On Gevrey asymptotics for linear singularly perturbed equations with linear fractional transforms

Alberto Lastra and Stéphane Malek (joint work with Guoting Chen)

ATDDE2020, January 28–February 3, 2021

Main problem

The main problem under study is a family of linear singularly perturbed Cauchy problems of the form

$$Q(\partial_z)u(t,z,\epsilon) = L(z,\epsilon,t,\partial_t,\partial_z)u(t,z,\epsilon) + \epsilon^{\delta_0}\left((t^2\partial_t)^{\delta_0}R_0(\partial_z)u\right)\left(\frac{1}{1+k_0\epsilon t},z,\epsilon\right) + f(t,z,\epsilon),$$

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under null initial data $u(0,z,\epsilon)\equiv 0$.

$L \in (\mathcal{O}(H\times D))[t,\partial_t,\partial_z]$, $f\in\mathcal{O}(D_1\times H\times D)$,

where D, D_1 are discs at the origin, H is a horizontal strip.

$Q(X), R_0(X)\in\mathbb{C}[X]$, and δ_0, k_0 are positive integers.

The main motivation of the present study is the previous work [1] where we consider the Cauchy problem

$$P(\epsilon t^2\partial_t)\partial_z^S u(t,z,\epsilon) = \sum_{k=(k_0,k_1,k_2)\in\mathcal{A}} c_k(z,\epsilon)((t^2\partial_t)^{k_0}\partial_z^{k_1}u)\left(\frac{t}{1+k_2\epsilon t},z,\epsilon\right)$$

for the Cauchy data $(\partial_z^j u)(t,0,\epsilon)=\varphi_j(t,\epsilon)$, $0\leq j\leq S-1$,

[1] A. Lastra, S. Malek, Parametric Borel summability for linear singularly perturbed Cauchy problems with linear fractional transforms, Electron. J. Differ. Equ. 2019, Paper No. 55, 75 p. (2019).

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$P(X)\in\mathbb{C}[X]$ with all its roots in $\{z\in\mathbb{C} : \mathrm{Re}(z)>0\}$

$c_k(z,\epsilon)\in\mathcal{O}\{z,\epsilon\}$

$\mathcal{A}$ a finite subset of $\mathbb{N}^3$ under certain technical conditions.

The action of a linear fractional map

$$t \mapsto \frac{t}{1 + k_2 \epsilon t}$$

is motivated by the fact that the change of variable $t = \dfrac{1}{s}$ turns the problem into a singularly perturbed linear PDE through the change of function

$$u(t, z, \epsilon) = X(\frac{1}{t}, z, \epsilon)$$

with small shifts

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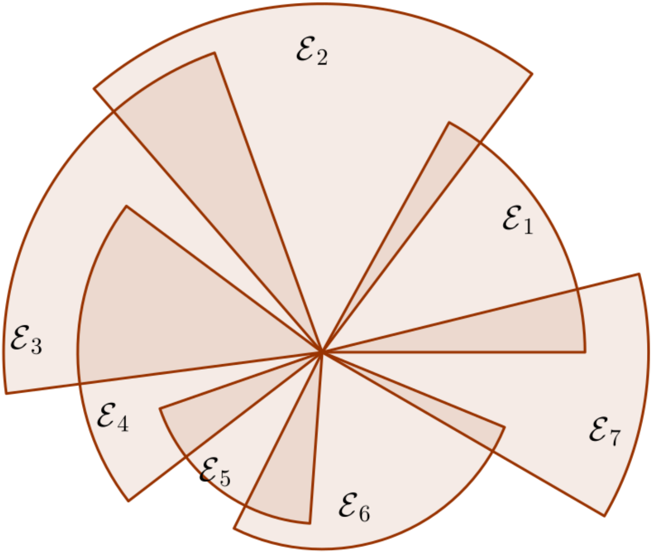
Equations involving such operator appear in applications such as reaction-diffusion equations with small delays [2,3].

[2] X. Zhang, Singular perturbation of initial boundary value problems of reaction diffusion equations with delay, Applied Mathematics and Mechanics, vol. 5, no. 3, 1994.

[3] J. Mo, Z. Wen, Singularly perturbed reaction diffusion equations with time delay, Applied Math. Mech. 31 (6), 2010.

In this previous work, two families of actual solutions are constructed.

