# An algebraic characterization of the affine three space

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# Notations

For a ring $R$, $B = R^{[n]}$ will mean $B \cong R[X_1, X_2, \ldots, X_n]$

$k$ : an algebraically closed field of characteristic zero.

$B$ : an affine $k$-domain.

$B^*$ : group of units of $B$.

# Characterization Problem

Affine algebraic geometry is a branch of commutative algebra and algebraic geometry which deals with the study of affine algebraic varietes (i.e. finite generated algebras over fields).

One of the important problems in affine algebraic geometry is the Characterization Problem.

Characterization Problem :
Find necessary and sufficient conditions on the affine domain $B$ so that $B = k^{[n]}$

The Characterization Problem is closely related to other challenging problems on the affine space like the Zariski Cancellation Problem.

# A necessary and sufficient condition

Let $dim\ B = n$. Then

$$B = k^{[n]} \Leftrightarrow \begin{cases} \text{there exists a coordinate system of } B \\ \text{consisting of } n \text{ elements.} \end{cases}$$

**Remark** :

- Given an arbitrary set of generators of $B$, it is not always easy to decide whether they form a coordinate system.
- Even when $B = k^{[n]}$, given an arbitrary $f \in B$, it is not always easy to decide whether $f$ is a coordinate.
- $f = X + X^2Y + Z^2 + T^3 \in k[X, Y, Z, T]$ is not a coordinate ( Makar-Limanov [6]).
- Is $f = Y + X(XZ + Y(YT + Z^2))$ a coordinate in $k[X, Y, Z, T]$ ?
  (Open; c.f Bhatwadekar, Dutta [13] and Lewis [14])

# Some necessary conditions

- $B$ is a UFD
- $B^* = k^*$.
- $B$ is regular.
- $Der_k(B)$ is a free $B$-module.
- If $k = \mathbb{C}$, $X = Spec\ B$ is contractible.

**Question : Are these also sufficient ?**

An algebraic characterization :

$$(A1) \quad B = k^{[1]} \Leftrightarrow \begin{cases} B \text{ is a UFD} \\ B^* = k^* \end{cases}$$

A topological characterization :

(T1) If $k = \mathbb{C}$, the affine line $\mathbb{A}^1_{\mathbb{C}}$ is the only acyclic normal curve.

# $n = 2$ and dim $B = 2$

A topological characterization (**C. P. Ramanujam (1971)** [10]) :
$X$ is a smooth affine algebraic surface over $\mathbb{C}$. Then

$$(T2)\quad X \cong \mathbb{C}^2 \Leftrightarrow \begin{cases} X \ \text{ is topologically contractible} \\ \pi_1^{\infty} = (1) \end{cases}$$

An Algebraic Characterization (**Miyanishi (1975)** [7]) :

$$(A2)\quad B = k^{[2]} \Leftrightarrow \begin{cases} B \ \text{ is a UFD} \\ B^* = k^* \\ \textit{Spec } B \ \text{ contains a cylinder-like open set} \\ U \times \mathbb{A}^1, \text{i.e., there exists a subalgebra} \\ A \text{ of B and } f \in A \text{ such that} \\ B_f = A_f{}^{[1]}. \end{cases}$$

# A major application

## Zariski Cancellation Problem (ZCP) for $n \leq 2$

**Question (ZCP):** Does $B^{[1]} = k^{[n+1]}$ imply $B = k^{[n]}$?

- $(A1)$ immediately solves ZCP for $n = 1$.
- $(A2)$ was a key tool in solving ZCP for $n = 2$ (Miyanishi, Sugie (1980) [8] and Fujita (1979) [4]).

**Remark :** For $n \geq 3$, the answer is

- Open, when *char* $k = 0$.
- Negative, when *char* $k > 0$. (Neena Gupta (2014) [9]).

