Quotient Singularities in Positive Characteristic

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bibliography

joint work with Gebhard Martin (Bonn)

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Linearly Reductive Quotient Singularities

51 pages, arXiv:2102.01067

table of contents

Felix Klein's work

Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade (1884)

(Lectures on the icosahedron and the solution of equations of fifth degree)

1. determine finite subgroups (up to conjugacy)

$$G \subset \mathrm{SL}_2(\mathbb{C})$$

2. compute ring of invariants $\mathbb{C}[[x, y]]^G$

Felix Klein's work

cyclic

$$\begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{n-1} \end{pmatrix}$$

abstract group	C_n
invariant subring	$\mathbb{C}[[a,b,c]]/(c^n - ab)$
rational double point of type	A_{n-1}

Felix Klein's work

binary dihedral

$$\begin{pmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{2n-1} \end{pmatrix}, \quad \begin{pmatrix} 0 & \zeta_4 \\ \zeta_4 & 0 \end{pmatrix}$$

abstract group $\quad 1 \to C_{2n} \to \mathrm{BD}_n \to C_2 \to 1$

(dicyclic, metacyclic)

invariant subring $\quad \mathbb{C}[[a,b,c]]/(a^2 + b^2 c + c^{n+1})$

rational double point of type $\quad D_{n+2}$

Felix Klein's work

binary tetrahedral $\mathbb{C}[[a,b,c]]/(a^2+b^3+c^4)$

rational double point of type E_6

binary octahedral $\mathbb{C}[[a,b,c]]/(a^2+b^3+bc^3)$

rational double point of type E_7

binary icosahedral $\mathbb{C}[[a,b,c]]/(a^2+b^3+c^5)$

rational double point of type E_8

rational double point singularities

$x \in X$ normal surface singularity

rational double point (Du Val singularity, Kleinian singularity, canonical singularity,…)

$\Leftrightarrow$ rational + Gorenstein

$\Leftrightarrow$ $\pi : Y \to X$ minimal resolution of singularities

$\pi^* K_X = K_Y$

$\Leftrightarrow$ isolated hypersurface singularity of multiplicity 2

observations and questions

all complex rational double points arise in this way

- how much of this is true in positive characteristic?

Artin classified rational double points in positive characteristic:

- to what extent are all of them ``quotient singularities''?

Remark: we will (and have to) allow possibly non-reduced group schemes from the very beginning

observations and questions

good / tame

linearly reductive group schemes
linear actions

linearly reductive quotient singularities

structure results, invariants,…

bad / wild

how much can we actually say?

can we at least understand the wild case of rational double points?

finite group schemes

k algebraically closed field

of characteristic $p \geq 0$

G finite group scheme over k

(automatically flat)

characteristic zero

$p = 0$ equivalence of categories between

finite group schemes over k

and

finite (abstract) groups

given by $G \mapsto G(k)$

characteristic zero

$p = 0$

the category of linear representations

$$\rho : G \to \mathrm{GL}_{d,k}$$

is semi-simple

semi-simplicity

$p > 0$

Example 1 let G be the group scheme associated to C_p

$$\begin{array}{cccc} \rho & : & G & \to & \mathrm{GL}_2 \\ & & 1 & \mapsto & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{array}$$

is *not* semi-simple

connected group schemes

$p > 0$

Example 2 let $G := \mu_p$ be the subgroup scheme of $\mathbb{G}_m$ of the p^{th} roots of unity

G is a finite group scheme of length p over k

$\operatorname{Spec} k[\varepsilon]/(\varepsilon^p)$ (non-reduced)

associated abstract group

$G(k) = \{1\}$

connected-étale sequence

$p > 0$ G finite group scheme over k

canonical and split exact sequence

$$1 \to G^{\circ} \to G \to G^{\mathrm{et}} \to 1$$

G° connected (infinitesimal)

G^{et} étale (comes from an abstract group)

Nagata's theorem

Theorem (Nagata, 1961)

G a finite group scheme over an algebraically closed field k of characteristic $p > 0$.