First family:
Let $(\mathcal{E}_p)_{0 \leq p \leq \iota - 1}$ be a family of open bounded sectors with vertex at 0 which cover a punctured disc.


$\mathcal{E}_2$
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$\mathcal{E}_6$
$\mathcal{E}_7$

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$\mathcal{T}$ is a bounded sector at 0.

The main problem admits holomorphic solutions

$$u_p(t,z,\epsilon)=\int_{L_{\gamma_p}}\omega_p(\tau,z,\epsilon)e^{-\frac{\tau}{\epsilon t}}\frac{d\tau}{\tau}.$$

on $\mathcal{T}\times D(0,\rho)\times\mathcal{E}_p$.

$L_{\gamma_p}=[0,\infty)e^{i\gamma_p}$, for some $\gamma_p\in\mathbb{R}$, and $\omega_p(\tau,z,\epsilon)$ is a holomorphic function near the origin in (τ,z), and a punctured disc in ϵ, which can be extended to an infinite sector on the first variable with exponential growth.

Second family: Let $(\mathcal{E}_q^-)_{-n \leq q \leq n}$ be a family of open bounded sectors with vertex at 0 which cover a punctured semicircle $D(0,\epsilon_0) \cap \{\epsilon \in \mathbb{C} : \mathrm{Re}(\epsilon) < 0\}$.

Assume that for each $0 \leq j \leq S-1$, the initial data $\varphi_j(t,\epsilon)$ consists of certain family of holomorphic functions $\varphi_{j,q}(t,\epsilon)$ on $\mathcal{T} \times \mathcal{E}_q^-$ for $-n \leq q \leq n$.

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on $\mathcal{T}\times D(0,\rho)\times\mathcal{E}_q^-$.

$\omega_q^-(\tau,z,\epsilon)$ is a holomorphic function in $H\times D(0,\rho)\times D(0,\epsilon_0)\setminus\{0\}$, where

$$H=\{\tau\in\mathbb{C}:\mathrm{Re}(\tau)<0,|\mathrm{Im}(\tau)|<\beta\}$$

for some $\beta>0$ with

- super-exponential decay on some substrips $(H_q)_{-n\leq q\leq n}$,
- super-exponential growth on some substrips $(J_q)_{-n\leq q\leq n}$.

$$|\omega_n^-(\tau,z,\epsilon)| \leq C|\tau| \exp\left(\frac{\sigma_1}{|\epsilon|}|\tau| - \sigma_2 e^{\sigma_3|\tau|}\right),$$

for $\tau \in H_q$, $z \in D(0,\rho)$, $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$.

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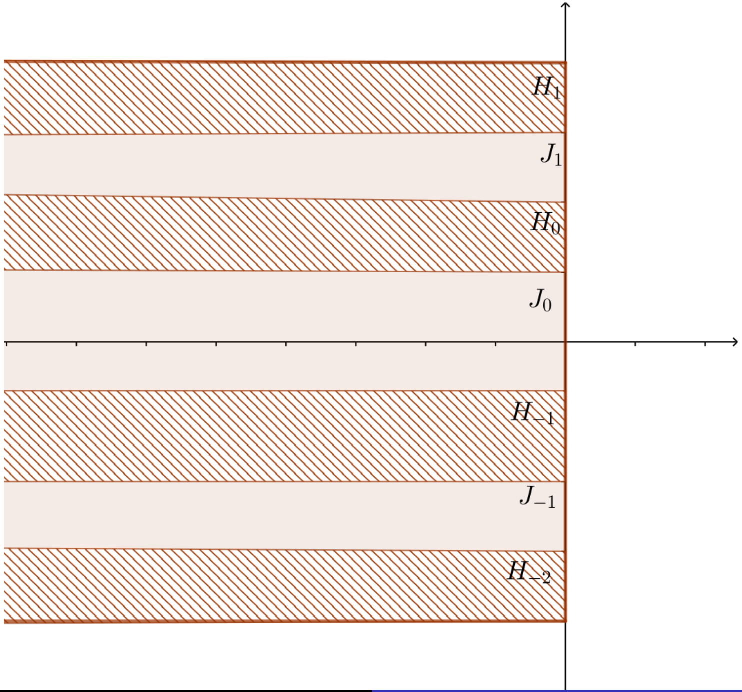
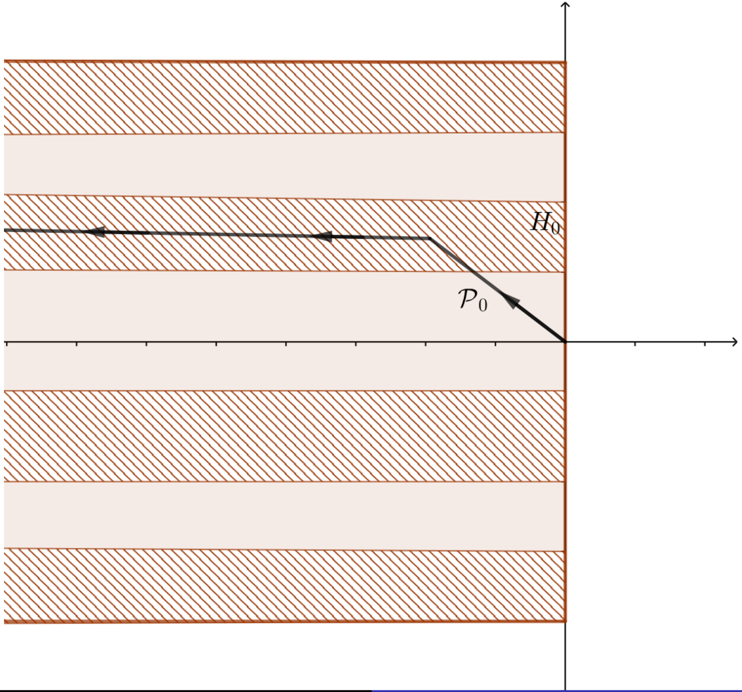

