Right inverses for the asymptotic Borel map in ultraholomorphic classes on sectors

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Sectors and weight sequences

$\mathcal{R}$ will denote the Riemann surface of the logarithm.

Given $\gamma > 0$, we consider unbounded sectors

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$\mathbb{N}_0 = \{0, 1, 2, ...\}$.

Let $\mathbb{M} = (M_n)_{n\in\mathbb{N}_0}$ be a sequence of positive real numbers, with $M_0 = 1$.

$\mathbb{M}$ is said to be logarithmically convex or (lc) if $M_n^2 \le M_{n-1}M_{n+1},\ n \ge 1$; equivalently, the sequence of quotients of $\mathbb{M}$, $\boldsymbol{m} = (m_n := \frac{M_{n+1}}{M_n})_{n\in\mathbb{N}_0}$, is nondecreasing.

We always assume that $\mathbb{M}$ is (lc) and $\lim_{n\to\infty} m_n = \infty$: we say $\mathbb{M}$ is a weight sequence.

Ultraholomorphic (Carleman-Roumieu) classes

Given $\mathbb{M}$, $A > 0$ and a sector S, we consider

$$\mathcal{A}_{\{\mathbb{M}\},A}(S) = \left\{ f \in \mathcal{H}(S) \colon \|f\|_{\mathbb{M},A} := \sup_{z\in S, n\in\mathbb{N}_0} \frac{|f^{(n)}(z)|}{A^n M_n} < \infty \right\}.$$

$(\mathcal{A}_{\{\mathbb{M}\},A}(S), \| \ \|_{\mathbb{M},A})$ is a Banach space (A may be called the type).

$\mathcal{A}_{\{\mathbb{M}\}}(S) := \bigcup_{A>0} \mathcal{A}_{\{\mathbb{M}\},A}(S)$ is an (LB) space.

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For $f\in\mathcal{A}_{\{\mathbb{M}\}}(S)$ and for every $n\in\mathbb{N}_0$, there exists

$$f^{(n)}(0):=\lim_{z\to0,z\in S}f^{(n)}(z),$$

and the formal Taylor series at 0, $\widehat{f}=\sum_{n=0}^{\infty}\dfrac{f^{(n)}(0)}{n!}z^n$ satisfies $|f^{(n)}(0)|\le CA^nM_n$ for some $C,A>0$.

Asymptotics

$f : S \to \mathbb{C}$ (holomorphic in a sector S) admits the series $\hat{f} = \sum_{n=0}^{\infty} a_n z^n$ as its $\mathbb{M}$-uniform asymptotic expansion at 0 if there exist $C, A > 0$ such that for every $z \in S$ and every $n \in \mathbb{N}_0$, we have

$$\Big| f(z) - \sum_{k=0}^{n-1} a_k z^k \Big| \le C A^n M_n |z|^n. \qquad [f \in \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\},A}(S)]$$

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The norm

$$\|f\|_{\mathbb{M},A,\widetilde{u}} := \sup_{z \in S, n \in \mathbb{N}_0} \frac{|f(z) - \sum_{k=0}^{n-1} a_k z^k|}{A^n M_n |z|^n}$$

makes it a Banach space.

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Asymptotics and Carleman-Roumieu classes

Let $\mathbb{M}$ be a sequence and S be a sector. Put $\widehat{\mathbb{M}} := (p!M_p)_{p\in\mathbb{N}_0}$. Then,

(i) By Taylor's formula, $\mathcal{A}_{\{\widehat{\mathbb{M}}\},A}(S) \hookrightarrow \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\},A}(S)$ and $\mathcal{A}_{\{\widehat{\mathbb{M}}\}}(S) \hookrightarrow \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\}}(S)$ continuously.