TFAE

1) G is linearly reductive, that is, every representation $\rho : G \to \mathrm{GL}_d$ is semi-simple

2) a) G^{et} is of length prime to p
   b) G° is a finite subgroup scheme of some $\mathbb{G}_m^N$

Nagata's theorem

concerning 2)

G^{et} is of length prime to p

$\Leftrightarrow$

$G^{\mathrm{et}}(k)$ is a finite group of order prime to p

G° is a finite subgroup scheme of some $\mathbb{G}_m^N$

$$\Leftrightarrow \qquad G \cong \prod_{i=1}^{N} \mu_{p^{n_i}}$$

linearly reductive group schemes

Definition (finite) group schemes G over k, such that all finite-dimensional representations are semi-simple are called **linearly reductive**

characteristic zero: all are linearly reductive

positive characteristic: see Nagata's theorem

positive characteristic

$p > 0$ equivalence of categories between

finite and linearly reductive group schemes over k

and

finite (abstract) groups with a
unique and abelian p-Sylow subgroup

quotient singularities

Definition A (finite and isolated) **quotient singularity** over k

a k-scheme that is formally isomorphic to

$$\left(\operatorname{Spec} k[[x_1, ..., x_d]]\right)/G$$

where G is a finite k-group scheme that

acts on

$$\operatorname{Spec} k[[x_1, ..., x_d]]$$

freely outside the closed point.

linearisation

Definition

G a finite k-group scheme

a G-action on $k[[x_1, ..., x_d]]$

is **linear** if it factors through

$$G \to \mathrm{GL}_{d,k}$$

w.r.t. the usual $\mathrm{GL}_{d,k}$ - action on

$$k[[x_1, ..., x_d]]$$

linearisation

Definition let G be a finite k-group scheme

a G-action on $k[[x_1, ..., x_d]]$

is **linearisable** if it becomes

linear after a change of coordinates

linearisation

Proposition G a finite k-group scheme

that acts on $k[[x_1, ..., x_d]]$

such that the action is free outside the closed

point and fixes the closed point

TFAE:

1) the G-action is linearisable

2) G is linearly reductive

linearisation

if $k = \mathbb{C}$ then all finite group schemes are linearly reductive and this result is a classical lemma of Cartan

the implication $2) \Rightarrow 1)$ is ``well-known to the experts''

lrq singularities

Observation let G be a finite and linearly reductive k-group scheme

given a G-action on $\mathrm{Spec}\, k[[x_1, ..., x_d]]$

that is free outside the closed point

$\Rightarrow$

without loss of generality, G acts linearly,

say via the representation $\rho : G \to \mathrm{GL}_{d,k}$

lrq singularities

given a representation

$$\rho : G \to \mathrm{GL}_{d,k}$$

the action is free outside the closed point

$$\Leftrightarrow \qquad \dim\left((k^d)^{\mu_n}\right) = 0$$

for all $\mu_n \subset G$ with $n > 0$

(``no eigenvalue 1'')

lrq singularities

Definition a representation

$$\rho : G \to \mathrm{GL}_{d,k}$$

is **very small** if

$$\dim\left((k^d)^{\mu_n}\right) = 0$$

for all $\mu_n \subset G$ with $n > 0$

lrq singularities

Definition a representation

$$\rho \,:\, G \,\to\, \mathrm{GL}_{d,k}$$

is **small** or **contains no pseudo-reflection**

$$\dim\left((k^d)^{\mu_n}\right) \,\leq\, d-2$$

for all $\mu_n \subset G$ with $n > 0$

very small representations

$$\rho : G \to \mathrm{GL}_d(\mathbb{C})$$

very small representation

of a finite group G

1) ρ is faithful

2) if $H \subset G$ is an abelian subgroup of G,

then H is cyclic

very small representations

Theorem (classical? Zassenhaus? Cartan-Eilenberg?)