H_1
J_1
H_0
J_0
H_{-1}
J_{-1}
H_{-2}



H_0
$\mathcal{P}_0$

The asymptotic behavior of the solutions $u_p(t,z,\epsilon)$ and $u_q^-(t,z,\epsilon)$ is also studied for $\epsilon \to 0$.

We obtain:

$$u_p(t,z,\epsilon) = u_p^1(t,z,\epsilon) + u_p^2(t,z,\epsilon),$$
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There exist

$$\hat{u}^1(t,z,\epsilon), \hat{u}^2(t,z,\epsilon) \in \mathcal{O}_b(\mathcal{T} \times D(0,\rho))[[\epsilon]]$$

such that u_p^1 (resp. u_q^{-1}) admits $\hat{u}^1$ as its Gevrey asymptotic expansion of order 1 in $\mathcal{E}_p$ for all $0 \leq p \leq \iota - 1$ (resp. in $\mathcal{E}_q$ for all $-n \leq q \leq n$).

such that u_p^2 (resp. u_q^{-2}) admits $\hat{u}^2$ as its Gevrey asymptotic expansion of order 1^- in $\mathcal{E}_p$ for all $0 \leq p \leq \iota - 1$ (resp. in $\mathcal{E}_q$ for all $-n \leq q \leq n$). (with level 1^+)

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- Some more generality on the Cauchy problems can be stated. Such problems are written in the form of coupled equations for given Cauchy data.

Main problem

We write the main problem under study in the form

$$
\begin{aligned}
Q(\partial_z)u(t,z,\epsilon) = \epsilon^{\delta_D}(t^2\partial_t)^{\delta_D}R_D(\partial_z)u(t,z,\epsilon) \\
+ \epsilon^{\delta_0}\left((t^2\partial_t)^{\delta_0}R_0(\partial_z)u\right)\left(\frac{t}{1+k_0\epsilon t},z,\epsilon\right) \\
+ \sum_{\ell\in I}\epsilon^{\Delta_\ell}t^{\delta_\ell}\partial_t^{d_\ell}c_\ell(z,\epsilon)R_\ell(\partial_z)u(t,z,\epsilon) + f(t,z,\epsilon),
\end{aligned}
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\end{aligned}
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$u(0,z,\epsilon)\equiv 0.$

$Q(X), R_D(X), R_0(X), R_\ell(X) \in \mathbb{C}[X]$ for $\ell \in I$, under additional assumptions.

$\delta_D, \delta_0, k_0 \geq 1$ and $\Delta_\ell, \delta_\ell, d_\ell \geq 0$ for $\ell \in I$, under technical conditions.

$f(t,z,\epsilon)$ is holomorphic in a neighborhood of the origin w.r.t. (t,ϵ) and on a horizontal strip with respect to z

$$H_\beta = \{z \in \mathbb{C} : |\mathrm{Im}(z)| < \beta\},$$

for some $\beta > 0$.

