

q -Stokes phenomenon of basic hypergeometric equations and the Painlevé equations

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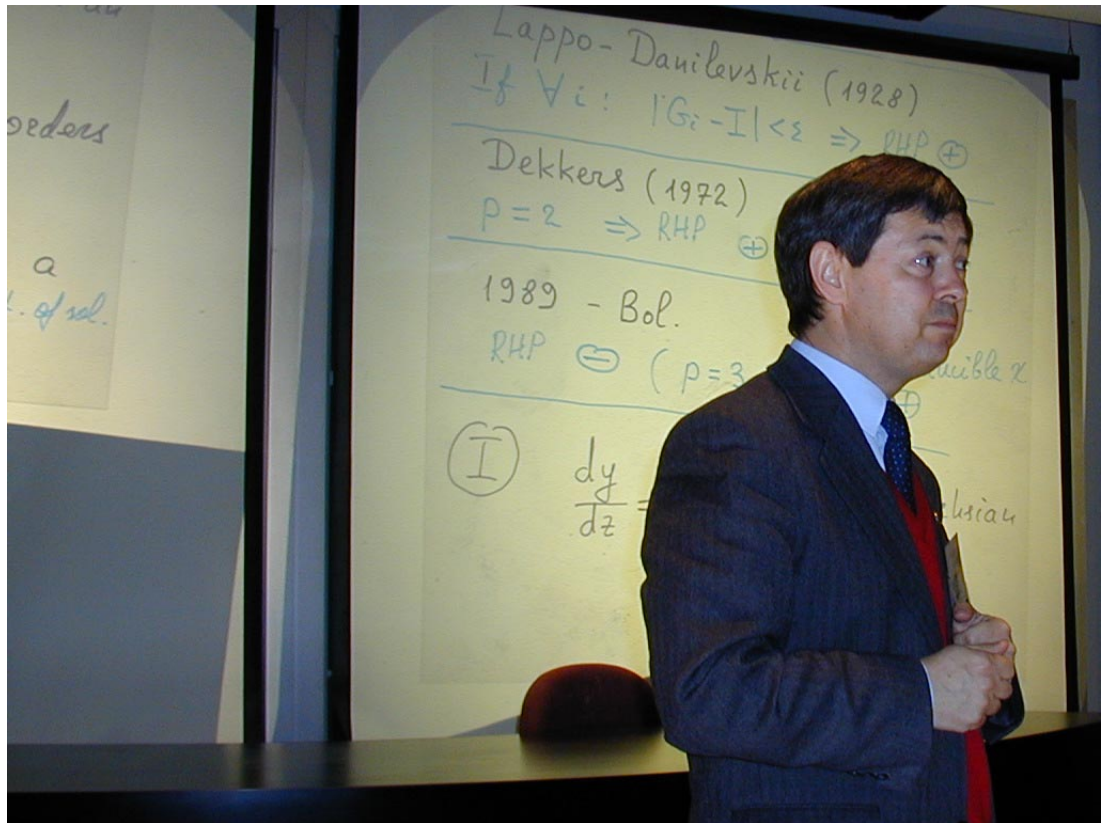


“Analytic theory of differential and difference equations”
dedicated to the memory of Andrey Bolibrukh
online, Moscow

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“Inverse problems for linear differential equations
with meromorphic coefficients”



“Isomonodromic Deformations and Applications in Physics”

1-6 May 2000, CRM, Montreal

<http://www.crm.umontreal.ca/~harnad/home.dir/ISOMON.dir/isomon-workshop.html>

1. Andrey Bolibrukh's works

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Bolibrukh's works on the 21st Hilbert problem:

a **negative solution** of the 21st Hilbert problem

positive solution when the monodromy group is irmagentaucible

The Riemann-Hilbert-Birkhoff problem

Correspondence between **connection data** and **difference equations**

G. D. Birkhoff solved generalized Riemann problems
for linear difference and q -difference equations

Future Problem

Solve the generalized Riemann problems in modern sense.

Single equations or Matrix systems

Counterexamples in magentaucible cases

Construct good moduli space of connections

We consider q -difference equations.

additive: $Y(x+1) = A(x)Y(x),$
 $Y_{+\infty}(x) = Y_{-\infty}(x)C(x), \quad C(x+1) = C(x)$

multiplicative: $Y(xq) = A(x)Y(x), \quad 0 < |q| < 1$
 $Y_{\infty}(x) = Y_0(x)P(x), \quad P(xq) = P(x)$

We do not have enough examples of connections on q -difference equations.

Differential case

Hypergeometric equations ${}_nF_{n-1}$ (Gauss, Thomae, Orr)

other **rigid systems**, middle convolutions

(Kac, Simpson, Dettweiler-Reiter, . Crawley-Boevey, ..)

irregular cases ${}_rF_s$ (Barnes, Meijer, Duval-Mitchi,...)