G a finite group

TFAE

1) every abelian subgroup of G is cyclic

2) G has periodic cohomology $\widehat{H}^*(G, \mathbb{Z})$

3) every p-Sylowsubgroup is cyclic

or $p = 2$ and generalised quaternionic

very small representations

finite groups that admit a very small representation

have been essentially classified by work of

Zassenhaus (1935 - in connection with near fields)

Suzuki

differential topologists in connection with the Clifford-Klein spherical space form problem (Milnor, Thomas, Wall, Wolf)

very small representations

solvable case (Zassenhaus)

cyclic

metacyclic

groups related to the binary tetrahedral group

groups related to the binary octahedral group

very small representations

non-solvable case (Suzuki and differential topologists)

groups related to the binary icosahedral group

a perfect group of order 120 isomorphic to $\mathrm{SL}_2(\mathbb{F}_5)$

Remark: one can use this classification to recover the classification of finite and very small subgroups of
$\mathrm{SL}_2(\mathbb{C})$ (Klein)
$\mathrm{GL}_2(\mathbb{C})$ (Brieskorn?)
(admittedly, a complete overkill)

lrq singularities

quotient singularities by linearly reductive group schemes

arise from

very small representations of linearly reductive group schemes

lrq singularities

given an lrq singularity $x \in X$

$\Rightarrow$ there exists G linearly reductive group scheme
$\rho : G \to \mathrm{GL}_d$ very small representation

such that $X \cong \mathbb{A}^d/G$

Questions 1) is G unique (up to isomorphism)?

2) is ρ unique (up to conjugation)?

invariants of lrq singularities

Proposition $\rho : G \to \mathrm{GL}_d$ very small representation of a linearly reductive group scheme

$X := \mathbb{A}^d/G$ associated lrq singularity

1) normal, Cohen-Macaulay, log terminal, $\mathbb{Q}$ -Gorenstein,

2) $\mathrm{Cl}(X) \cong \mathrm{Hom}(G, \mathbb{G}_m)$

3) $\pi_{\mathrm{loc}}^{\mathrm{et}}(X) \cong G^{\mathrm{et}}$

invariants of lrq singularities

if $p > 0$

4) F-regular

5) F-signature $s(X) = \dfrac{1}{|G|}$

6) Hilbert-Kunz multiplicity

$$e_{\mathrm{HK}}(X) = \frac{1}{|G|} \operatorname{length} k[[x_1, ..., x_d]]/\mathfrak{m}_R$$

where $R := k[[x_1, ..., x_d]]^G$

reminder on F-invariants

R a local ring over algebraically closed field of positive characteristic p

F-signature

define a_{p^e} to be the maximal rank of a free summand of R over considered as a module over itself via the e-fold Frobenius

$$s(R) := \lim_{e \to \infty} \frac{a_{p^e}}{p^{ne}}$$

reminder on F-invariants

F-signature $$s(R) := \lim_{e \to \infty} \frac{a_{p^e}}{p^{ne}}$$

properties

the limit exists (Tucker)

R regular $\Rightarrow$ F is flat

$\Rightarrow s(R) = 1$

in general $s(R) \leq 1$

reminder on F-invariants

R a local ring over algebraically closed field of positive characteristic p

Hilbert-Kunz multiplicity

$$e_{\mathrm{HK}}(R) := \lim_{e \to \infty} \frac{\mathrm{length}(R/\mathfrak{m}^{[p^e]})}{p^{e \dim(R)}}$$

lrq singularities

$\rho : G \to \mathrm{GL}_d$ very small representation of a linearly reductive group scheme G

$X := \mathbb{A}^d/G$ associated lrq singularity

$\pi_{\mathrm{loc}}^{\mathrm{et}}(X) \cong G^{\mathrm{et}}$ $\Rightarrow$ if p=0 or if G is étale,

then X determines G and

the G-action on the

universal étale cover

lrq singularities

$\rho : G \to \mathrm{GL}_d$ very small representation of a linearly reductive group scheme G

$X := \mathbb{A}^d/G$ associated lrq singularity

assume p>0

$$s(X) = \frac{1}{|G|} \quad \Rightarrow$$ the length of G is determined by X

lrq singularities

assume p>0

$$s(X) = \frac{1}{|G|} \quad \Rightarrow$$

the length of G is determined

by X

$$\mathrm{Cl}(X) \cong \mathrm{Hom}(G, \mathbb{G}_m)$$

$\Rightarrow$ if, for example, $G \cong \mu_{p^n}$

then $\mathrm{Cl}(X) \cong \mathbb{Z}/p^n\mathbb{Z}$

lrq singularities

if, for example, $G \cong \mu_{p^n}$

then $\mathrm{Cl}(X) \cong \mathbb{Z}/p^n\mathbb{Z}$

moreover, then, X is a toric singularity and thus,

uniqueness of the G-action follow from

Demushkin's theorem

uniqueness theorem

Theorem $\mathbb{A}^d/G_1$ an lrq singularity (in particular, G_1 is linearly reductive and acts linearly)

assume $X \cong \mathbb{A}^d/G_2$ where G_2 is a finite k-group scheme that acts freely outside the closed point and fixes the origin