$c_\ell(z,\epsilon)$ are holomorphic functions on $H_\beta \times D(0,\epsilon_0)$.

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- A second leading term
$$\epsilon^{\delta_0}\left((t^2\partial_t)^{\delta_0} R_0(\partial_z)u\right)\left(\frac{t}{1+k_0\epsilon t}, z, \epsilon\right)$$
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- Other fractional linear maps $\frac{t}{1+k_\ell \epsilon t}$ could be included in the linear part, assuming $0<k_\ell<k_0$. This has been omitted for the sake of clarity.

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- The main equation does not factorize into a set of coupled equations, a condition imposed to extend our previous results, which is now relaxed.

Let $(\mathcal{E}_p)_{0 \leq p \leq \varsigma - 1}$ be a finite family of bounded sectors with vertex at 0 which covers a neighborhood of the origin in $\mathbb{C}^{\star}$.

Let $\mathcal{T}$ be a bounded sector with vertex at 0.

We split the set $\{0, 1, \dots, \varsigma - 1\}$ into a disjoint union of indices as follows.

$$\{0, 1, \dots, \varsigma - 1\} = J_1 \uplus J_2$$

$$\boxed{J_2}$$

There exists $\rho > 0$ such that for every $p_2 \in J_2$ one can select an infinite sector $S_{d_{p_2}}$ with bisecting direction $d_{p_2} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that

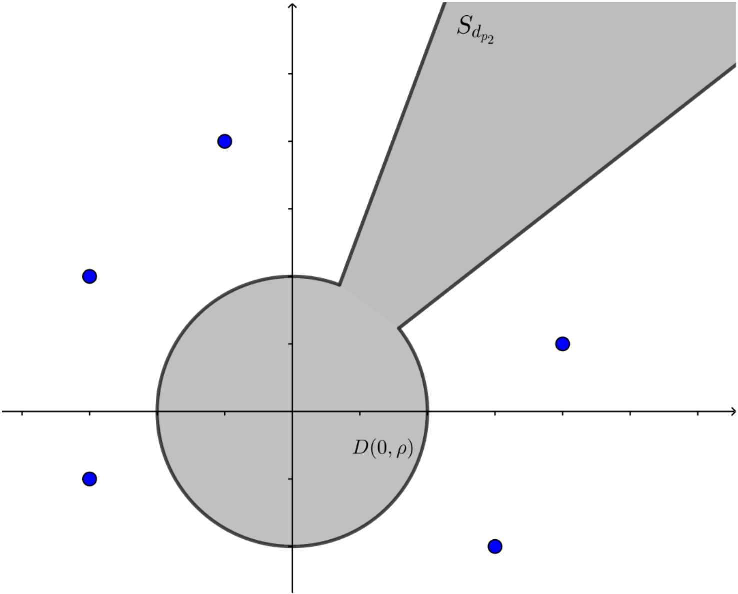
$$S_{d_{p_2}} \cup D(0,\rho)$$

avoids all the roots of the polynomial

$$\tau \mapsto Q(im) - \tau^{\delta_D} R_D(im),$$

for all $m \in \mathbb{R}$.

This condition comes from the first leading term of the main equation.


$S_{d_{p2}}$
$D(0, \rho)$

$$\boxed{J_1}$$

There exists a convex neighborhood $\mathcal{U}$ of 0 such that for every $p_1 \in J_1$ one can choose a horizontal halfstrip

$$L_{p_1} = \{\tau \in \mathbb{C} : \beta_{1,p_1} < \mathrm{Im}(\tau) < \beta_{2,p_1}, \mathrm{Re}(\tau) < 0\}$$

such that $L_{p_1} \cup \mathcal{U}$ avoids the zeros of the function

$$\tau \mapsto Q(im) - \tau^{\delta_0} e^{-\tau k_0} R_0(im),$$

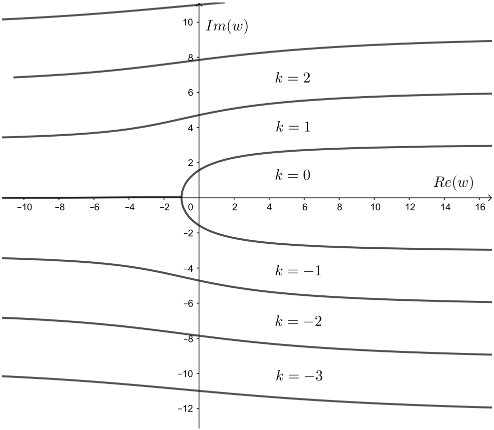
for all $m \in \mathbb{R}$.