# $n = 3$ and dim $B = 3$

An algebro-topological characterization (**Kaliman (2002)**[5])

Let $X = Spec\ B$ be an affine algebraic variety of dim 3 over $\mathbb{C}$. Then

$$(AT3)\ X \cong \mathbb{C}^3 \Leftrightarrow \begin{cases} B^* = \mathbb{C}^* \\ B \text{ is a UFD} \\ H_3(X;\mathbb{Z}) = (0) \\ X \text{ contains a cylinder-like open set } U \times \mathbb{A}^2 \\ \quad \text{such that each irreducible} \\ \quad \text{component of the complement} \\ X \setminus (U \times \mathbb{A}^2) \text{ has at most isolated} \\ \text{singularities.} \end{cases}$$

# Locally Nilpotent Derivations

Definitions and properties :

- A $k$-linear map $D : B \to B$ is called a derivation of $B$ if

$$D(ab) = aD(b) + bD(a) \text{ for all } a, b \in B.$$

  $D$ is said to be *locally nilpotent* if for each $b \in B$, there exists $n \in \mathbb{N}$ such that $D^n(b) = 0$.
  $LND(B) :=$ set of all locally nilpotent derivations on $B$.
- $Ker\ D := \{a \in B \mid D(a) = 0\}$
  is the *kernel* of the locally nilpotent derivation $D$.
- $Ker\ D$ is always *factorially closed* (hence algebraically closed) in $B$ and $\operatorname{tr.deg}_{Ker\ D} B = 1$.
- If $B$ is affine, is $Ker\ D$ always affine ?
  For $B = k^{[n]}$, yes if $n \leqslant 3$, no if $n \geq 5$ and open for $n = 4$ unless $Ker\ D$ contains a coordinate.

# Locally Nilpotent Derivations

Definitions and Properties(contd) :

- There is a well-known one to one correspondence [15, Section 1.5]

$$LND(B) \longleftrightarrow \{\text{regular } \mathbb{G}_a \text{ } actions \text{ on } X = Spec\ B\}$$

- An affine $k$-domain $B$ is defined to be rigid if it does not have any non-zero locally nilpotent derivation.
- $B$ is defined to be almost rigid if there exists a non-zero locally nilpotent derivation $D$ on $B$ such that $LND(B) = \{fD \mid f \in Ker\ D\}$.
- The *Makar-Limanov invariant* of $B$, denoted by $ML(B)$, is defined to be

$$ML(B) := \bigcap_{D \in LND(B)} Ker\ D.$$

$ML(k^{[n]}) = k$ and for a rigid ring $B$, $ML(B) = B$.

# Locally Nilpotent Derivations with slices

Definitions and properties :

- Any $D \in LND(B)$ is said to have a slice is there exists $s \in B$ such that $Ds = 1$. It means that the corresponding $\mathbb{G}_a$ *action* is equivariantly trivial.
- Slice Theorem : [15, Corollary 1.26]
  Let $D \in LND(B)$ has a slice $s$ and $A := Ker\ D$. Then
  - i) $B = A[s] = A^{[1]}$ and $D = \frac{d}{ds}$.
  - ii) $A = \pi_s(B)$ and $Ker\ \pi_s = sB$, where the Dixmier map $\pi_s : B \to A$ is defined as

$$\pi_s(b) = \sum_{i \geqslant 0} \frac{(-1)^i}{i!}(D^i b)s^i.$$

  - iii) If $B$ is affine, then $Y$ is affine.

# $ML^*$

Consider the subset $LND^*(B) \subseteq LND(B)$ defined by

$$LND^*(B) = \{D \in LND(B) \mid Ds = 1 \text{ for some } s \in B\}.$$

Then we define

$$ML^*(B) := \bigcap_{D \in LND^*(B)} Ker\ D.$$

If $LND^*(B) = \emptyset$, we define $ML^*(B)$ to be $B$.

Remarks :

- $ML^*(k^{[n]}) = ML(k^{[n]}) = k$.
- $ML(B) \subseteq ML^*(B)$ (can be a proper subset!).
- $ML^*(B^{[1]}) \subseteq ML(B)$ (**Dasgupta, Gupta** [2, Lemma 3.1])

# dim $B = 1$

A characterization of $k^{[1]}$ in terms of the Makar-Limanov invariant

**Theorem** :(**Crachiola, Makar-Limanov (2005)**[1]) : Let $B$ be an affine $k$-domain such that dim $B = 1$. Then

$$B = k^{[1]} \Leftrightarrow ML(B) = k.$$

**Remark :** When dim $B = 2$, $ML(B) = k \nRightarrow B = k^{[2]}$ (see Danielewski Surfaces).

# Danielewski Surfaces

Let

$$B := \frac{k[X, Y, Z]}{(X^n Z - p(Y))}$$

where $n \in \mathbb{N}$ and $p(Y) \in k[Y]$. Let $x$ be the image of $X$ in $B$.