Remark we do *not* assume that G_2 is linearly reductive *nor* that it acts linearly

uniqueness theorem

then

1) $G_1 \cong G_2$ as finite k-group schemes

2) the G_2 - action is linearisable

3) with respect to linearisations, G_1 and G_2

are conjugate as subgroup schemes of GL_d

rigidity of lrq singularities

Schlessinger over the complex numbers,

(isolated and finite) quotient singularities

in dimension at least 3

are infinitesimally rigid

Remark also true for finite quotient singularities by groups of order prime to the characteristic

rigidity of lrq singularities

Proposition lrq singularities

are infinitesimally rigid

1) in dimension at least 4 or

2) in dimension 3 if the quotient is by an

étale group scheme

rigidity of lrq singularities

what about infinitesimal rigidity of lrq singularities

in dimension 3?

$$X \cong \mathbb{A}^3/G \qquad\qquad G \subset \mathrm{GL}_3$$

if $G^\circ = \{e\}$, then X is infinitesimally rigid,

thus, we may assume G° is non-trivial

rigidity of lrq singularities

G° non-trivial

$\Rightarrow \quad G^\circ \cong \mu_{p^n}$

embedded into GL_3 via

$$\begin{pmatrix} \zeta_{p^n} & & \\ & \zeta_{p^n}^{a_2} & \\ & & \zeta_{p^n}^{a_3} \end{pmatrix}$$

weights $(1, a_2, a_3)$

rigidity of lrq singularities

1) adjoint representation

$$\chi_{\mathrm{ad}} \; : \; G \to \underbrace{\mathrm{GL}(\mathrm{Lie}(G))}_{\cong \mathbb{G}_m}$$

2) determinant of the action

$$\chi_{\det} \; : \; G \xrightarrow{\rho} \mathrm{GL}_3 \xrightarrow{\det} \mathbb{G}_m$$

rigidity of lrq singularities

Theorem a 3-dimensional lrq singularity $\mathbb{A}^3/G$

is infinitesimally rigid if and only if

1) G° is trivial

2) $G^\circ \cong \mu_{p^n}$ and $\chi_{\mathrm{ad}} \neq \chi_{\mathrm{det}}$

rigidity of lrq singularities

Example

$\frac{1}{n}(1, a_2, a_3)$ is infinitesimally rigid

if and only if

1) $p \nmid n$ or

2) $p \mid n$ and $n \nmid (1 + a_2 + a_3)$

rigidity of lrq singularities

Theorem if a 3-dimensional lrq singularity

$$X := \mathbb{A}^3/G$$

is not infinitesimally rigid, then

$G^\circ \cong \mu_{p^n}$ for some $n > 0$ and then

$$\mathrm{Def}_X \cong \operatorname{Spec} W(k)[\varepsilon]/(\varepsilon^2, p^n\varepsilon)$$

dimension 2

for a normal surface singularity $x \in X$ over the complex numbers

TFAE

1) X is a quotient singularity

   by a finite subgroup of $\mathrm{GL}_2(\mathbb{C})$

2) X is klt

dimension 2

for a normal surface singularity $x \in X$ over the complex numbers

TFAE

1) X is a quotient singularity

   by a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$

2) X is canonical

3) X is a rational double point

dimension 2

Theorem Let X be a normal surface singularity

in characteristic $p > 0$

TFAE

1) X is an lrq singularity by a finite subgroup scheme of GL_2

2) X is an F-regular singularity

moreover, if $p \geq 7$ then this is equivalent to

3) X is klt

dimension 2

Theorem Let X be a normal surface singularity in characteristic $p > 0$

TFAE

1) X is an lrq singularity by a finite subgroup scheme of SL_2
2) X is an F-regular Gorenstein singularity

moreover, if $p \geq 7$ then this is equivalent to

3) X is canonical

to be continued…

joint work with Gebhard Martin (Bonn)

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Torsors over the Rational Double Points,

(in preparation)