Extension of plurisubharmonic functions with subharmonic singularities

Żywomir Dinew [1] and Sławomir Dinew [1]

[1] Faculty of Mathematics and Computer Science
Jagiellonian University

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- $u(x) \leqslant const \cdot d(x, S)^{\alpha+2-n}$, E contained in a C^2 submanifold S of dimension $n-1$, $H^{\alpha}(E) = 0$- Hausdorff measure, $0 < \alpha < n-2$ (Gardiner' 91)

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Other results: Harvey-Polking' 70, Kaufman-Wu' 80, Tamrazov' 86, Riihentaus-Tamrazov' 91 and ' 93, Yarmetov' 94, Abdullaev-Imomkulov' 97, Pokrovskiĭ' 17

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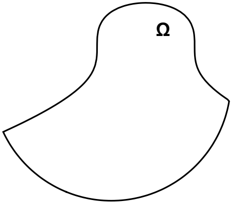
Other results: Grauert-Remmert' 56, Pflug' 80, Favorov' 81, Abidi' 99 and ' 10

Extension through bigger sets (but with compatibility assumptions). What is known?



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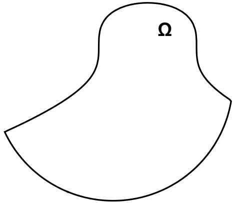

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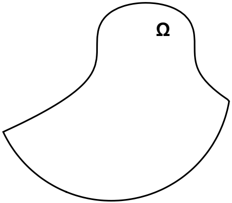
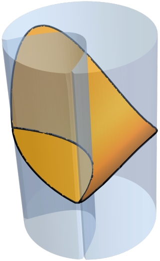
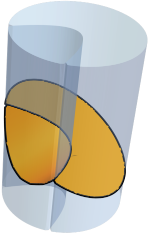
- $u \in C^0(\Omega) \cap C^2(\Omega_1 \cup \Omega_2)$, E hypersurface of class C^1 which divides Ω to two subdomains Ω_1 and Ω_2, $u|_{\Omega_i} = u_i \in C^1(\Omega_i \cup E)$,

$$\frac{\partial u_i}{\partial \vec{\mathbf{n}}_k} \geqslant \frac{\partial u_k}{\partial \vec{\mathbf{n}}_k} \quad \text{on } E, \quad i \neq k, i = 1, 2, k = 1, 2,$$

  where $\vec{\mathbf{n}}_k$ are the outward unit normal vectors of Ω_k on E (Blanchet' 95)







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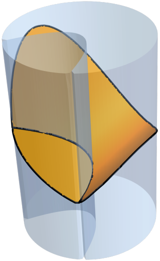

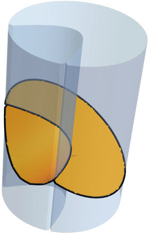

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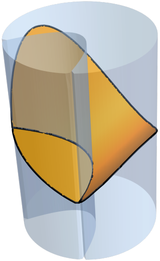

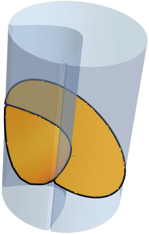

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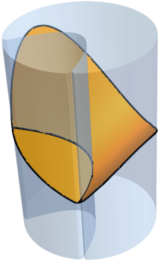

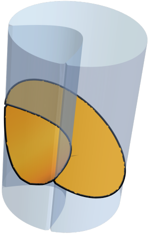

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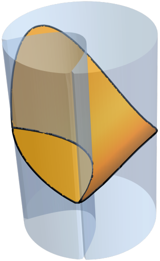

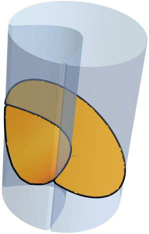

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Subharmonicity on Ω plays the role of the compatibility conditions.Without the subharmonicity on Ω one has the counterexample:

$$u(z) := \begin{cases} \|z\|^2 & \text{if } \|z\| \leqslant 1 \\ 1 & \text{if } \|z\| > 1 \end{cases}$$

Definition

Let Ω be a domain in $\mathbb{C}^n$ and $E \subseteq \Omega$ be a closed subset of Lebesgue measure zero. Let u be a subharmonic function in Ω which is furthermore plurisubharmonic in $\Omega \backslash E$. Then u is said to be a plurisubharmonic function with **subharmonic singularities** along E.

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Theorem (D.-D.'21)

Let $E \subseteq \Omega$ be a closed subset of Lebesgue measure zero. Then any subharmonic function u in Ω which is plurisubharmonic in $\Omega \setminus E$ is actually plurisubharmonic in the whole Ω.

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recommended reference: Crandall-Ishii-Lions'92

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General second-order PDE:

$$F(x, u, Du, D^2u) = 0,$$

where $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}(n) \to \mathbb{R}$ and $\mathcal{S}(n)$ is the set of all $n \times n$ symmetric real matrices.

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Classical solutions:

$u \in C^2$ and pointwise $F(x, u(x), Du(x), D^2u(x)) = 0$.

In general too hard to obtain, problems with existence, uniqueness etc. $\Longrightarrow$ one should look for another notion of solutions.