→ the **Stokes phenomenon**

Additive difference case

P. M. Batchelder, Introduction to Linear Difference Equations, 1927

N. Nørlund, Vorlesungen über Differenzenrechnung 1924

2.1 Strategy

Before we construct general theory on the Riemann-Hilbert correspondence, we **construct many examples of connection formulae** of q -difference equations

Known example

J. Thomae (1869) ${}_2\phi_1(a, b; c; q, x)$

J. Thomae (1870) ${}_r\phi_{r-1}(\mathbf{a}; \mathbf{b}; q, x)$

G. N. Watson (1910) [cannot treat divergent series]

${}_r\phi_{r-1}(\mathbf{a}_r; \mathbf{b}_s, \mathbf{0}; q, x)$

${}_{s+1}\phi_s(\mathbf{a}_r, \mathbf{0}; \mathbf{b}_s; q, x)$

${}_r\phi_s(\mathbf{a}_r; \mathbf{b}_s; q, x)$ ($s + 1 > r$) (not complete)

See Slater's book in detail

Modern works

After 1990s, J.-P. Ramis, C. Zhang, J. Sauloy are constructing modern theory on q -asymptotics [RSZ]

q -Borel-Laplace method

Application to **global analysis on q -Painlevé equations**

2.2 From hypergeometric to Painlevé

Monodromy/connection/Stokes data are

determined exactly for **rigid systems**.

rigid systems : no accessory parameters

Ex. hypergeometric, Simpson's even/odd system

For **Painlevé**, the Lax pair has accessory parameters (two in Painlevé)

We **cannot determine connection data** exactly.

But we can determine **the space of connection**

$\mathcal{RM}$: Moduli of connections $\rightarrow$ monodromy space

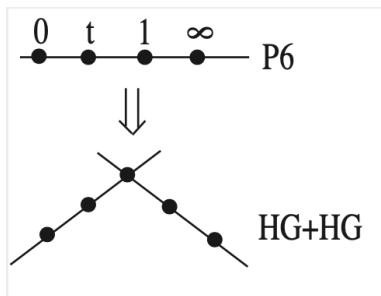
(Okamoto IVS)

(Fricke cubic for PVI)

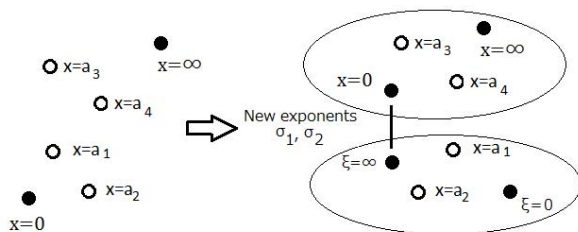
The **connection space of q -Painlevé VI** : Ramis, Sauloy, O

In order to study other q -Painlevé, we need **q -confluent cases**

Differential P6



q -P6



Painlevé (2×2 matrix)

$$\frac{dY}{dx} = \left[\frac{A_0}{x} + \frac{A_t}{x-t} + \frac{A_1}{x-1} \right] Y(x)$$

$$Y(xq) = [A_0 + A_1x + A_2x^2] Y(x)$$

$$\det A(x) = c \prod_{j=1}^4 (x - a_j)$$

Hypergeometric

$$\frac{dY}{dx} = \left[\frac{A_0 + A_t}{x} + \frac{A_1}{x-1} \right] Y(x)$$

$$\frac{dY}{d\xi} = \left[\frac{A_0}{\xi} + \frac{A_t + A_1}{\xi-1} \right] Y(\xi)$$

$$Y(xq) = x [\Lambda + A_2x] Y(x)$$

$$Y(\xi q) = [A'_0 + \Lambda\xi] Y(\xi)$$

3. Single q -difference equation

A q -difference linear equation with polynomial coefficients:

$$\sum_{j=0}^n a_j(x) u(q^j x) = 0$$

If $a_0(0)a_n(0) \neq 0$, $x = 0$ is a **regular singular point**.

characteristic equation at the origin

$$a_n(0)\lambda^n + a_{n-1}(0)\lambda^{n-1} + \cdots + a_1(0)\lambda + a_0(0) = 0$$

Let $\lambda_j = q^{c_j}$ be a solution of the characteristic equation.

Proposition [Adams] The q -difference equation has convergent solutions

$$u_j = x^{c_j} \sum_{n=0}^{\infty} u_{j,n} x^n \quad (j = 1, \dots, n),$$

when $c_j - c_m \notin \mathbb{Z}$.

3.1 Basic notations

0) **q -shifted factorial:**

$$(a_1, \dots, a_r; q)_n = \prod_{i=1}^n (a_i; q)_n, \quad (a; q)_n = (1-a)(1-qa) \cdots (1-q^{n-1}a).$$

1) **q -shift operator**

$$\sigma_q[f(x)] = f(xq)$$

2) **generalized q -hypergeometric series:**

$$\begin{aligned} {}_r\varphi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, z) \\ = \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_s; q)_n (q; q)_n} \left[(-1)^n q^{n(n-1)/2} \right]^{1+s-r} z^n. \end{aligned}$$

3) **bilateral q -hypergeometric series:**

$$\begin{aligned} {}_r\psi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, z) \\ = \sum_{n \in \mathbb{Z}} \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_s; q)_n (q; q)_n} \left[(-1)^n q^{n(n-1)/2} \right]^{1+s-r} z^n. \end{aligned}$$

3) **Theta function:**

$$\theta_q(x) := \sum_{n=-\infty}^{\infty} q^{n(n-1)/2} x^n = (q, -x, -q/x; q)_{\infty}.$$

$$e_a(x) := \frac{\theta(x)}{\theta(ax)}, \text{ for } a \in \mathbb{C}^{\times}.$$

$$x\sigma_q[\theta_q(x)] = \theta_q(x), \quad \sigma_q[e_a(x)] = ae_a(x);$$

Remark. For $u_c(x) = x^c$, $\sigma_q[u_c(x)] = q^c u_c(x)$.
 $u_c(x)$ and $e_a(x)$ satisfy the same q -difference equation if $q^c = a$.