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(ii) If S is unbounded and T is a proper subsector of S, by Cauchy's formula there exists a constant $c = c(T,S) > 0$ such that, by restriction, $\widetilde{\mathcal{A}}^u_{\{\mathbb{M}\},A}(S) \hookrightarrow \mathcal{A}_{\{\widehat{\mathbb{M}}\},cA}(T)$ and $\widetilde{\mathcal{A}}^u_{\{\mathbb{M}\}}(S) \hookrightarrow \mathcal{A}_{\{\widehat{\mathbb{M}}\}}(T)$ continuously.

The asymptotic Borel map

$\mathbb{C}[[z]]$ formal complex power series.

$$\mathbb{C}[[z]]_{\{\mathbb{M}\},A} = \Big\{ \widehat{f} = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}[[z]] : \Big|\widehat{f}\Big|_{\mathbb{M},A} := \sup_{p\in\mathbb{N}_0} \frac{|a_p|}{A^p M_p} < \infty \Big\}.$$

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We consider the asymptotic Borel map (continuous homomorphism of algebras)

$$\begin{aligned} \widetilde{\mathcal{B}} : \mathcal{A}_{\{\widehat{\mathbb{M}}\}}(S),\ \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\}}(S) \quad &\longrightarrow \quad \mathbb{C}[[z]]_{\{\mathbb{M}\}} \\ f \quad &\mapsto \hat{f} = \sum_{n=0}^{\infty} a_n z^n. \end{aligned}$$

It may also be considered from $\mathcal{A}_{\{\widehat{\mathbb{M}}\},A}(S)$ or $\widetilde{\mathcal{A}}^u_{\{\mathbb{M}\},A}(S)$ into $\mathbb{C}[[z]]_{\{\mathbb{M}\},A}$.

Surjectivity intervals and its non-triviality

$$S_{\{\widehat{\mathbb{M}}\}} := \{\gamma > 0; \quad \widetilde{\mathcal{B}} : \mathcal{A}_{\{\widehat{\mathbb{M}}\}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\{\mathbb{M}\}} \text{ is surjective}\},$$
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$\mathbb{M}$ is strongly non-quasianalytic (snq) if there exists $B > 0$ such that

$$\sum_{k \geq n} \frac{M_k}{(k+1)M_{k+1}} \leq B \frac{M_n}{M_{n+1}}, \quad n \in \mathbb{N}_0.$$

H.-J. Petzsche, On E. Borel's theorem, Math. Ann. 282 (1988), no. 2, 299–313.

V. Thilliez (2003)

If $\mathbb{M}$ does not satisfy (snq), $S_{\{\widehat{\mathbb{M}}\}} = \widetilde{S}^u_{\{\mathbb{M}\}} = \emptyset$.

Thilliez's index and regular variation

V. Thilliez (2003) introduces a growth index $\gamma(\mathbb{M})$. Now we know:

(i) A sequence $(c_p)_{p\in\mathbb{N}_0}$ is almost increasing if there exists $a>0$ such that for every $p\in\mathbb{N}_0$ we have that $c_p\le ac_q$ for every $q\ge p$.
We have proved that

$$\gamma(\mathbb{M}) = \sup\{\gamma>0 : (m_p/(p+1)^{\gamma})_{p\in\mathbb{N}_0} \text{ is almost increasing}\}$$
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(ii) For any $\beta>0$ we say that $\boldsymbol{m}$ satisfies the condition (γ_β) if there exists $A>0$ such that

$$\sum_{\ell=p}^{\infty}\frac{1}{(m_\ell)^{1/\beta}}\le\frac{A(p+1)}{(m_p)^{1/\beta}},\qquad p\in\mathbb{N}_0. \qquad (\gamma_\beta)$$

(For $\beta=1$, H. Komatsu (1973) and H.-J. Petzsche (1988), and for $\beta\in\mathbb{N}$, J. Schmets and M. Valdivia (2000).)

$$\gamma(\mathbb{M})=\sup\{\beta>0;\ \boldsymbol{m} \text{ satisfies } (\gamma_\beta)\}.$$

Moreover, $\gamma(\mathbb{M})>0$ if and only if $\mathbb{M}$ is (snq).

