Isomonodromic deformations of irregular singularities

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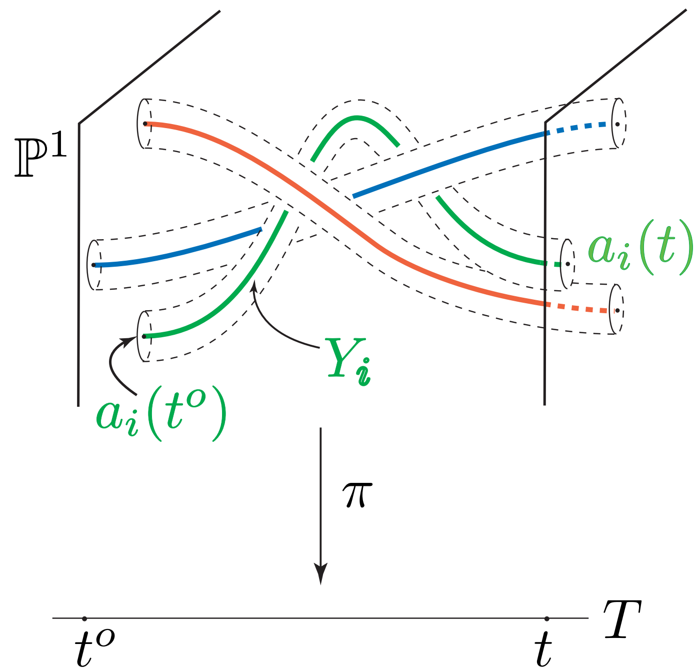
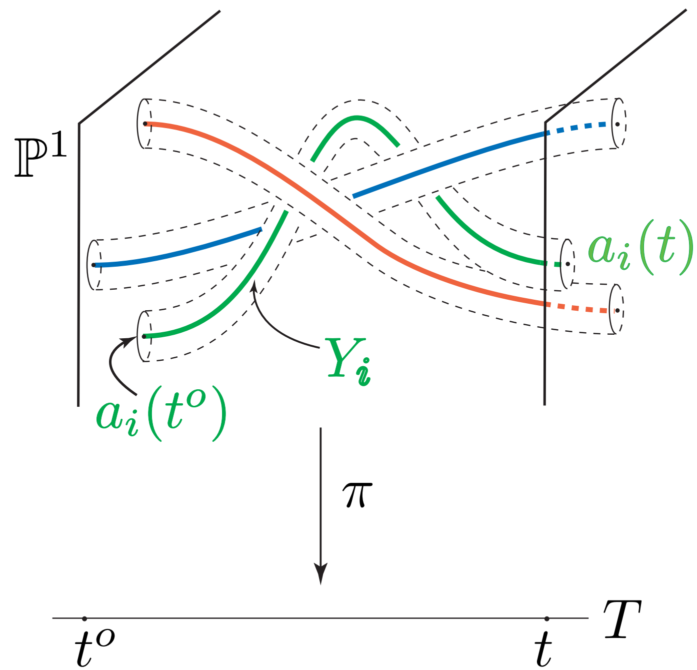
Isomono. deform.: reg. sing. case



Theorem (Malgrange). *If ∇^o has simple poles on E^o and T is 1-conn., there exists a unique vector bundle E on $\mathbb{P}^1 \times T$ equipped with an integrable logarithmic connection ∇ having poles along the hypersurfaces Y_i, and with an identification $(E, \nabla)_{|\mathbb{P}^1 \times \{t^o\}} \xrightarrow{\sim} (E^o, \nabla^o)$.*

Main property used. The log. connection deforms uniquely in each tube $\mathrm{nb}(Y_i)$.