We take $e_a(x)$ instead of x^c .
 Since $e_a(x)$ is **single-valued**, all solutions are also single-valued.
 Therefore the connection matrix $P(x)$ is **elliptic on $\mathbb{C}^{\times}/q^{\mathbb{Z}}$** :

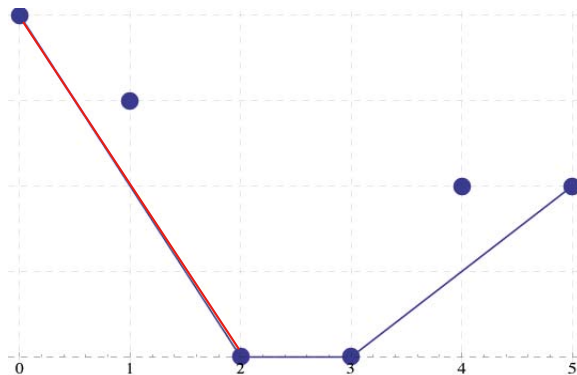
$$Y_{\infty}(x) = Y_0(x)P(x)$$

$$P(xq) = P(x), \quad P(xe^{2\pi i}) = P(x)$$

3.2 The Newton-Puiseux diagram

The **Newton-Puiseux diagram** of a linear q -difference equation at the origin is a **lower convex hull** of

$$\{(j, \text{ord } a_j(x)) \in \mathbb{R}^2 \mid 0 \leq j \leq n\}$$



segment : a line jointed with $(j, \text{ord } a_j(x))$ and $(k, \text{ord } a_k(x))$ ($j < k$)

slope of s segment:

$$\mu = \frac{\text{ord } a_k(x) - \text{ord } a_j(x)}{k - j}$$

length of s segment: $m = k - j$

General Theorem (Adams [A])

1) If partial characteristic exponents λ_j are non-resonant for any **segment with the slope μ and the length m** , the q -difference equation has m formal solutions

$$u_j(x) = \theta(x)^\mu \frac{\theta_q(x)}{\theta_q(\lambda_j x)} \sum_{k \geq 0} u_k x^k,$$

for each segment

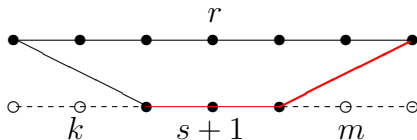
2) When the slope of a segment is not integer $\mu = r/s$ ($s > 0$), formal solutions are given by formal power series of $x^{1/s}$.

3) When the segment contains **(0, ord $a_0(x)$)**, the power series are convergent.

4. Basic hypergeometric equations (with Zhang)

$0 \leq s+1 \leq r$, $k+s+1 \leq r$; $k+s+1+m=r$. $b_0=q$, $a_j \neq 0$:

$$\left[x \prod_{j=1}^r (1 - a_j \sigma_q) - \left(-\frac{\sigma_q}{q} \right)^k \prod_{j=0}^s \left(1 - \frac{b_j}{q} \sigma_q \right) \right] y(x) = 0,$$



I) Around $x = \infty$, we have r **convergent hypergeometric** solutions

II) Around $x = 0$, we have

- (1) k **convergent non-hypergeometric** solutions with ramification
- (2) $s+1$ **divergent hypergeometric** solutions
- (3) m **divergent non-hypergeometric** solutions with ramification

The cases $k=0$, $s=-1$ or $m=0$ may appear.

If $k=0$, there exist $s+1$ convergent solutions around $x=0$.

If $k=0$ and $s=-1$, there exist $m(=r)$ convergent solutions around $x=0$.

4.1 Local solutions

Local solutions around the infinity ($j = 1, \dots, r$)

$$y_{j,\infty}(x) = \frac{\theta(-a_j x)}{\theta(-x)^{k+s+1}} \phi_{r-1} \left(qa_j/b_0, qa_j/b_1, \dots, qa_j/b_s, \mathbf{0}_k \right. \\ \left. qa_j/a_1, \dots, qa_j/a_{j-1}, qa_j/a_{j+1}, \dots, qa_j/a_r, ; q, \frac{q^m a_j^m b_0 b_1 \cdots b_s}{a_1 a_2 \cdots a_r x} \right)$$

Ramified convergent solutions at the origin ($j = 1, \dots, k$)

$$x = z^k \text{ and } q = p_0^k. \ \omega^k = 1.$$

$$y_{j,a}(z) = \frac{1}{\theta_{p_0}(-p_0^{(m+1)/2} \omega^j z)} w_j(z), \quad w_j(z) = \sum_{n=0}^{\infty} w_n^{(k)} z^n.$$

Formal divergent solutions at the origin ($j = 0, \dots, s$)

$$y_{j,b}(x) = \frac{\theta(-x)}{\theta(-qx/b_j)^r} \phi_{s+m} \left(qa_1/b_j, qa_2/b_j, \dots, qa_r/b_j \right. \\ \left. qb_1/b_j, \dots, qb_{j-1}/b_j, qb_{j+1}/b_j, \dots, qb_s/b_j, \mathbf{0}_m ; q, x \right)$$

Ramified divergent solutions at the origin ($j = 1, \dots, m$)

$$x = z^m \text{ and } q = p^m. \ \omega^m = 1.$$

$$y_{j,c}(z) = \theta_p(-cp^{(m+1)/2} \omega^j z) u_j(z), \quad v_j(z) = \sum_{n=0}^{\infty} u_n^{(k)} z^n.$$

4.2. Transformations

0) **Ramification**: $x = z^n$, $q = p^n$.