Surjectivity intervals for strongly regular sequences

$\mathbb{M}$ is strongly regular if it is (lc), (snq) and has moderate growth (mg): there exists $A > 0$ such that $M_{n+p} \leq A^{n+p} M_n M_p$, $n, p \in \mathbb{N}_0$.

Example: $\mathbb{M}_{\alpha,\beta} = \left(n!^{\alpha} \prod_{m=0}^{n} \log^{\beta}(e+m)\right)_{n\in\mathbb{N}_0}$, $\alpha > 0$, $\beta \in \mathbb{R}$.

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Theorem (V. Thilliez, 2003; J. Jiménez-Garrido, J. S., G. Schindl, 2019)

Let $\mathbb{M}$ be a strongly regular sequence. Then, $\gamma(\mathbb{M}) \in (0, \infty)$. Moreover, each of the following statements implies the next one:

(i) $0 < \gamma < \gamma(\mathbb{M})$,

(ii) *there exists $c \geq 1$, depending on $\mathbb{M}$ and γ, such that for every $A > 0$ there exists a right inverse for $\widetilde{\mathcal{B}}$, $U_{\mathbb{M},A,\gamma} : \mathbb{C}[[z]]_{\{\mathbb{M}\},A} \to \mathcal{A}_{\{\widehat{\mathbb{M}}\},cA}(S_\gamma)$,*

(iii) $\widetilde{\mathcal{B}} : \mathcal{A}_{\{\widehat{\mathbb{M}}\}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}}$ *is surjective,*

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Remark: If $\gamma(\mathbb{M}) \in \mathbb{Q}$, (i)-(iii) are equivalent, and this is our conjecture in general

Results for regular sequences in the sense of E. M. Dyn'kin

E. M. Dyn'kin, Pseudoanalytic extension of smooth functions. The uniform scale, Amer. Math. Soc. Transl. (2) 115 (1980), 33–58.

$\mathbb{M}$ is derivation closed (dc) if there exists a constant $A > 0$ such that

$$M_{n+1} \leq A^{n+1} M_n, \quad n \in \mathbb{N}_0.$$

If $\mathbb{M}$ is a weight sequence and satisfies (dc), $\widehat{\mathbb{M}} := (p!M_p)_{p\in\mathbb{N}_0}$ is regular.
If $\mathbb{M}$ is strongly regular, the corresponding $\widehat{\mathbb{M}}$ is regular.

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Theorem (J. Jiménez-Garrido, J. S., G. Schindl, 2019)

Suppose $\widehat{\mathbb{M}}$ is regular. One has $\widetilde{S}^u_{\{\mathbb{M}\}} \subseteq (0, \lfloor\gamma(\mathbb{M})\rfloor + 1)$; if moreover $\gamma(\mathbb{M}) \in \mathbb{N}$, then $\widetilde{S}^u_{\{\mathbb{M}\}} \subseteq (0, \gamma(\mathbb{M}))$.

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No proof of surjectivity had been given for regular $\widehat{\mathbb{M}}$, except for the q-Gevrey sequences $\mathbb{M}_q = (q^{p^2})_{p \in \mathbb{N}_0}$, $q > 1$, see
C. Zhang, Développements asymptotiques q-Gevrey et séries Gq-sommables, Ann. Inst. Fourier 49 (1999), 227–261.

Connection with the Stieltjes moment problem

A. Debrouwere, J. Jiménez-Garrido, J. S., Injectivity and surjectivity of the Stieltjes moment mapping in Gelfand-Shilov spaces, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 113 (2019), 3341–3358, doi: 10.1007/s13398-019-00693-6.

By a suitable application of the Fourier transform, there exists a close connection between this problem and the surjectivity or injectivity of the asymptotic Borel map in ultraholomorphic classes in a half-plane, and so our results in JMAA19 could be transferred.

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A. Debrouwere, Solution to the Stieltjes moment problem in Gelfand-Shilov spaces, Studia Math., online first April 2020, DOI: 10.4064/sm190627-8-10.