(i) If $n = 1$ or if $\deg p(Y) = 1$, then $ML(B) = k$.

(ii) If $n \geq 2$ and $\deg p(Y) \geq 2$, then $ML(B) = k[x]$.
Moreover,
$Ker\ D = k[x]$ for every non-zero $D \in LND(B)$.

# A Characterization of $k^{[2]}$

**Theorem 1** : (**Dasgupta, Gupta** [2, Theorem 3.6])
Let $B$ be an affine $k$-domain such that dim $B = 2$. Then the following are equivalent :

(i) $B = k^{[2]}$

(ii) $ML^*(B) = k$

(iii) $ML(B) = k$ and $ML^*(B) \neq B$

Thus

$$(A2)' \quad B = k^{[2]} \Leftrightarrow ML^*(B) = k.$$

**Remark :**

(i) In Theorem 1, the hypothesis that $k$ is algebraically closed can be dropped.

(ii) Theorem 1 does not hold when dim $B = 3$ ( see Example 1).

# Example 1

Let $k$ be a field of characteristic zero, $R := \frac{k[X,Y,Z]}{(XY-Z^2-1)}$ and $B := R[T]$. Then the following hold:

(i) If $k$ is an algebraically closed field, then $B$ is not a UFD.

(ii) If $k = \mathbb{R}$, then $B$ is a UFD.

(iii) $ML(B) = ML^*(B) = k$ and $ML^*(R) = R$.

(iv) $B \neq k^{[3]}$.

# A Characterization of $k^{[3]}$

We begin with some interesting results. Let $B$ be an affine $k$-domain such that $B$ is a UFD and dim $B = 3$.

**Lemma 1** : (**Dasgupta, Gupta**[2, Lemma 4.2])
If $B$ admits two non-zero distinct locally nilpotent derivations with slices, then there exists a $k$-subalgebra $R$ of $B$, such that $R$ is a UFD and $B = R^{[2]}$.

**Remark** : Example 6 shows that such a result does not extend to a four-dimensional affine UFD.

# A Characterization of $k^{[3]}$ (contd)

**Proposition 1** :(**Dasgupta, Gupta**[2, Proposition 4.3])
Let $ML(B) = ML^*(B)$. Then the following hold:

(i) If $\operatorname{tr.deg}_k ML(B) = 3$, then $B$ is rigid.

(ii) If $\operatorname{tr.deg}_k ML(B) = 2$, then there exists an affine $k$-subalgebra $C$ of $B$ such that $C$ is rigid, $C$ is a UFD and $B = C^{[1]}$.

(iii) If $\operatorname{tr.deg}_k ML(B) = 1$, then there exists a $k$-subalgebra $S$ of $B$ such that $S$ is rigid, $S$ is a UFD and $B = S^{[2]}$. Moreover, there exists $t \in B$, such that $S = k[t, \frac{1}{p(t)}]$, where $k[t] = k^{[1]}$ and $p(t) \in k[t] \setminus k$. In particular, $k^* \subsetneqq B^*$.

(iv) If $\operatorname{tr.deg}_k ML(B) = 0$, then $B = k^{[3]}$.

# A Characterization of $k^{[3]}$ (contd)

**Lemma 2** :(**Dasgupta, Gupta** [2, Lemma 4.4])
If $ML^*(B) \neq B$, then $ML(B) = ML^*(B)$.

**Theorem 2** :(**Dasgupta, Gupta** [2, Theorem 4.5])
The following are equivalent:

(i) $B = k^{[3]}$

(ii) $ML^*(B) = k$

(iii) $ML(B) = k$ and $ML^*(B) \neq B$

Thus

$$(A3) \quad B = k^{[3]} \Leftrightarrow ML^*(B) = k.$$

# A Characterization of $k^{[3]}$ (contd)

**Remark :**

(i) In Lemma 2, the hypothesis that $ML^*(B) \neq B$ is necessary. For a three-dimensional affine UFD $B$, containing an algebraically closed field of characteristic zero, it may happen that $ML^*(B) = B$ but $\operatorname{tr.deg}_k ML(B)$ is zero (Example 2), one (Example 3), or two (Example 4) i.e. $ML^*(B) \neq ML(B)$.