1) Gauge transform by **θ -functions**:

$$x^m \sigma_q^n [\theta(cx)^{\pm 1} f(x)] = c^{\mp n} q^{\mp n(n-1)/2} x^{m \mp n} \theta(cx)^{\pm 1} \sigma_q^n f(x).$$

2) **q -Borel transformation** $\mathcal{B}_q^{\pm} : \mathbb{C}[[t]] \rightarrow \mathbb{C}[[\tau]]$:

$$\mathcal{B}_q^{\pm} \left[\sum_{n=0}^{\infty} a_n t^n \right] := \sum_{n=0}^{\infty} a_n q^{\pm n(n-1)/2} \tau^n.$$

The q -Borel transformation $\mathcal{B}_q^{\pm}$ satisfies

$$\mathcal{B}_q^{\pm}(t^m \sigma_q^n f) = q^{\pm m(m-1)/2} \tau^m \sigma_q^{n \pm m} \mathcal{B}_q^{\pm}(f).$$

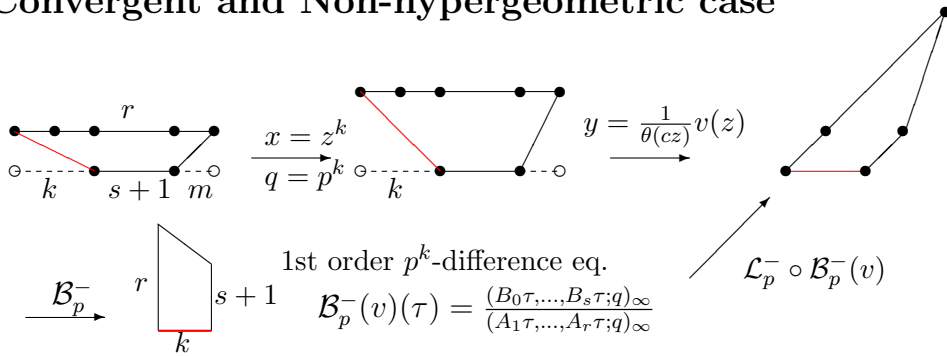
3) The **q -Laplace transform** $\mathcal{L}_{q;1}^{[\lambda]}$ (**a formal inverse of $\mathcal{B}_q^+$**)

$$\mathcal{L}_{q;1}^{[\lambda]}(\varphi)(x) = \frac{1}{1-q} \int_0^{\lambda \infty} \frac{\varphi(\tau)}{\theta_q(\tau/x)} \frac{d_q \tau}{\tau} = \sum_{n \in \mathbb{Z}} \frac{\varphi(q^n \lambda)}{\theta_q(q^n \lambda/x)}.$$

3') The **q -Laplace transform** $\mathcal{L}_q^-$ (**an inverse of $\mathcal{B}_q^-$**)

$$\mathcal{L}_q^-(\varphi)(x) = \frac{1}{2\pi i} \int_{|\tau|=\varepsilon} \varphi(\tau) \theta_p(x/\tau) \frac{d\tau}{\tau}$$

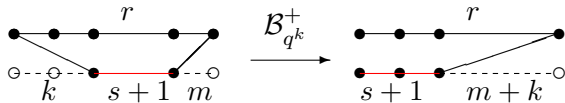
5. Convergent and Non-hypergeometric case



Lemma 1. $m > 0$, $p^m = q$. We assume that $1 + s + m \leq r$.

$$\begin{aligned}
 I &= \int_{|\tau|=\varepsilon} \frac{\prod_{j=0}^s (B_j\tau; q)_\infty}{\prod_{k=1}^r (A_k\tau; q)_\infty} \theta_p(z/\tau) \frac{d\tau}{\tau} \\
 &= \frac{(B_0/A_1, \dots, B_s/A_1; q)_\infty}{(q, A_2/A_1, \dots, A_r/A_1; q)_\infty} \theta_p(A_1 z) \\
 &\quad \times {}_{1+s+m}\phi_{r-1} \left(\begin{matrix} qA_1/B_0, \dots, qA_1/B_s, \mathbf{0}_m; \\ qA_1/A_2, \dots, qA_1/A_r \end{matrix}; q, \frac{(-1)^r q^{r-s-(1+m)/2} B_0 \cdots B_s}{A_1^{m-r+1} A_2 \cdots A_r z^m} \right) \\
 &\quad + \text{idem}(A_1; A_2, \dots, A_r).
 \end{aligned}$$

6. Divergent and Hypergeometric case



The case $k = 0$ is shown by **Watson** 1910.

Theorem 2 (Watson). *We assume that $0 \leq s < r$. Then*

$$\begin{aligned}
 {}_r\phi_{r-1} \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s, 0, \dots, 0 \end{matrix}; q, z \right) &= \frac{(a_2, \dots, a_{r+1}, b_s/a_1, \dots, b_s/a_1; q)_\infty}{(b_1, \dots, b_s, a_2/a_1, \dots, a_r/a_1; q)_\infty} \\
 &\times \frac{\theta_q(-a_1 z)}{\theta_q(-z)} {}_{s+1}\phi_{r-1} \left(\begin{matrix} a_1, qa_1/b_1, \dots, qa_1/b_s \\ qa_1/a_2, \dots, qa_1/a_r \end{matrix}; q, (a_1 q)^{r-s-1} \frac{qb_1 \cdots b_s}{a_1 \cdots a_r z} \right) \\
 &\quad + \text{idem}(a_1; a_2, \dots, a_r).
 \end{aligned}$$

The right handside is convergent when $|z| < 1$ and RHS is convergent when $s < r - 1$ or $s = r - 1$ and $|qb_1 \cdots b_s/a_1 \cdots a_r z| < 1$.