Isomono. deform.: reg. sing. case



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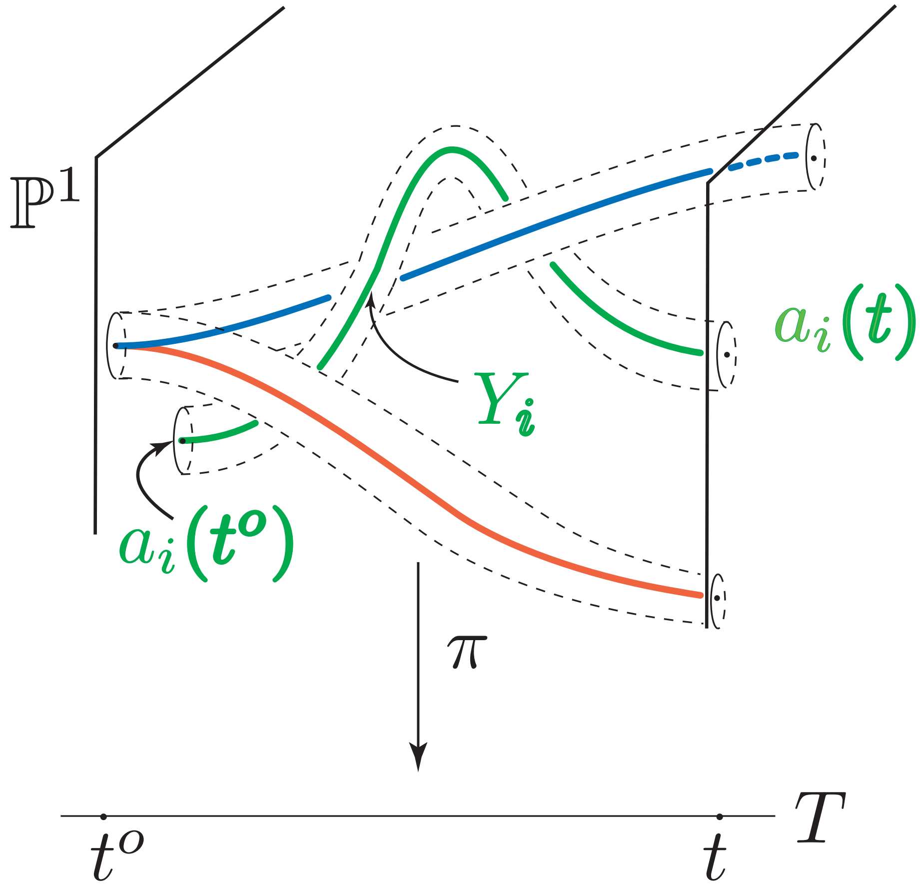
By Laplace transform w.r.t. x. $\rightsquigarrow$ Irreg. sing. along $\{\infty\} \times T$, ***non-coalescing*** exponents (Jimbo-Miwa-Ueno, Malgrange).

Isomono. deform.: splitting reg. sing. case



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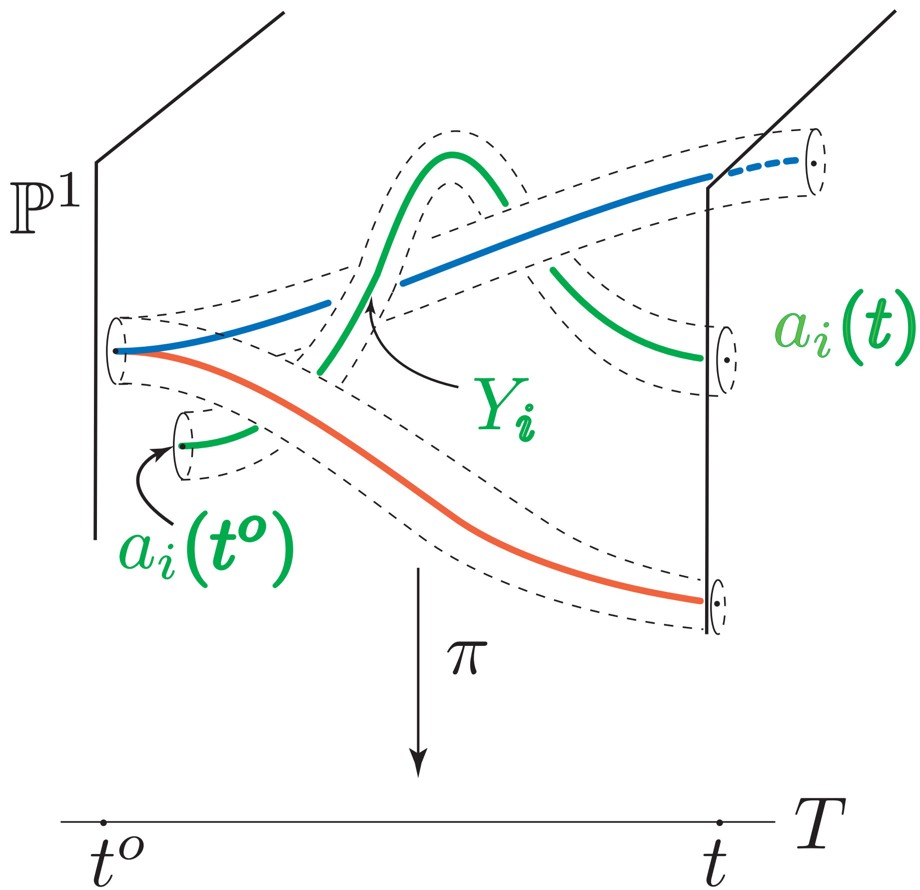

