{kk} \in \mathbb{Z},$$

Main idea of the proof. To give main ideas, let us just consider case the simplest case: $\underline{A_{kk} \notin \mathbb{Z}}$.

$$\vec{Y}_k(z, u \mid 0) := \frac{1}{2\pi i} \int_{\gamma_k(\eta)} e^{z\lambda} \vec{\Psi}_k^{(sing)}(\lambda, u \mid \eta) d\lambda \underset{A_{kk}\notin\mathbb{Z}}{=} \frac{1}{2\pi i} \int_{\gamma_k(\eta)} e^{z\lambda} \vec{\Psi}_k(\lambda, u \mid \eta) d\lambda$$

- $\lambda = \infty$ regular singularity $\Rightarrow$ the integral converges in a sector of amplitude π.

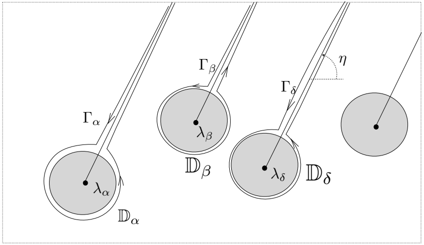
If u varies in $\mathbb{D}(u^c)$:

- $\vec{\Psi}_k(\lambda, u \mid \nu)$ is holomorphic at all $\lambda = u_j$ such that $u_j^c = u_k^c$ but $j \neq k$.

Therefore we change the path
$\gamma_k(\eta) \longmapsto \Gamma_\alpha(\eta)$

$$\vec{Y}_k(z, u \mid 0) = \frac{1}{2\pi i} \int_{\Gamma_\alpha(\eta)} e^{z\lambda} \vec{\Psi}_k(\lambda, u \mid \eta) d\lambda,$$

$$u_k^c = \lambda_\alpha.$$

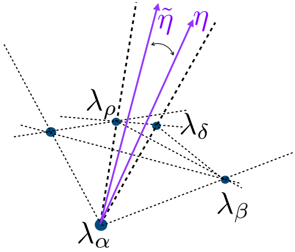


As a consequence, now u can vary in $\mathbb{D}(u^c)$ and the integral converges for sector

$$\mathcal{S}(\eta): \quad \pi/2 - \eta < \arg z < 3\pi/2 - \eta.$$

Enlarge sector.

$\vec{\Psi}(\lambda, u|\ \eta) = \vec{\Psi}(\lambda, u|\ \tilde{\eta})$ for:



This gives the sector $\bigcup_{\eta_- < \eta < \eta_+} \mathcal{S}(\eta)$ of amplitude $> \pi$.

- **Desired asymptotic behaviour**. It is standard computation

$$\vec{Y}_k(z, u\ |0) = \frac{1}{2\pi i} \int_{\gamma_k(\eta)} e^{z\lambda} \left(\Gamma(A_{jj}+1)\vec{e}_j + \sum_{l\geq 1} \vec{b}_l^{(k)}(u)(\lambda - u_k)^l \right) (\lambda - u_k)^{-A_{kk}-1} d\lambda.$$

$$\sim \left(\vec{e}_k + \sum_{\ell=1}^{\infty} \vec{f}_\ell^{\ (k)}(u) z^{-\ell} \right) z^{A_{kk}} e^{u_k z}, \quad \vec{f}_\ell^{\ (k)}(u) := \frac{\vec{b}_\ell^{(k)}(u)}{\Gamma(A_{kk}+1-l)}.$$

Use $\int_{\gamma_k(\eta)} (\lambda - \lambda_k)^a e^{z\lambda} d\lambda = z^{-a-1} e^{\lambda_k z} / \Gamma(-a)$

- Computation of Stokes matrices.

$$\text{Changing } \eta - \nu\pi: \underset{\text{Laplace}}{\longrightarrow} \text{ we find } Y_\nu(z,u),\ \nu \in \mathbb{Z}.$$

For example, for fixed $u \in \tau$-cell, we analyse change

$$\vec{\Psi}^{(sing)}(\lambda, u|\ \eta) \leftrightarrow \vec{\Psi}^{(sing)}(\lambda, u|\ \eta - \pi) \leftrightarrow \vec{\Psi}^{(sing)}(\lambda, u|\ \eta - 2\pi). \quad (**)$$

and obtain $\mathbb{S}_0(u)$ and $\mathbb{S}_0(u)$ in

$$Y_1(z,u) = Y_0(z,u)\mathbb{S}_0(u), \qquad Y_2(z,u) = Y_1(z,u)\mathbb{S}_1(u)$$

We find

$$(\mathbb{S}_0(u))_{jk} = \begin{cases} e^{2\pi i A_{kk}}\alpha_k\ c_{jk} & \text{for } j \prec k, \\ 1 & \text{for } j = k, \\ 0 & \text{for } j \succ k, \end{cases} \quad (\mathbb{S}_1^{-1}(u))_{jk} = \begin{cases} 0 & \text{for } j \prec k, \\ 1 & \text{for } j = k, \\ -e^{2\pi i (A_{kk} - A_{jj})}\alpha_k\ c_{jk} & \text{for } j \succ k. \end{cases}$$

Ordering relation $j \prec k \iff \Re(z(u_j - u_k))|_{\arg z = \tau} < 0$ well defined for u in the τ-cell.

But... several technical issues must be solved in studying $(**)$ ... See paper

- **Next step: Observe an important fact:** If size of $\mathbb{D}(u^c)$ small, then $j \prec k$ may change to $j \succ k$ from one τ-cell to another only for j, k such that $u_j^c = u_k^c$.