6.1 q^k -Laplace transform

We can apply q^k -Laplace transform to Watson's connection formula:

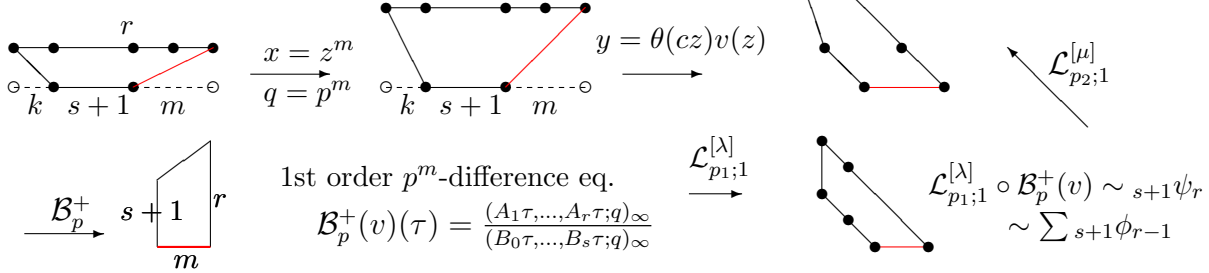
Lemma 3. *We take a positive integer m . We consider*

$$\varphi(\xi) = \frac{\theta_q(a\xi)}{\theta_q(b\xi)} \sum_{n \geq 0} c_n \xi^{-n}.$$

Then we obtain

$$\mathcal{L}_{q^k;1}^{[\lambda]} \varphi(x) = \frac{\theta_q(a\lambda)\theta_{q^k}(q^k a^k x/b^k \lambda)}{\theta_q(b\lambda)\theta_{q^k}(q^k x/\lambda)} \sum_{n \geq 0} c_n (q^k)^{-\frac{n(n-1)}{2}} \left(\frac{b^k}{a^k q^k x} \right)^n.$$

7. Divergent and Non-hypergeometric case



We set $\varphi(\tau) = \mathcal{B}_p^+(v)(\tau)$.

$$\varphi(\tau) = \frac{(A_1\tau, \dots, A_r\tau; q)_\infty}{(B_0\tau, \dots, B_s\tau; q)_\infty}$$

We set $p_1 = p^{m/m+k} = q^{1/m+k}$. ($m + k = r - s - 1$)

$$\mathcal{L}_{p_1;1}^{[\lambda]}(\phi)(\xi) \sim \frac{(A_1\lambda, \dots, A_r\lambda; q)_\infty}{(B_0\lambda, \dots, B_s\lambda; q)_\infty} {}_{s+1}\psi_r \left[\begin{matrix} B_0\lambda, \dots, B_s\lambda \\ A_1\lambda, \dots, A_r\lambda \end{matrix}; q, \frac{c}{\xi^{m+k}} \right]$$

Theorem 4 (Slater [Sl]). *We set $d = a_1 a_2 \dots a_r / c_1 c_2 \dots c_r$.*

For $\left| \frac{b_1 b_2 \dots b_s}{a_1 a_2 \dots a_r} \right| < |z| < 1$, we have

$$\begin{aligned} & \frac{(b_1, b_2, \dots, b_r, q/a_1, q/a_2, \dots, q/a_r, dz, q/dz; q)_\infty}{(c_1, c_2, \dots, c_r, q/c_1, q/c_2, \dots, q/c_r; q)_\infty} {}_r\psi_r \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_r \end{matrix}; q, z \right] \\ &= \frac{q}{c_1} \frac{(c_1/a_1, c_1/a_2, \dots, c_1/a_r, qb_1/c_1, qb_2/c_1, \dots, qb_r/c_1, dc_1 z/q, q^2/dc_1 z; q)_\infty}{(c_1, q/c_1, c_1/c_2, \dots, c_1/c_r, qc_2/c_1, \dots, qc_r/c_1; q)_\infty} \\ & \quad \times {}_r\psi_r \left[\begin{matrix} qa_1/c_1, qa_2/c_1, \dots, qa_r/c_1 \\ qb_1/c_1, qb_2/c_1, \dots, qb_r/c_1 \end{matrix}; q, z \right] + \text{idem}(c_1; c_2, \dots, c_r). \end{aligned}$$

Theorem 5. *We set $d' = a_1 a_2 \dots a_s / b_1 b_2 \dots b_r$. When $s < r$, we have*

$$\begin{aligned} & \frac{(q/a_1, q/a_2, \dots, q/a_s; q)_\infty}{(q/b_1, q/b_2, \dots, q/b_r; q)_\infty} {}_s\psi_r \left[\begin{matrix} a_1, a_2, \dots, a_s \\ b_1, b_2, \dots, b_r \end{matrix}; q, x \right] \\ &= \frac{q}{b_1} \frac{(b_1/a_1, b_1/a_2, \dots, b_1/a_s, q; q)_\infty}{(b_1, q/b_1, b_1/b_2, \dots, b_1/b_r; q)_\infty} \cdot \frac{\theta(-d' b_1 x/q)}{\theta(-d' x)} \\ & \quad \times {}_s\phi_{r-1} \left[\begin{matrix} qa_1/b_1, qa_2/b_1, \dots, qa_s/b_1 \\ qb_2/b_1, \dots, qb_r/b_1 \end{matrix}; q, \left(\frac{q}{b_1} \right)^{r-s} x \right] + \text{idem}(b_1; b_2, \dots, b_r), \end{aligned}$$