By Laplace transform w.r.t. $\boldsymbol{x}$**.** $\rightsquigarrow$ Irreg. sing. along $\{\infty\} \times T$, ***coalescing*** exponents (Cotti-Dubrovin-Guzzetti).

Irregular singularity in dim. one

Theorem (Levelt-Turrittin). *For any* $\Omega(z) = A(z)\mathrm{d}z$ *(*$A(z) =$ *Laurent series),* $\exists$ *formal gauge transf.* $\widehat{P}(z) \in \mathrm{GL}_d(\mathbb{C}((z)))$ *s.t.* $\widehat{P}[\Omega] =$ *Jordan normal form* Ω^{norm}*; up to ramif.,* Ω^{norm} *is block-diag., with blocks*

$$\boxed{\mathrm{d}\varphi_i(z) + C_i \cdot \mathrm{d}z/z, \quad \varphi_i \in \mathbb{C}((z)), \quad C_\varphi \text{ is cst.}}$$

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Fix a formal normal form, e.g. without ramification

$$\Omega^{\mathrm{norm}}(z) = \mathrm{diag}\big((\mathrm{d}\varphi_i(z) + C_i \cdot \mathrm{d}z/z)_i\big).$$

Theorem (Malgrange-Sibuya).

$$\boxed{\big\{\big(\Omega(z), \widehat{P}(z)\big)\big\} = H^1\big(S^1, \underbrace{\mathrm{St}(\Omega^{\mathrm{norm}}(z))}_{\text{Stokes cocycles}}\big)}$$

Example (case $\varphi_i = c_i/z \ \forall\, i$). A Stokes cocycle can be represented by a pair of Stokes matrices (S_+, S_-).

Integrable deform. of an irreg sing.

- Consider the pb in a tube $\mathrm{nb}(Y_i)$ of one hypersurface,
  $\rightsquigarrow$ can assume the total space is $T \times \mathrm{nb}(0)_z$.
- $\rightsquigarrow$ Need to ***be given*** a formal model before stating the problem.
- $\rightsquigarrow$ Better to considered objects with a ***given*** formal isomorphism with the formal model, in order to eliminate automorphisms.

Generalization of Malgrange-Sibuya

- Fix $\Omega^{(0)}(t, z)$:
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$$\boxed{\Omega^{(0)} = \frac{A(t,z)}{z^p} \cdot \frac{\mathrm{d}z}{z} + \sum_i \frac{\Omega_i(t,z)}{z^{p_i}} \,\mathrm{d}t_i, \quad \left\{\begin{matrix} A(t,z) \\ \Omega_i(t,z) \end{matrix}\right. \quad \text{holom.}}$$

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  - merom. connection matrix, poles on $T \times \{0\}$,
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- Consider pairs $(\Omega, \widehat{P})$ such that
  - $\widehat{P} \in \mathrm{GL}_d(\mathcal{O}_T((z)))$,
  - $\widehat{P}[\Omega] = \Omega^{(0)}$.
- Isomorphism of pairs:

$$\boxed{(\Omega^{(1)}, \widehat{P}^{(1)}) \simeq (\Omega^{(2)}, \widehat{P}^{(2)}) \overset{\text{def.}}{\iff} (\widehat{P}^{(2)})^{-1}\widehat{P}^{(1)} \textit{ cvg.}}$$

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Problem (Generalization of Malgrange-Sibuya).
Given $\Omega^{(0)}$, to classify pairs $(\Omega, \widehat{P})$ up to isom.