He has got a characterization of the surjectivity of the Stieltjes moment mapping for regular sequences by using only functional-analytic methods.

Theorem (A. Debrouwere, 2020)

Let $\widehat{\mathbb{M}}$ be regular. The following are equivalent:

(i) $\widetilde{\mathcal{B}}\colon \mathcal{A}_{\{\widehat{\mathbb{M}}\}}(S_1) \to \mathbb{C}[[z]]_{\{\mathbb{M}\}}$ *is surjective.*

(ii) $\gamma(\mathbb{M}) > 1$.

Surjectivity intervals for regular sequences

By using Balser's moment summability methods, with associated Laplace and Borel transforms, we prove

Theorem (J. Jiménez-Garrido, J. S., G. Schindl, submitted)

Let $\widehat{\mathbb{M}}$ *be a regular sequence. Then,*

$$(0,\gamma(\mathbb{M}))\subseteq S_{\{\widehat{\mathbb{M}}\}}\subseteq \widetilde{S}^{u}_{\{\mathbb{M}\}}\subseteq (0,\gamma(\mathbb{M})].$$

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In general it is not known whether $\gamma(\mathbb{M})$ belongs or not to the surjectivity intervals.
If $\gamma(\mathbb{M}) \in \mathbb{N}$, then $S_{\{\widehat{\mathbb{M}}\}} = \widetilde{S}^u_{\{\mathbb{M}\}} = (0,\gamma(\mathbb{M}))$.

Conjecture: $S_{\{\widehat{\mathbb{M}}\}} = \widetilde{S}^u_{\{\mathbb{M}\}} = (0,\gamma(\mathbb{M}))$ in general.

Global extension operators in a half-plane

One may ask about the existence of global extension operators, right inverses for the asymptotic Borel map.
In the ultradifferentiable setting, H.-J. Petzsche (1988) introduced the condition

$$\forall\varepsilon>0,\ \exists k\in\mathbb{N},\ k>1:\limsup_{p\to\infty}\left(\frac{M_{kp}}{M_p}\right)^{\frac{1}{(k-1)p}}\frac{1}{m_{kp-1}}\le\varepsilon, \qquad (\beta_2)$$

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Theorem (A. Debrouwere, 2020)

Suppose $\widehat{\mathbb{M}}$ is a regular sequence. The following are equivalent:

(i) *There exists a global extension operator $U_{\mathbb{M}} : \mathbb{C}[[z]]_{\{\mathbb{M}\}} \to \mathcal{A}_{\{\widehat{\mathbb{M}}\}}(S_1)$.*

(ii) $\gamma(\mathbb{M}) > 1$, *and* (β_2) *is satisfied.*

Global extension operators in a fixed sector

The use of Laplace and Borel transforms of arbitrary positive order allows us to generalize this statement.

Theorem (J. Jiménez-Garrido, J. S., G. Schindl)

Suppose $\widehat{\mathbb{M}}$ *is a regular sequence, and let* $r>0$. *Each of the following statements implies the next one:*

(i) $r<\gamma(\mathbb{M})$, *and* (β_2) *is satisfied.*

(ii) *There exists a global extension operator* $V_{\mathbb{M},r}:\mathbb{C}[[z]]_{\{\mathbb{M}\}}\to\widetilde{\mathcal{A}}^u_{\{\mathbb{M}\}}(S_r)$.

(iii) $r\le\gamma(\mathbb{M})$, *and* (β_2) *is satisfied.*

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(i) $r<\gamma(\mathbb{M})$, *and* (β_2) *is satisfied.*

(ii) *There exists a global extension operator* $V_{\mathbb{M},r}:\mathbb{C}[[z]]_{\{\mathbb{M}\}}\to\widetilde{\mathcal{A}}^u_{\{\mathbb{M}\}}(S_r)$.