7.2 q -Laplace transform to series on x^m

We take a positive integer m . We consider

$$\varphi(\xi) = \frac{\theta_q(a\xi)}{\prod_{j=1}^m \theta_q(b_j \xi)} \sum_{n \geq 0} c_n \xi^{-mn}.$$

Then we obtain

$$\mathcal{L}_{q;1}^{[\lambda]} \varphi(x) = \frac{\theta_q(a\lambda) \theta_{q^m}(q^m a x / \lambda^m \prod_{j=1}^m b_j)}{\theta_q(qx/\lambda) \prod_{j=1}^m \theta_q(b_j \lambda)} \sum_{n \geq 0} c_n (q^m)^{-\frac{n(n-1)}{2}} \left(\frac{\prod_{j=1}^m b_j}{a q^m x} \right)^n.$$

We set $p_2 = p^{k/m+k}$. $p_1 \cdot p_2 = p$.

We apply p_2 -Laplace transformation to ${}_s\phi_{r-1}(1/x^{m+k})$, then we have a power series on x^m .

8. q -Painlevé equations: q - P_{VI} , q - P_{V}

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$$y = y(t), z = z(t), \bar{y} = y(qt), \bar{z} = z(qt).$$

$a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$: parameter

$$P(A_3) \sim q\text{-}P_{\text{VI}} : \frac{y\bar{y}}{a_3a_4} = \frac{(\bar{z} - b_1t)(\bar{z} - b_2t)}{(\bar{z} - b_3)(\bar{z} - b_4)}, \quad \frac{z\bar{z}}{b_3b_4} = \frac{(y - a_1t)(y - a_2t)}{(y - a_3)(y - a_4)},$$

$$\frac{b_1b_2}{b_3b_4} = q \frac{a_1a_2}{a_3a_4}.$$

$$P(A_4) \sim q\text{-}P_{\text{V}} : \frac{y\bar{y}}{a_3a_4} = -\frac{(\bar{z} - b_1t)(\bar{z} - b_2t)}{\bar{z} - b_3}, \quad \frac{z\bar{z}}{b_3} = -\frac{(y - a_1t)(y - a_2t)}{a_4(y - a_3)},$$

$$\frac{b_1b_2}{b_3} = q \frac{a_1a_2}{a_3a_4}.$$

$$P(A_5)^{\sharp} \sim q\text{-}P_{\text{III}}(D_6) : \frac{y\bar{y}}{a_3a_4} = -\frac{\bar{z}(\bar{z} - b_2t)}{\bar{z} - b_3}, \quad \frac{z\bar{z}}{b_3} = -\frac{y(y - a_1t)}{a_4(y - a_3)}, \quad \frac{b_1b_2}{b_3} = q \frac{a_1}{a_3a_4}.$$

$$P(A_6)^{\sharp} \sim q\text{-}P_{\text{III}}(D_7) : \frac{y\bar{y}}{a_3a_4} = -\frac{\bar{z}(\bar{z} - b_2t)}{\bar{z} - b_3}, \quad \frac{z\bar{z}}{b_3} = -\frac{y^2}{a_4(y - a_3)}, \quad \frac{b_1b_2}{b_3} = q \frac{1}{a_3a_4}.$$

9. Connection Preserving Deformation

by Jimbo-Sakai, $P(A_3)$, by M. Murata (other)

$$Y(qx, t) = A(x, t)Y(x, t),$$

$$Y(x, qt) = B(x, t)Y(x, t).$$

$$A(x, t) = A_0(t) + xA_1(t) + x^2A_2,$$

$$B(x, t) = f(x, t)(xI + B_0(t)),$$

$$f(x, t) = \begin{cases} \frac{x}{(x - a_1qt)(x - a_2qt)} & P(A_3), P(A_4) \\ \frac{1}{x - a_1qt} & P(A_5)^\sharp \\ \frac{1}{x} & P(A_6)^\sharp \end{cases}$$

The **compatibility condition** leads to q -Painlevé equations

$$A(x, qt)B(x, t) = B(qx, t)A(x, t).$$

10. The linearized equation

$P(A_3)$:

Eigenvalues: $A_0(t) \sim \text{diag}(\theta_1 t, \theta_2 t)$, $A_2 = \text{diag}(\kappa_1, \kappa_2)$
 $\det A(x, t) = \kappa_1 \kappa_2 (x - a_1 t)(x - a_2 t)(x - a_3)(x - a_4).$

$P(A_4)$:

Eigenvalues: $A_0(t) \sim \text{diag}(\theta_1 t, \theta_2 t)$, $A_2 = \text{diag}(\kappa_1, \mathbf{0})$
 $\det A(x, t) = \kappa_1 \kappa_2 (x - a_1 t)(x - a_2 t)(x - a_3).$

$P(A_5)^\sharp$:

Eigenvalues: $A_0(t) \sim \text{diag}(\theta_1 t, \mathbf{0})$, $A_2 = \text{diag}(\kappa_1, \mathbf{0})$
 $\det A(x, t) = \kappa_1 \kappa_2 x(x - a_1 t)(x - a_3).$

$P(A_6)^\sharp$:

Eigenvalues: $A_0(t) \sim \text{diag}(\theta_1 t, \mathbf{0})$, $A_2 = \text{diag}(\kappa_1, \mathbf{0})$
 $\det A(x, t) = \kappa_1 \kappa_2 x^2(x - a_3).$

10.1 Nonlinear Asymptotics and linear connection

A generic q -Painlevé function has a **wild singularity** at $x = 0, \infty$.