But in this case $c_{jk} = 0$ whenever $u_j^c = u_k^c$.

Thus, at every fixed u in every τ-cell,

$$(\mathbb{S}_0(u))_{jk} = \begin{cases} e^{2\pi i A_{kk}}\alpha_k\ c_{jk}, & j \prec k,\ u_j^c \neq u_k^c, \\ 1 & j = k, \\ 0 & j \succ k,\ u_j^c \neq u_k^c, \\ 0 & j \neq k\ ,\ u_j^c = u_k^c, \end{cases}$$

$$(\mathbb{S}_1^{-1}(u))_{jk} = \begin{cases} 0 & j \neq k,\ u_j^c = u_k^c, \\ 0 & j \prec k,\ u_j^c \neq u_k^c, \\ 1 & j = k, \\ -e^{2\pi i (A_{kk}-A_{jj})}\alpha_k\ c_{jk} & j \succ k,\ u_j^c \neq u_k^c, \end{cases}$$

with relation $j \prec k$, defined for $u_j^c \neq u_k^c$, when $\Re(z(u_j^c - u_k^c))\Big|_{\arg z = \tau} < 0$.

• Final step. The matrices $\mathbb{S}_\nu(u)$ are holomorphic in $\mathbb{D}(u^c)$ (by def.) and the $c_{jk}^{(\nu)}$ are constant in $\mathbb{D}(u^c)$. We conclude that the $\mathbb{S}_\nu$ are constant.

□

SYSTEM AT THE CENTRAL COALESCENCE POINT

$$(u_1^c,\dots,u_n^c)=(\underbrace{\lambda_1,...,\lambda_1}_{p_1\text{ times}},\underbrace{\lambda_2,...,\lambda_2}_{p_2\text{ times}},\dots,\underbrace{\lambda_s,...,\lambda_s}_{p_2\text{ times}}),\quad p_1+p_2+\cdots+p_s=n.$$

$$\frac{d\Psi}{d\lambda}=\left(\frac{\sum_{j=1}^{p_1}B_j(u^c)}{\lambda-\lambda_1}+\frac{\sum_{j=p_1+1}^{p_1+p_2}B_j(u^c)}{\lambda-\lambda_2}+\cdots+\frac{\sum_{j=p_1+\ldots+p_{s-1}+1}^{n}B_j(u^c)}{\lambda-\lambda_s}\right)\Psi$$

Gauge transformation $\Psi(\lambda)=G^{(p_1)}\widetilde{\Psi}(\lambda)$ to reduce $\sum_{j=1}^{p_1}B_j(u^c)$ to Jordan $T^{(p_1)}$.

$$\frac{d\widetilde{\Psi}}{d\lambda}=\left(\frac{T^{(p_1)}}{\lambda-\lambda_1}+\sum_{\alpha=2}^{s}\frac{D_\alpha^{(p_1)}}{\lambda-\lambda_\alpha}\right)\widetilde{\Psi}$$

Fundamental solutions have form

$$\mathring{\Psi}^{(p_1)}(\lambda)=G^{(p_1)}\left(I+\sum_{j=1}^{\infty}\mathfrak{G}_j(\lambda-\lambda_1)^j\right)(\lambda-\lambda_1)^{T^{(p_1)}}(\lambda-\lambda_1)^{R^{(p_1)}},$$

If $A_{jj}-A_{kk}\in\mathbb{Z}$ for some j,k, then the $\mathfrak{G}_j$ contain free parameters (a class of solutions).

Recall that

$$d\Psi = \Big(\sum_{k=1}^{n} \frac{B_k(u)}{\lambda - u_k} d(\lambda - u_k) + \sum_{k=1}^{n} \gamma_k(u) du_k\Big)\Psi.$$

admits the fundamental matrix solution

$$\Psi^{(p_1)}(\lambda, u) = G^{(p_1)} U^{(p_1)}(\lambda, u) \cdot \prod_{j=1}^{p_1} (\lambda - u_j)^{T^{(j)}} \cdot \prod_{j=q_1+1}^{p_1} (\lambda - u_j)^{R^{(j)}},$$

Take

$$\Psi^{(p_1)}(\lambda, u^c).$$

It is equal to <u>one element of the class</u>

$$\mathring{\Psi}^{(p_1)}(\lambda) = G^{(p_1)}\Big(I + \sum_{j=1}^{\infty} \mathfrak{G}_j(\lambda - \lambda_1)^j\Big)(\lambda - \lambda_1)^{T^{(p_1)}}(\lambda - \lambda_1)^{R^{(p_1)}},$$

Laplace transforms of $\mathring{\Psi}^{(p_1)}(\lambda)$ and analogous $\mathring{\Psi}^{(p_\alpha)}(\lambda)$, $\alpha = 1, ..., s$, $\longrightarrow$ find a class of matrix solutions $\mathring{Y}_\nu(z)$ of

$$\frac{dY}{dz} = \left(\Lambda(u^c) + \frac{A(u^c)}{z}\right) Y$$

such that

$$\mathring{Y}_\nu(z) \sim \mathring{Y}_F(z) = \Big(I + \sum_{l=1}^{\infty} \mathring{F}_l z^{-l}\Big) z^B e^{\Lambda(u^c)}.$$

So, $\mathring{Y}_F(z)$ is a parameter class of formal solutions. $Y_F(z, u^c)$ ia one element.

Thank you !