A theorem of J.-B. Teyssier (2018)

Theorem (J.-B. Teyssier). *If* $T = \mathrm{nb}(t^o)$*,*

$$\boxed{\begin{array}{c}(\Omega^{(1)}, \widehat{P}^{(1)}) \simeq (\Omega^{(2)}, \widehat{P}^{(2)}) \\ \Updownarrow \\ (\Omega^{(1)}, \widehat{P}^{(1)})_{|t=t^o} \simeq (\Omega^{(2)}, \widehat{P}^{(2)})_{|t=t^o}\end{array}}$$

In other words, ***uniqueness****:* $\exists$ ***at most one*** $(\Omega, \widehat{P})$ *extending* $(\Omega, \widehat{P})_{|t=t^o}$ *up to iso.*

Remark (Important condition). $\widehat{P}$ holom., i.e., ***not formal***, w.r.t. t

$$\boxed{\widehat{P} \in \mathrm{GL}_d(\mathcal{O}_T(\!(z)\!))}$$

Main tool 1: Higher dim. Levelt-Turrittin

- $X = \mathbb{C}^m$, coord. $x_1, \dots, x_m$, $x^o = 0$
- D: $x_1 \cdots x_\ell = 0$,
- $S(x^o) = \{x_1 = \cdots = x_\ell = 0\}$, $\mathcal{O}_S = \mathbb{C}\{x_{\ell+1}, \dots, x_m\}$
- Normal form:

$$\Omega^{\mathrm{norm}} = \bigoplus_{a \in A} \Big(\mathrm{d}\varphi_a \, \mathrm{Id}_{r_a} + \sum_{i=1,\dots,\ell} C_{a,i} \frac{\mathrm{d}x_i}{x_i} \Big) \qquad \begin{cases} \varphi_a \in \mathcal{O}(*D)/\mathcal{O} \\ C_{a,i} \text{ constant} \\ \quad\quad\quad \text{size } r_a \end{cases}$$

- Good normal form ($\Leftrightarrow$ non-coalescence):

$$\boxed{\forall\, a, b \in A, \quad \varphi_a - \varphi_b = \frac{u_{a,b}(x)}{x_1^{m_1} \cdots x_\ell^{m_\ell}} \qquad \begin{cases} u_{a,b}(x) \text{ holom.} \\ u_{a,b}(0) \neq 0 \end{cases}}$$

Higher dim. Levelt-Turrittin

Definition. Ω integrable satisfies Levelt-Turrittin at x^o if

- $\exists$ a ***good*** normal form Ω^{norm},
- $\exists\ \widehat{P}_{x^o} \in \mathrm{GL}_d\big(\mathcal{O}_S[\![x_1, \ldots, x_\ell]\!][x_1^{-1}, \ldots, x_\ell^{-1}]\big)$

s.t. $\Omega = \widehat{P}_{x^o}[\Omega^{\mathrm{norm}}]$.

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s.t. $\Omega = \widehat{P}_{x^o}[\Omega^{\mathbf{norm}}]$.

Theorem (Kedlaya-Mochizuki, 2008-11). *Given Ω integrable with poles on $T \times \{0\}$, $\exists$ a sequence of blowing ups $e : X \to T \times \mathbb{C}_z$ such that:*

- *X is smooth and $D := e^{-1}(T \times \{0\})$ is a divisor with normal crossings;*
- *Near each $x^o \in D$, the higher dimensional Levelt-Turrittin theorem applies to $e^*\Omega$.*

Main tool 2: Majima

Theorem (Majima, 1984), C.S., Mochizuki). *If* Ω *satisfies Levelt-Turrittin at* $x^o \in D$,
$\widehat{P}_{x^o}$ *can be lifted in any small polysectors of dir.*
$\theta^o = (\theta_1^o, \ldots, \theta_\ell^o)$ *to* P_{x^o,θ^o} *having asympt. exp.*

- $\rightsquigarrow$ Higher dim. Stokes multipliers
- $\rightsquigarrow$ Higher dim. Malgrange-Sibuya

Existence statement

- Given $\Omega^{(0)}(t,z)$ and $\big(\Omega(t^o,z),\widehat{P}(t^o,z)\big)$,

$$\exists\,\big(\Omega(t,z),\widehat{P}(t,z)\big)\,?$$

- I.e., can we ***deform*** $\big(\Omega(t^o,z),\widehat{P}(t^o,z)\big)$ so that it matches $\Omega^{(0)}(t,z)$?