- We can take **formal double power series solutions** around $x = 0$ (for $P(A_3)$, $P(A_4)$, $P(A_5)^\sharp$, $P(A_6)^\sharp$):

$$y(t) = \sum_{n=1}^{\infty} \sum_{m=-n}^n b_{nm} t^{n+m\sigma},$$

$$z(t) = \sum_{n=1}^{\infty} \sum_{m=-n}^n c_{nm} t^{n+m\sigma}.$$

- The formal double power series solutions are **convergent for a small domain**, and we can take a limit $t \rightarrow 0$.
- By taking the limit $t \rightarrow 0$, we can **determine the Connection matrix**

[Remark] The τ functions are described by **conformal blocks** in **differentiable** cases (Lisovyy, et al.).

10.2 Double asymptotic solutions of q -Painlevé

This type of expansions are studied by **R. Fuchs** at first.

Jimbo also gives the same type of asymptotic solutions for P_{III} , P_{V} and P_{VI} . He also showed that the double asymptotic series **converges in a small angle domain**.

Mano studied the case of $P(A_3)$ (q - P_{VI}).

Recently **Lisovyy** et al show that asymptotic solutions for P_{III} , P_{V} and P_{VI} can be expressed by **conformal blocks**.

Remark For other q -Painlevé equations, the equation has the form

$$y\bar{y} = t^\varepsilon F(t, \bar{z}), \quad z\bar{z} = G(t, y)$$

where $\varepsilon = \pm 1$. These systems **do not have double power series** solutions. (q -difference Nonlinear irregular singular points)

11. Connection formula

1. The first limit

We take a limit $t \rightarrow 0$. Then $A(x, t)$ goes to $x\Lambda + x^2A_2$

2. The second limit

We set $x = \xi t$. We take a new connection:

$$\tilde{A}(\xi, t) = t^{-1}t^{-\log_q \Lambda} A(t\xi, t)t^{\log_q \Lambda}$$

We take a limit $t \rightarrow 0$. Then $\tilde{A}(\xi, t)$ goes to $M + \xi\Lambda$. $M \sim A_0/t$

The limit equations

$$Y_1(xq) = [x(\Lambda + xA_2)]Y_1(x), \quad Y_2(\xi q) = (M + \xi\Lambda)Y_2(\xi)$$

are magentauced to hyperegeometric.

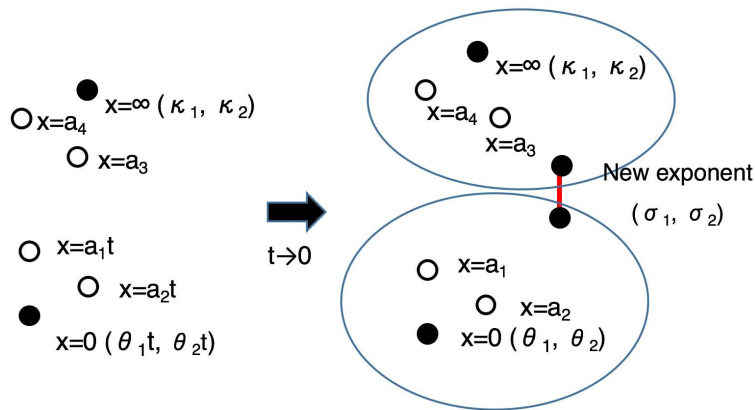
We set eigenvalues of Λ as σ_1, σ_2 . (assume that $\sigma_1\sigma_2 \neq 0$)

$$C^{-1}\Lambda C = \text{diag}(\sigma_1, \sigma_2).$$

We write $D = \text{diag}(\log_q \sigma_1, \log_q \sigma_2)$. Then $t^{\log_q \Lambda} = C^{-1}t^D C$.

12.1 Naive proof

In $P(A_3)$ case:



$$\det A(x, t) = \kappa_1 \kappa_2 (x - a_1 t)(x - a_2 t)(x - a_3)(x - a_4).$$

$$\rightarrow \det A_1(x, t) = \kappa_1 \kappa_2 x^2 (x - a_3)(x - a_4)$$

$$\rightarrow \det A_2(x, t) = \sigma_1 \sigma_2 (x - a_1)(x - a_2)$$

Theorem 6.

$$P(x) = P_2(x/t)P_1(x)$$

12.2 Limit from Painlevé to hypergeometric

$$P(A_3): \det A(x, t) = \kappa_1 \kappa_2 (x - a_1 t)(x - a_2 t)(x - a_3)(x - a_4).$$

$$\rightarrow \det A_1(x, t) = \kappa_1 \kappa_2 x^2 (x - a_3)(x - a_4) \quad \text{Heine}$$

$$\rightarrow \det A_2(x, t) = \sigma_1 \sigma_2 (x - a_1)(x - a_2) \quad \text{Heine}$$

$$P(A_4): \det A(x, t) = \kappa_1 \kappa_2 (x - a_1 t)(x - a_2 t)(x - a_3).$$

$$\rightarrow \det A_1(x, t) = \kappa_1 \kappa_2 x^2 (x - a_3) \quad q\text{-Kummer}$$

$$\rightarrow \det A_2(x, t) = \sigma_1 \sigma_2 (x - a_1)(x - a_2) \quad \text{Heine}$$

