String topology & Homotopy Skein Modules
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Uwe Kaiser

Department of Mathematics
Boise State University, 1910 University Drive
Boise, ID 83725-1555, ukaiser@boisestate.edu

References

[1] U. Kaiser, Presentations of q-homotopy skein modules of oriented 3-manifolds, *J. Knot Theory Ramifications*, **10** (2001), no. 3, 461–491

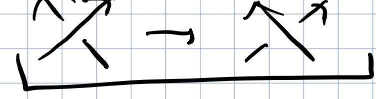
[2] U. Kaiser, Deformation of string topology into homotopy skein modules, *Algebr. Geom. Topol.* , **3** (2003), 1005–1035

Przytycki, Kalfagianni, Lin,
Vassiliev

survey

- h. skein modules
- string topology $\leftrightarrow$ h. skein modules
- examples, appl.

M compact connected oriented 3-mfd.
$\pi_1 M$ "essentially" determines the topology of M
Milnor 1954: or. links in M, allow homotopies keeping components disjoint
only for same components



(1) homotopy skein modules
Comm. ring with 1, R

$$\frac{\text{free } R\text{-modules on } \mathcal{H}(M)}{\text{submodule gen. by skein relations}}$$

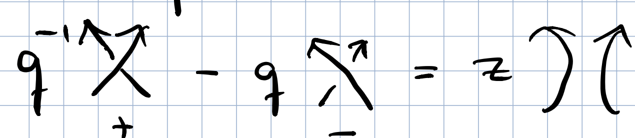
$\mathcal{H}(M)$ = set of link htp. class of oriented, unordered links in M

$$\begin{tikzcd}
\mathbb{Z}[q^{\pm1}, z] \arrow[r, "z=0"] \arrow[d, "q=1"'] & \mathbb{Z}[q^{\pm1}] \arrow[d, "q=1"] \\
\mathbb{Z}[z] \arrow[r, "z=0"] & \mathbb{Z}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A}(M) \arrow[r] \arrow[d] & \mathcal{L}(M) \arrow[d] \\
\mathcal{C}(M) \arrow[r] & \text{free ab. grp on } \mathcal{H}(M)
\end{tikzcd}$$

$K_+ - K_-$

$= \mathbb{Z} \operatorname{mon} \hat{\pi}$

Przytycki: q-homotopy skein relation



crossing of distinct cpts.

$\mathcal{C}(M)$ = Hoste-Przytycki skein module

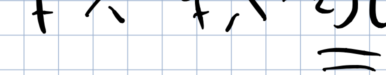
$\hat{\pi} = \hat{\pi}(M)$ = conj. classes in $\pi_1 M$

$\mathcal{H}(M) \to \operatorname{mon} \hat{\pi}$

monomials of elements in $\hat{\pi}$

free comm. monoid on $\hat{\pi}$.

$\mathcal{A}(M) = \dfrac{\text{free } \mathbb{Z}[q^{\pm1}, z]\text{-module on } \mathcal{H}(M)}{\text{subm. gen. by}}$



$\mathcal{C}(M)$

Ex: $M = S^3$.

free module generated by unlinks

with r components, usually
$r \geq 0$ (includes an empty link)

$$\boxed{R(\text{mon}\,\hat{\pi}) = S R \hat{\pi}}$$

empty link,
empty mon.

$\mathcal{H}(M)$ $\qquad$ $\mathcal{H}(M)$

$$\boxed{\mathcal{H}(S^3) \cong R[q^{\pm 1}, z, u]}$$

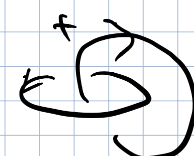
Przytycki

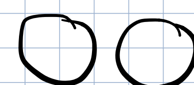
$$\mathcal{H}(M) \xrightarrow{\omega} \text{mon}(\hat{\pi})$$

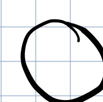
Problems:

- understand underlying link homotopy invts.
- find suitable presentation. in terms of a minimal set of generators, and relate the relation to the top. of M
- torsion ? free ?