Existence statement–Example

- $T = \mathbb{C}^n$, coordinates $t_1, \dots, t_n$,
- $\Omega^{(0)}(t,z) = \bigoplus_{i=1}^{n} \Big(\mathrm{d}(t_i/z)\, \mathrm{Id}_{r_i} + C_i \dfrac{\mathrm{d}z}{z} \Big)$
- C_i constant of size r_i, $\sum_i r_i = d$.
- $\Delta : \bigcup_{i \neq j} \{t_i = t_j\}$.
- Strata S of Δ: coalescence indices fixed.

Theorem (C.S. 2017). *For any stratum S and any $t^o \in S$, the restriction map Restr$_{t^o}$*

$$\big\{ \big(\Omega(t,z), \widehat{P}(t,z) \big) \mid t \in \mathrm{nb}(S) \big\} \longrightarrow \big\{ \big(\Omega(t^o,z), \widehat{P}(t^o,z) \big) \big\}$$

is a bijection.

Proof of existence 1

- $\Omega^{(0)}(t,z) = \bigoplus_{i=1}^{n}\Big(\mathrm{d}(t_i/z)\,\mathrm{Id}_{r_i} + C_i\dfrac{\mathrm{d}z}{z}\Big)$
- work near $t^o \in S \subset \Delta \subset T \times \{0\}$

Theorem (existence). *For any stratum S and any $t^o \in S$, the restriction map Restr$_{t^o}$*

$$\big\{\big(\Omega(t,z), \widehat{P}(t,z)\big) \mid t \in \mathrm{nb}(S)\big\} \longrightarrow \big\{\big(\Omega(t^o,z), \widehat{P}(t^o,z)\big)\big\}$$

is a surjective.

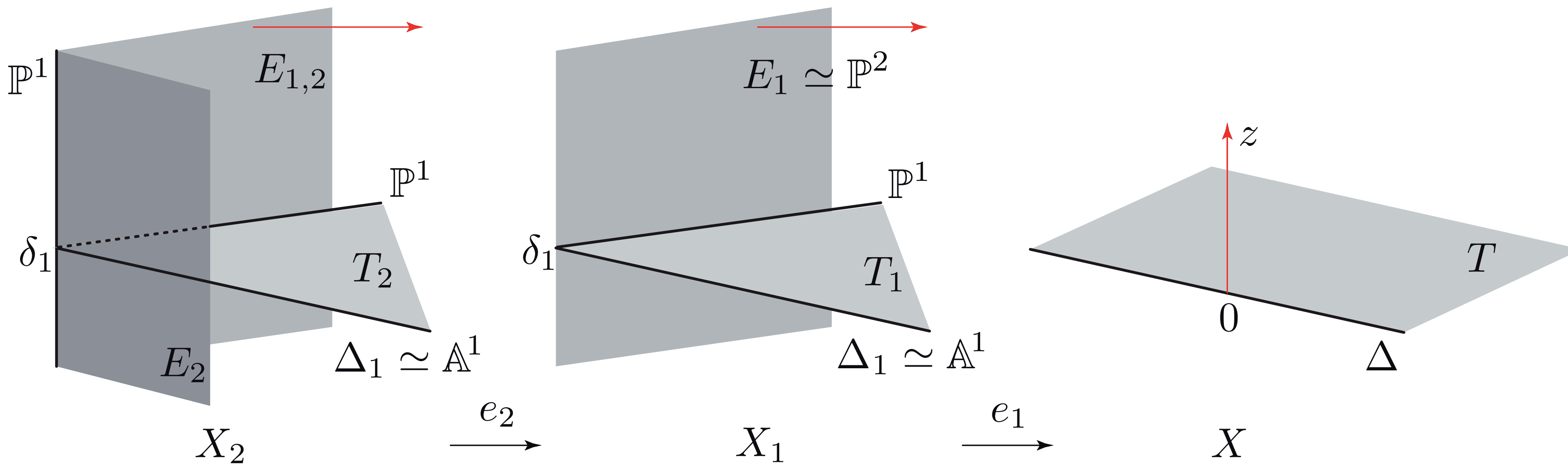
First step. Sequence of blowing-ups $e : X \to T \times \mathbb{C}_z$ such that the family $e^*\mathrm{d}(t_i/z)$ is ***good*** at every $x^o \in D$.