$$P(A_5)^\sharp: \det A(x, t) = \kappa_1 \kappa_2 x (x - a_1 t)(x - a_3).$$

$$\rightarrow \det A_1(x, t) = \kappa_1 \kappa_2 x^2 (x - a_3) \quad q\text{-Kummer}$$

$$\rightarrow \det A_2(x, t) = \sigma_1 \sigma_2 (x - a_1) \quad q\text{-Kummer}$$

$$P(A_6)^\sharp: \det A(x, t) = \kappa_1 \kappa_2 x^2 (x - a_3).$$

$$\rightarrow \det A_1(x, t) = \kappa_1 \kappa_2 x^2 (x - a_3) \quad q\text{-Kummer}$$

$$\rightarrow \det A_2(x, t) = \sigma_1 \sigma_2 x^2 \quad \text{Hahn-Exton}$$

12.3 How to calculate connection matrix

1) The original equation:

$$Y^{(\infty)}(xq, t) = A(x, t)Y^{(\infty)}(x, t)$$

$$Y^{(\infty)}(x, t) = q^{u(u-1)}\hat{Y}^{(\infty)}(x, t)x^K, \quad (u = \log x, \hat{Y}^{(\infty)}(0) = I)$$

2) The first limit of solution: From $Y^{(\infty)}(x, t)$ to $Y_1(x)$:

$$\lim_{t \rightarrow 0} Y^{(\infty)}(x, t) =: Y_1^{(\infty)}(x).$$

$$Y_1^{(\infty)}(xq) = [x(\Lambda + xA_2)]Y_1^{(\infty)}(x)$$

3) Connection formula for $Y_1(x)$:

$$Y_1^{(\infty)}(x) = Y_1^{(0)}(x)P_1(x),$$

$$Y_1^{(0)}(x) = q^{u(u-1)/2} \textcolor{violet}{C} \hat{Y}^{(0)}(x)x^D, \quad (\hat{Y}_0(0) = I)$$

where $C^{-1}\Lambda C = \text{diag}(\sigma_1, \sigma_2)$, and $D = \text{diag}(\log_q \sigma_1, \log_q \sigma_2)$.
 C has an anbiguity $C \rightarrow CG$, G is a diagonal matrix.

12.3 How to calculate connection matrix 2

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4) **The second limit** : Set $\xi = x/t$.

$$Y_2^{(\infty)}(\xi q) = (M + \xi \Lambda) Y_2^{(\infty)}(\xi)$$

$$Y_2^{(\infty)}(\xi) = q^{v(v-1)/2} \textcolor{violet}{C} \hat{Y}_2^{(\infty)}(x) x^D.$$

5) **Connection formula** for $Y_1(x)$

$$Y_2^{(\infty)}(\xi) = Y_2^{(0)}(\xi) P_2(\xi)$$

6) We can reconstruct $Y(x, t)$ from Y_1 by an **iteration operator**

$$U(x, t) = I + \sum_{n=1}^{\infty} \int_0^t d_q t_1 \int_0^{t_1} d_q t_2 \cdots \int_0^{t_{k-1}} d_q t_k F(x, t_1) F(x, t_2) \cdots F(x, t_k).$$

Here

$$F(x, t) = \frac{1}{t(q-1)} (B(x, t) - I),$$

$$B(x, t) = I + \frac{1}{x} \left(1 - \frac{qta_1}{x} \right) \left(1 - \frac{qta_2}{x} \right) [B_0(t) + qt(a_1 + a_2)I - x^{-1}q^2t^2a_1a_2I]$$

$$Y^{(\infty)}(x, t) = U(x, t)Y_1^{(\infty)}(x),$$

$$Y^{(0)}(x, t)P_2(x/t) = U(x, t)Y_1^{(0)}(x).$$

Proof is the standard way to solve an **integral equation** starting from the initial value $Y_1^{(\infty)}(x)$.

Proof of Theorem:

By the Lemma above, we have

$$\begin{aligned} Y^{(0)}(x, t)P_2(x/t)P_1(x) &= U(x, t)\underline{Y_1^{(0)}(x)}P_1(x) \\ &= U(x, t)\underline{Y_1^{(\infty)}(x)} \\ &= Y^{(\infty)}(x, t). \end{aligned}$$

Therefore

$$Y_1^{(\infty)}(x) = Y^{(0)}(x, t)P_2(x/t)P_1(x) = Y^{(0)}(x, t)P(x).$$

- We can determine the connection matrix of basic hypergeometric equations of **the second order**.
- We can determine the connection matrix of basic hypergeometric equations of **any order** when **one singular point is regular**
- For q -Painlevé equations, linear connection problems are solved for $P(A_3)$, $P(A_4)$ which correspond to q - P_{VI} , q - P_{V} , q - $P_{\text{III}}(D_6)$, q - $P_{\text{III}}(D_7)$. The connection matrix is a product of a connection matrix of basic hypergeometric functions:

$$P(A_3): \text{Heine} \times \text{Heine}, \quad P(A_4): \text{Heine} \times q\text{-Kummer},$$

$$P(A_5)^\sharp: q\text{-Kum.} \times q\text{-Kum.}, \quad P(A_5)^\sharp: q\text{-Kum.} \times \text{Hahn-Exton}$$

Future problems

- How about other q -Painlevé equations? (nonlinear irregular)
- Higher order cases? Elliptic Painlevé case?
- **The space of connection** of other q -Painlevé equations?

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