+

q^{-1} 

$-\, q$ 

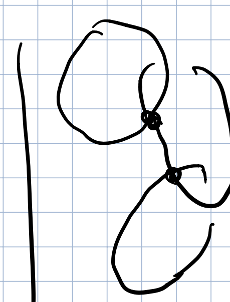
$=\, z$ 

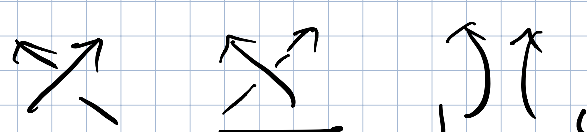
 $= q^2$ 

$+\, z$ 

$[K] \in \mathcal{H}(M)$

link

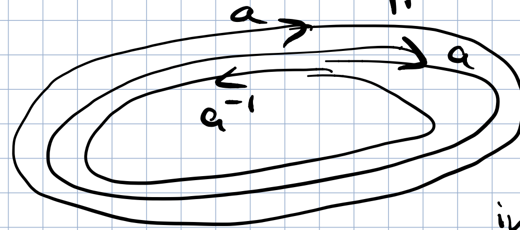




Minimal set of generators by picking a "standard link" for each $\alpha \in mon(\hat{\pi})$, $K_\alpha \in \mathcal{H}(M)$ bordism

pick $\{K_\alpha\}$ in a special way:

- every sublink of a standard link is a standard link
- if $a^{\pm 1} \in \hat{\pi}(M)$ appears in α



same free hom. class in standard

- trivial conj.

$K_{\alpha'}$

3-ball

$$R\,mon\,\hat{\pi} = SR\,\hat{\pi}(M) \xrightarrow{\sigma} \mathcal{S}(M) \leftarrow$$

any of $\mathcal{H}(M)$, $\mathcal{C}(M)$, $\mathcal{L}(M)$

$$\alpha \longmapsto [K_\alpha]$$

onto by induction

Find $\ker \sigma$ (in terms of top. of M)

Theorem:

(a) $\mathcal{H}(M)$ is free $\Longleftrightarrow$ $M \cong S^3$, $S^1 \times D^2$, $S^1 \times S^1 \times I$, $S^2 \times I$ ($\cong F \times I$ with

$\pi_1 F$ abelian)

(b) $\hat{M} = M$, cap off all 2-sph. in ∂M

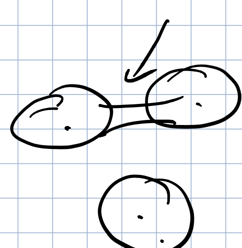
essential : inj. in π_*

If M is aspherical and atoroidal (all essential maps of S^2, $S^1 \times S^1$ in M are homotopic into ∂M)

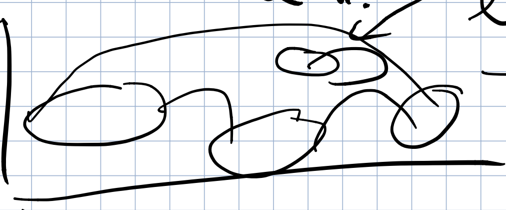
then $\mathcal{C}(M)$ is free

Conversely: If M has an essential S^2 not hom. into ∂M then $\mathcal{C}(M)$ has torsion. If M is not atoroidal then $\mathcal{C}(M)$ has torsion, except possibly when M is special Seifert fibred. (study conj. in cat. Fuchsian groups)

(c) $\mathcal{L}(M)$ is free $\Leftrightarrow$ each ess. sing. torus is homologous into ∂M.

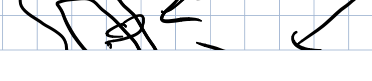


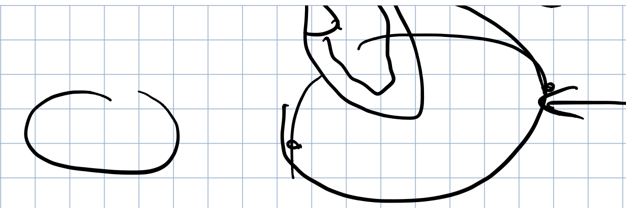
nullh. Links in a sub. Q-h.



lk #'s.