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$F(x, r, p, X) \leqslant F(x, r, p, Y)$ if $Y \leqslant X$ as matrices i.e., $X - Y$ is nonnegative definite

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uniform ellipticity:

if $P \geqslant 0$ as matrices then for some $\lambda, \Lambda > 0$

$$\lambda\, tr\, P \leqslant F(x, r, p, X - P) - F(x, r, p, X) \leqslant \Lambda\, tr\, P$$

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- **Constrained complex Hessian equation**: $u \in PSH$

$$-\det{}^{+}\left(\frac{\partial^2 v}{\partial z_j \partial \bar{z}_k}\right) + f(x) = 0,$$ where $f \geqslant 0$ and

$$\det{}^{+}(A) := \begin{cases} \det(A) & \text{if } A \geqslant 0 \\ -\infty & \text{otherwise} \end{cases}.$$ Here

$$F(x,r,p,X) = \begin{cases} -\sqrt{\det \frac{1}{2}\left(X + J^{\intercal} X J\right)} + f(x) & \text{if } X + J^{\intercal} X J \geqslant 0 \\ +\infty & \text{otherwise} \end{cases},$$

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. It is proper but not uniformly elliptic (Eyssidieux-Guedj-Zeriahi' 11).

Definition

A function φ defined on some neighborhood V of a point z_0 is called a **differential test from above** at z_0 for the upper-semicontinuous function u defined on a domain $\Omega \subseteq \mathbb{C}^n$, also containing z_0, if it is C^2 smooth on V, and

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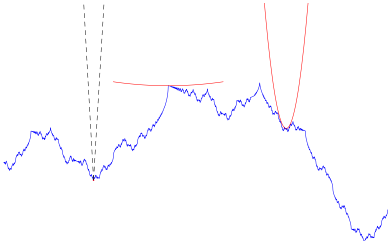
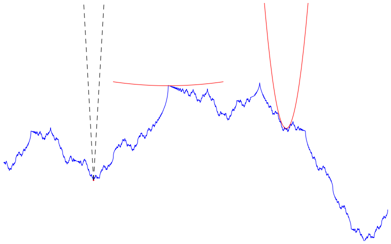
Definition

An upper-semicontinuous function u on a domain Ω is said to allow a differential test from above at $z_0 \in \Omega$ if there exists a neighborhood $V \ni z_0$ such that the set of differential tests from above for u at z_0 is non-empty.





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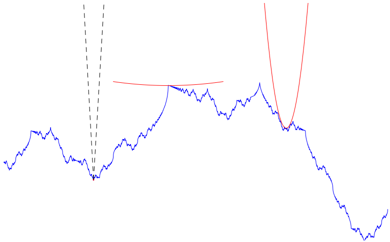

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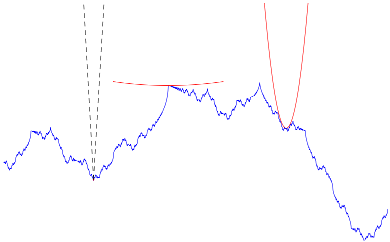
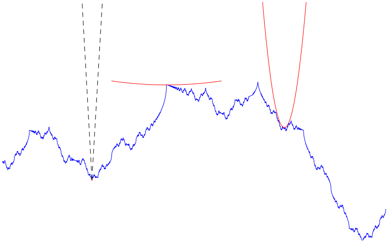


- if u is bounded above then local differential test can be made global
- the same function φ can be a differential test from above at several points
- u allows a differential test from above at any point at which it is twice differentiable



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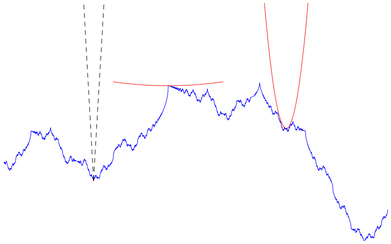

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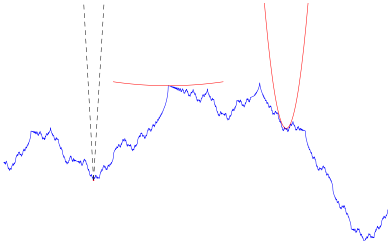

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REMARK

What we introduced is the so-called **continuous** *viscosity theory. There are other viscosity theories, for example the L^p viscosity theory, where the differential tests are assumed to be $W^{2,p}$ instead of C^2.*

An upper semicontinuous function u is called a viscosity **subsolution** to $F(x, u, Du, D^2u) = 0$ in Ω if for every $x_0 \in \Omega$ and every differential test from above φ at x_0 one has

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Let $\Omega \subseteq \mathbb{C}^n$ be a domain. An upper semicontinuous function u on Ω is

- *subharmonic if for every $z_0 \in \Omega$ and every local C^2-smooth function φ defined near z_0 and satisfying $\varphi(z) \geqslant u(z)$ with equality at z_0 we have $\Delta\varphi(z_0) \geqslant 0$*

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u is plurisubharmonic $\Leftrightarrow$ u is "viscosity plurisubharmonic" $\Leftrightarrow$ it is a viscosity subsolution to the constrained complex Hessian equation $-\det^+\left(\frac{\partial^2 v}{\partial z_j \partial \bar{z}_k}\right) = 0$ (Eyssidieux-Guedj-Zeriahi' 11).