Remark. Easy to find e for this example.

Proof of an example

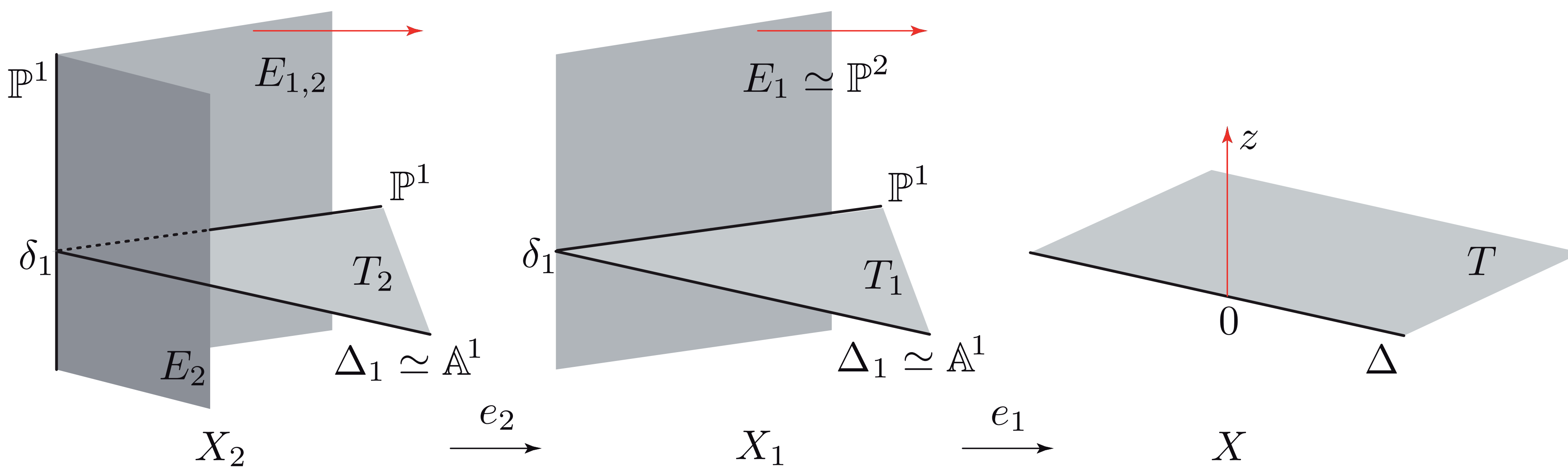
- $T = \mathbb{C}^2, \quad t = (t_1, t_2), \quad \Delta = S = \{t_2 = 0\},$
- $\Omega^{(0)}(t,z) = \bigoplus_{i=1}^{4}\Big(\mathrm{d}(f_i(t)/z)\,\mathrm{Id}_{r_i} + C_i \dfrac{\mathrm{d}z}{z}\Big),$
- $f_1(t) = t_1, \quad f_2(t) = t_1 + t_2, \quad f_3(t) = t_1 + a,$
  $f_4(t) = t_2 + b,$
- $a,\ b,\ a - b \neq 0$
- only coalescence: $f_1 - f_2$ vanishes on Δ.

Blowing-ups



- $e = e_2 \circ e_1, \quad e^*\Omega^0(t, z)$ ***good***,
- Have replaced coalescence with smooth divisor T with non-coalescence (goodness) with non-smooth divisor (i.e., ncd).
- General. Malgrange-Sibuya thm: if we know $e^*\Omega$ near one point x^o of D, can extend $e^*\Omega$ to a neighbd of the stratum of x^o.

Blowing-ups



- Start with $\Omega_{|z\text{-axis}}$ given, same as $e^*\Omega_{|\text{red line}}$
- $\exists\, e^*\Omega$ on a neighbd of $E_{1,2} \setminus (T_2 \cup E_2)$ (easy)
- Main Lemma: $e^*\Omega$ extends to a neighbd of $E_{1,2}$
- $\rightsquigarrow e^*\Omega$ extends to each stratum of D
- Check compatibility of the various extensions
- Descent: $e^*\Omega \rightsquigarrow \Omega$. $\square$