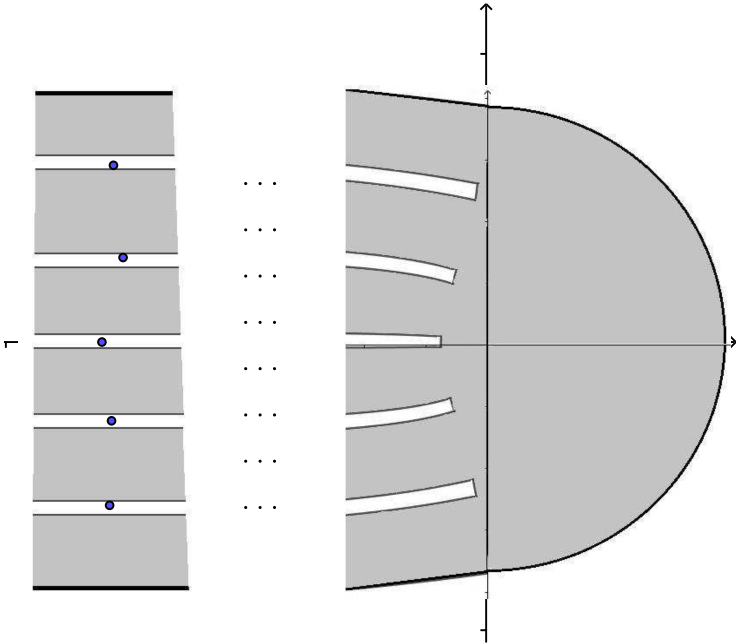
This condition comes from the second leading term of the main equation.

These zeroes are quite related with Lambert W function which is defined by

$$W(z)e^{W(z)} = z.$$

It is a multivalued function, whose branches split the $w = W(z)$ plane as follows.





For every $p_2 \in J_2$,

$$u_{p_2}(t,z,\epsilon) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{L_{\gamma_{d_{p_2}}}} \omega_{p_2}(\tau,m,\epsilon) e^{-\frac{\tau}{\epsilon t}} e^{imz} \frac{d\tau}{\tau} dm,$$

is an analytic solution of the problem, holomorphic on $\mathcal{T} \times H_\beta \times \mathcal{E}_{p_2}$.

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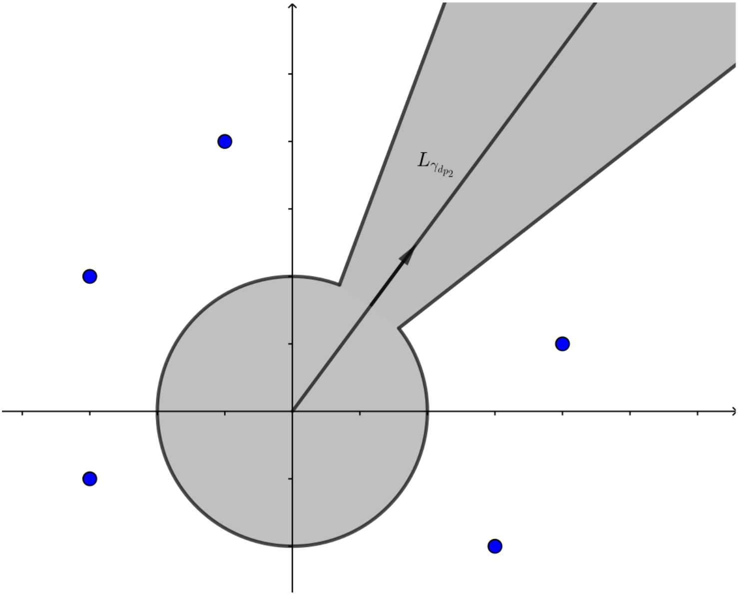
The integration path is $L_{\gamma_{d_{p_2}}} = [0,\infty)e^{i\gamma_{d_{p_2}}}$.

The function $\omega_{p_2}(\tau,m,\epsilon)$ satisfies exponential bounds w.r.t. τ and it is obtained from a fixed point argument on certain operator acting on some complex Banach spaces.