90's.





String topology

1999 Chas – Sullivan for loop space

$$H_*^{S^1}(\mathrm{map}(S^1,M)) = h_*(M)$$

S^1-action by rotating maps in the domain.

$$H_*^{S^1}(\mathrm{map}_\bullet(S^1,M)) = H_*\left(\mathrm{map}_\bullet(S^1,M)/S^1\right)$$

$$\gamma = \gamma_1 : h_1(M) \otimes h_0(M) \to h_0(M)$$

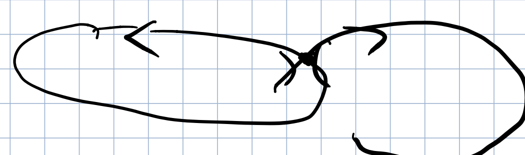
module over $h_1(M)$

Lie bracket $\gamma L : h_1(M) \otimes h_1(M) \to h_1(M)$

$H_1(\mathrm{map}(S^1,M))$, $\quad S^1 \times S^1 \to M$

part of a graded Lie algebra structure on h_*.

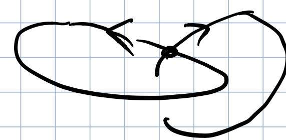
1-dim. $\underbrace{S^1 \times S^1} \times \underbrace{S^1 \times S^1} \to M \times M \supset \Delta$



$$S^1 \to \mathrm{map}_\bullet(S^1,M) \to \mathrm{map}_\bullet(S^1,M)/S^1$$

Gysin-sequence

$$(S^1 \times S^1) \times S^1 \longrightarrow M \times M \supset \Delta$$



$h_0(M)$

$\|$

γ: $h_1(M) \otimes \mathbb{Z}\hat{\pi}(M) \longrightarrow \mathbb{Z}\hat{\pi}(M)$

extends to

$h_1(M) \otimes S\mathbb{Z}\hat{\pi}(M) \longrightarrow S\mathbb{Z}\hat{\pi}(M)$

$$\gamma(a \otimes xy) = \gamma(a \otimes x)\, y + x\, \gamma(a \otimes y)$$

$R \xrightarrow{\varepsilon} \mathbb{Z}$ homomorphism.

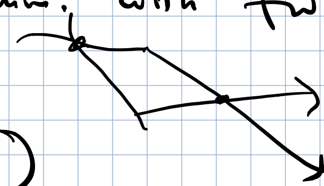
comm. with 1

Def: A quadruple $(A, \alpha_A, \beta_A, \gamma_A)$ with A an R-module a geom. R-deformation of γ if

$$\begin{tikzcd} R\mathcal{H}(M) \arrow[r, "\alpha_A"] \arrow[dr, "\varepsilon_*"'] & A \arrow[r, "\beta_A"] & S\mathbb{Z}\hat{\pi}(M) \\ & \mathbb{Z}\mathcal{H}(M) \arrow[ur] & \end{tikzcd}$$

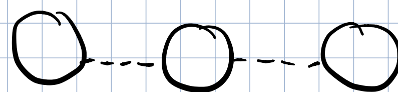
and γ_A : $h_1(M) \otimes A \longrightarrow A$ R-linear and defined by a "transversal" version of Chas-Sull. construction (A becomes a module over $h_1(M)$)

isotopy classes of imm. with two double points



$\mathcal{K}^i(M)$

$\mathcal{K}^2$ 

$\mathcal{K}^3$ 

$\mathcal{K}^4$ 

$$K_{**} \in \mathcal{K}^i(M)$$

$$K_{+0} - K_{-0} - K_{0+} + K_{0+}$$ integral elements

$$\in \mathbb{Z}\mathcal{H}(M)$$

$$0 \quad \text{for } \mathcal{K}^2(M)$$

$$W(M;\mathbb{Z}) = \mathbb{Z}\mathcal{H}(M) \Big/ \text{subgr. gen. by all } \mathcal{K}^3(M)$$

(universal $\mathbb{Z}$-def.)