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This will be achieved by looking for points near z_0 at which the smallest eigenvalue of $\frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}$, which may be negative, is controlled from below + the continuity of the smallest eigenvalue of the complex Hessian, as φ is C^2.

Translating if necessary we may assume that z_0 is the origin, that Ω contains a ball $B_{\delta_0} = \{z \in \mathbb{C}^n : \|z\| < \delta_0\}$ centered at the origin and that φ is defined on the whole B_{δ_0}.

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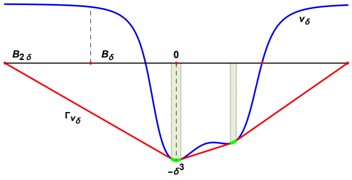
- v_δ is a lower semicontinuous viscosity supersolution to the Poisson equation $\Delta v = f$, where $f := \Delta\varphi + 4n\delta$. Note that f is continuous.

The convex envelope of v_δ in $B_{2\delta}$ is

$$\Gamma_{v_\delta}(z) := \sup\{l(z)|\ l - \text{affine},\ l \leqslant v_\delta \text{ on } B_\delta,\ l \leqslant 0 \text{ on } B_{2\delta} \backslash B_\delta\}.$$



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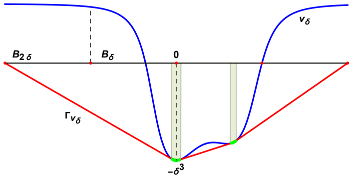

Lemma (Alexandrov-Bakelman-Pucci estimate, e.g. Caffarelli- Cabré, *Fully Nonlinear Elliptic Equations*, Theorem 3.2)

Let v_δ and Γ_{v_δ} be as above. Then for some universal constant C, dependent only on n we have

$$\delta^3 \leqslant \sup_{B_\delta} |v_\delta| \leqslant C\delta \left(\int_{\{B_\delta \cap \{v_\delta = \Gamma_{v_\delta}\}\}} \max\{f, 0\}^{2n} dV \right)^{\frac{1}{2n}}.$$

Here we crucially exploit the fact that the Laplacian is a uniformly elliptic operator.

The upshot is that for every $\delta \in \left(0, \frac{\delta_0}{2}\right)$ the function v_δ matches its convex envelope Γ_{v_δ} on a set of **positive measure** within B_δ.

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We pick a point $z_\delta \in \{B_\delta \cap \{v_\delta = \Gamma_{v_\delta}\}\} \setminus E$. As u is plurisubharmonic around z_δ for any $0 < r$ small enough and any unit vector $T \in \mathbb{C}^n$ we have

$$u(z_\delta) \leqslant \frac{1}{2\pi} \int_0^{2\pi} u(z_\delta + re^{i\theta} T) d\theta.$$

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The same inequality is valid for the convex (hence plurisubharmonic) function Γ_{v_δ}:

$$\Gamma_{v_\delta}(z_\delta) \leqslant \frac{1}{2\pi}\int_0^{2\pi} \Gamma_{v_\delta}(z_\delta + re^{i\theta}T)d\theta.$$

But then:

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LEMMA

For $\varphi \in C^2$ one has

$$\lim_{r\to 0+}\frac{\int_0^{2\pi}\varphi(z_\delta + re^{i\theta}T)d\theta - \varphi(z_\delta)}{r^2} = \sum_{j,k=1}^{n}\frac{\partial^2\varphi}{\partial z_j\partial\bar{z}_k}(z_\delta)T_j\bar{T}_k$$

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As T is an arbitrary unit vector this shows the non negative definiteness of the complex Hessian of φ at z_0.

Remark

It is well known that there exist closed sets of Lebesgue measure zero and of full Hausdorff dimension. Take for example a product of $2n$ *copies of* A*, where* $A = \{1\} \cup \bigcup_{j \geqslant 1} A_j$ *and* A_j *is a generalized Cantor set of Hausdorff dimension* $1 - 1/j$ *situated in the interval* $[1 - 1/j, 1 - 1/(j + 1)]$.

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Remark

Our theorem could be combined with some removable singularity theorems for (particular classes of) subharmonic functions. Now if u is subharmonic (of a particular class) on $\Omega \backslash F$, where F is removable (for this particular class) and plurisubharmonic on $\Omega \backslash (E \cup F)$, where E is closed and of measure zero, then u is plurisubharmonic on Ω.

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Remark

We note that the closedness assumption on E *is not really necessary, and is introduced only to ensure that plurisubharmonicity on* $\Omega \backslash E$ *makes sense. Alternatively, we could assume that* E *is any set of Lebesgue measure zero and if u is plurisubharmonic on some neighborhood of* $\Omega \backslash E$ *then u is plurisubharmonic on* Ω *if it is subharmonic there. This may be essential in some situations, as the closure of a null set can have positive measure.*

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