(iii) $r\le\gamma(\mathbb{M})$, *and* (β_2) *is satisfied.*

Conjecture: (i) and (ii) are equivalent.

Rapidly varying weight sequences

Aim: Determine the weight sequences for which (global) extension operators exist for sectors of arbitrary opening.

From the previous result, we should have $\gamma(\mathbb{M}) = \infty$.

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Theorem (J. Schmets, M. Valdivia, 2000)

Let $\mathbb{M}$ be a weight sequence such that

$$\text{for every } r \in \mathbb{N},\ (m_n/n^r)_{n\in\mathbb{N}} \text{ is eventually increasing.} \qquad (*)$$

The following are equivalent:

(i) *For every $r \in \mathbb{N}$, there exists a global extension operator $U_{\mathbb{M},r} : \mathbb{C}[[z]]_{\{\mathbb{M}\}} \to \mathcal{A}_{\{\widehat{\mathbb{M}}\}}(S_r)$.*

(ii) *For some $r \in \mathbb{N}$, there exists a global extension operator $U_{\mathbb{M},r} : \mathbb{C}[[z]]_{\{\mathbb{M}\}} \to \mathcal{A}_{\{\widehat{\mathbb{M}}\}}(S_r)$.*

(iii) *$\mathbb{M}$ satisfies (β_2).*

Rapidly varying sequences

J. Jiménez-Garrido, J. S., G. Schindl, Indices of O-regular variation for weight functions and weight sequences, RACSAM 113 (4) (2019), 3659–3697

Proposition (J. Jiménez-Garrido, J. S., G. Schindl)

Let $\mathbb{M}$ be a weight sequence. Each of the following statements implies the next one, and only the implication (ii) $\Longrightarrow$ (iii) may be reversed:

(i) $\mathbb{M}$ *satisfies* $(*)$.

(ii) $\gamma(\mathbb{M}) = \infty$.

(iii) *There exists* $k_0 \in \mathbb{N}$, $k_0 \geq 2$, *such that* $\lim_{n\to\infty} \dfrac{m_{k_0 n}}{m_n} = \infty$.

(iv) $\mathbb{M}$ *satisfies* (β_2).

(v) $\lim_{n\to\infty} \dfrac{m_n}{M_n^{1/n}} = \infty$.

(vi) $\omega(\mathbb{M}) := \liminf_{n\to\infty} \dfrac{\log(m_n)}{\log(n)} = \infty$ *(known as the lower order of* $\boldsymbol{m}$*).*

(vii) $\alpha(\boldsymbol{m}) = \infty$, *where* $\alpha(\boldsymbol{m})$ *is the upper Matuszewska index of* $\boldsymbol{m}$*. Equivalently,* $\mathbb{M}$ *does not satisfy (mg).*

Surjectivity and global extension operators for rapidly varying sequences

First consequence: For strongly regular sequences surjectivity does hold and local extension operators exist with an scaling in the type for small openings, but no global extension operator is possible.

Surjectivity and global extension operators for rapidly varying sequences

First consequence: For strongly regular sequences surjectivity does hold and local extension operators exist with an scaling in the type for small openings, but no global extension operator is possible.

Theorem (J. Jiménez-Garrido, J. S., G. Schindl)

Let $\mathbb{M}$ be a weight sequence. The following are equivalent:

(i) $\gamma(\mathbb{M}) = \infty$.

(ii) *For every $r > 0$, there exists a global extension operator* $U_{\mathbb{M},r} : \mathbb{C}[[z]]_{\{\mathbb{M}\}} \to \mathcal{A}_{\{\widehat{\mathbb{M}}\}}(S_r)$.

(iii) *For every $r > 0$, there exists a global extension operator* $V_{\mathbb{M},r} : \mathbb{C}[[z]]_{\{\mathbb{M}\}} \to \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\}}(S_r)$.

(iv) *All the surjectivity intervals are $(0, \infty)$.*

In this particular case condition (dc) is not relevant.

THANK YOU VERY MUCH FOR YOUR ATTENTION!