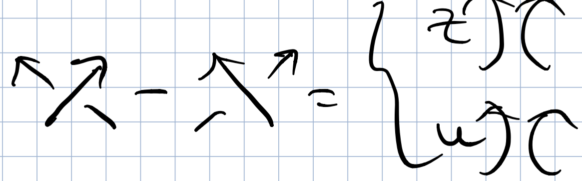
<u>Theorem</u>: $\mathcal{C}(M)$ is a geom. R-def. of γ for $\varepsilon : \underbrace{\mathbb{Z}[z]}_{= R} \xrightarrow[z \mapsto \infty]{} \mathbb{Z}$, in particular

$$\gamma_\infty : \underset{\substack{\| \\ h_1(M)\otimes R}}{h_1(M;R)} \otimes \mathcal{C}(M) \to \mathcal{C}(M)$$

$h_1(M)\otimes R$ Lie-algebra.

<u>Theorem</u>: $\mathcal{C}(M)$ is free $\Longleftrightarrow$

γ_∞ is trivial.



mixed cr.

self-cross.

$\mathbb{Z}[z,u]$

M subm. of $\mathbb{Z}$-homology 3-sphere

$H_1 M$ is free abelian

linking #'s in $\mathbb{Z}$ are defined

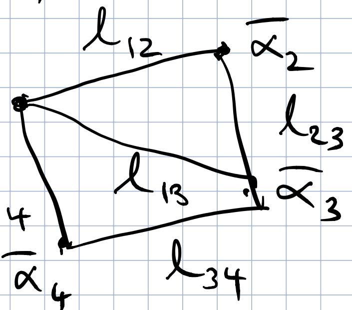
Suppose M is atoroidal.

$\hat{\pi}(M) \longrightarrow \underline{H_1 M}$

$\underline{L \text{ link in } M}$ $\rightsquigarrow$ $\underline{\text{graph } \Gamma}$

vertex for each comp.

$H_1 M \ni$



dichromatic polyn. of a graph.

$\gamma: \; SR\hat{\pi}(M) \longrightarrow SRH_1M$

$\mathcal{H}(M) \xrightarrow{\delta} SRH_1M$

q-homotopy

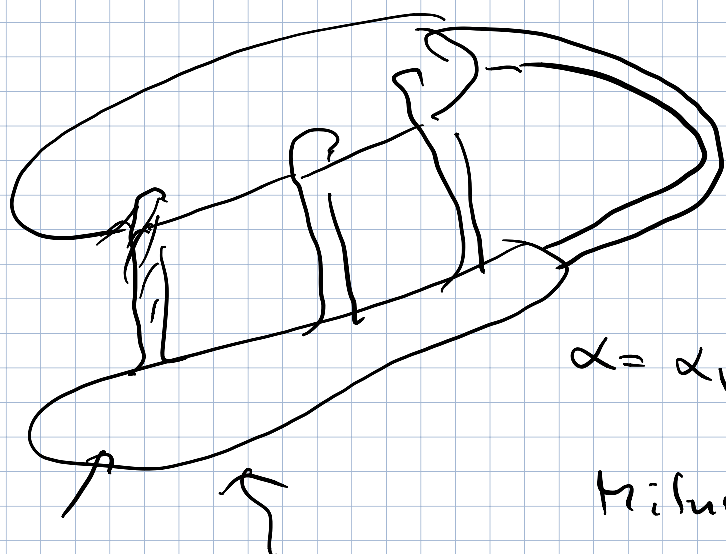
$$\underbrace{S[L]} = \sum_{\substack{S \subset \Gamma \\ \underline{\text{full forests}}}} q^{\sum_{e \in A_S} 2l_e} z^{|S|} \prod_{\substack{e \text{ edges} \\ \text{in } S}} q^{l_e} \underbrace{[l_e]}_{\gamma(S)}$$

π

<u>non $H_1 M$</u>

add all $\bar{\alpha}$'s in S

- What is in general $[L]$ for just 2-comp.



$\underline{K_\alpha}$ $\quad$ K_α

g

$\alpha = \alpha_1 \alpha_2$

Milnor

weighted linking #

Al. Pilz

$F \times \underline{I}$.

$\pi_1 F$.