$L_{\gamma_{dp2}}$

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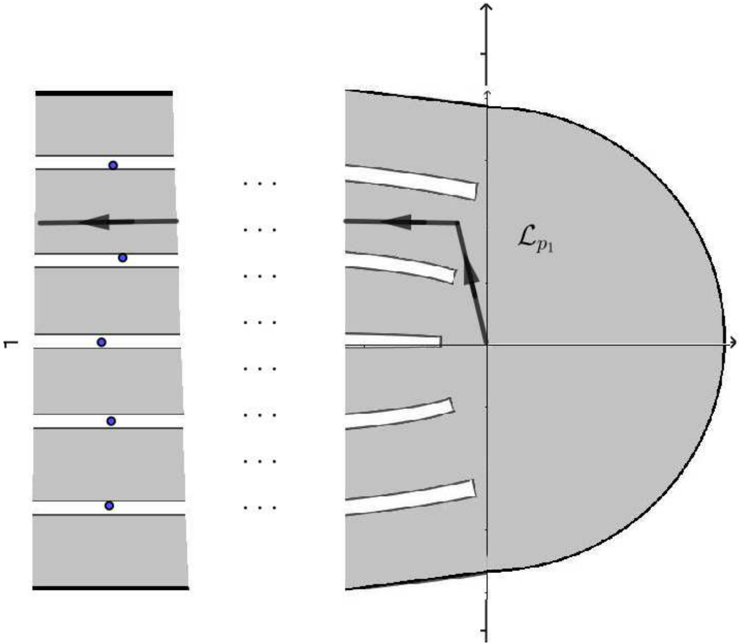
The integration path $\mathcal{L}_{d_{p_1}}$ is of the following form.



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$$\sup_{t \in \mathcal{T}, z \in H_{\beta'}} |u_p(t, z, \epsilon) - u_q(t, z, \epsilon)| \leq C \exp\left(-\frac{D}{|\epsilon|}\right),$$

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Let $\mathbb{E}$ be the Banach space of holomorphic and bounded functions defined on $\mathcal{T} \times H_{\beta'}$ for some fixed $0 < \beta' < \beta$.

Main problem. Asymptotic behavior of the solutions

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Let $\mathbb{E}$ be the Banach space of holomorphic and bounded functions defined on $\mathcal{T} \times H_{\beta'}$ for some fixed $0 < \beta' < \beta$.

There exists a formal power series $\hat{u}(t, z, \epsilon) \in \mathbb{E}[[\epsilon]]$ which is the common Gevrey asymptotic expansion of order 1 in $\mathcal{E}_p$ of the analytic solution $u_p(t, z, \epsilon)$, for all $0 \leq p \leq \varsigma - 1$.

The upper bounds regarding the difference of two consecutive analytic solutions

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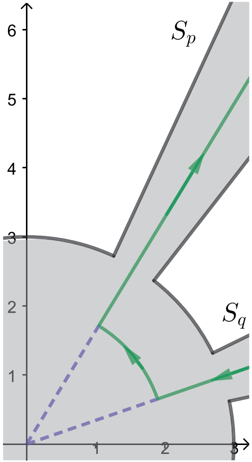
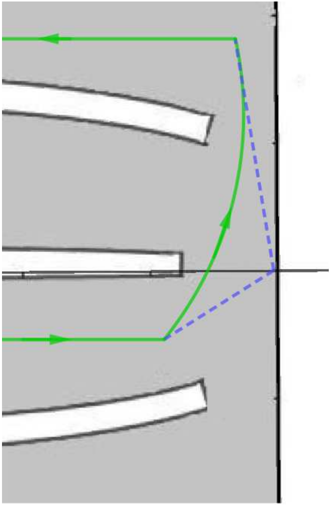
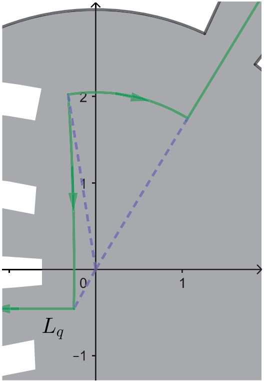
We take into account three different cases:

- Case 1: p and q belong to J_2.
- Case 2: p and q belong to J_1.
- Case 3: p belongs to J_1 and q belong to J_2.



S_p
S_q
6
5
4
3
2
1
1
2
3




2
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1
-1
L_q

Thank you for your attention

Keep healthy!