(ii) Theorem 2 does not extend to a four-dimensional affine regular UFD, i.e., a four-dimensional affine UFD $\widetilde{B}$ need not be $k^{[4]}$, even when $ML(\widetilde{B}) = ML^*(\widetilde{B}) = k$ (Example 5).

# Example 2 ($ML^*(B) = B$ but $ML(B) = k$)

Let

$$B := \frac{k[X, Y, Z, T]}{(XY - ZT - 1)}.$$

Let the images of $X$, $Y$, $Z$ and $T$ in $B$ be denoted by $x$, $y$, $z$ and $t$ respectively.

- $B$ is a UFD (by Nagata's criterion). Moreover $B$ is regular.
- $ML(B) = k$, i.e. $\operatorname{tr.deg}_k ML(B) = 0$.
  Consider four non-zero locally nilpotent derivations $D_1$, $D_2$, $D_3$ and $D_4$ on $B$ given by
  (i) $D_1x = 0, \quad D_1y = z, \quad D_1z = 0, \quad D_1t = x.$
  (ii) $D_2x = 0, \quad D_2y = t, \quad D_2z = x, \quad D_2t = 0.$
  (iii) $D_3x = z, \quad D_3y = 0, \quad D_3z = 0, \quad D_3t = y.$
  (iv) $D_4x = t, \quad D_4y = 0, \quad D_4z = y, \quad D_4t = 0.$

# Example 2 (contd)

Let $Ker\ D_i = A_i$ for each $i = 1, 2, 3, 4$. Now $k[x, z] \subseteq A_1 \subseteq B$. It follows that $A_1 = k[x, z]$ (since both $A_1$ and $k[x, z]$ are algebraically closed in $B$ and have the same transcendence degree over $k$). Similarly $A_2 = k[x, t]$, $A_3 = k[y, z]$, $A_4 = k[y, t]$ and $\bigcap_i A_i = k$. Thus $ML(B) = k$. Since the Whitehead group $K_1(B) \neq k^*$, $B \neq k^{[3]}$. Hence $ML^*(B) = B$ (by Theorem 2).

# Example 3 ($ML^*(B) = B$ but $\operatorname{tr.deg}_k ML(B) = 1$)

**Russell-Koras threefold :**

Let

$$B := \frac{\mathbb{C}[X, Y, Z, T]}{(X + X^2Y + Z^2 + T^3)}.$$

Let $x$ denote the image of $X$ in $B$.

- $B$ is a UFD (by Nagata's criterion).
- $ML(B) = \mathbb{C}[x] = \mathbb{C}^{[1]}$ (L.G. Makar-Limanov (1996) [2, Lemma 8]). In particular, $\operatorname{tr.deg}_k ML(B) = 1$.
- $ML(B) \neq ML^*(B)$ (by Proposition 1(iii) and since $B^* = \mathbb{C}^*$).
- $ML^*(B) = B$ (by Lemma 1).

# Example 4 ($ML^*(B) = B$ but $\operatorname{tr.deg}_k ML(B) = 2$)

Let

$$R := \frac{\mathbb{C}[X, Y, Z]}{(X^2 + Y^3 + Z^7)} \quad \text{and} \quad B := \frac{R[U, V]}{(X^2U - Y^3V - 1)}.$$

- $B$ is an almost rigid UFD of dim 3.
- $ML(B) = R$ (D.R. Finston and S. Maubach (2008) [3, Theorem 2]). In particular, $\operatorname{tr.deg}_k ML(B) = 2$.
- $ML^*(B) = B$ (since $B \neq R^{[1]}$).

# Example 5 (Theorem 2 does not extend to *dim* 4)

Let

$$B := \frac{k[X, Y, Z, T]}{(XY - ZT - 1)} \text{ and } \widetilde{B} := B[u] = B^{[1]}.$$

- $\widetilde{B}$ is a regular UFD of dim 4.
- $ML^*(\widetilde{B}) = ML(\widetilde{B}) = k$.
- $\widetilde{B} \neq k^{[4]}$.

**Proof :** For each $i = 1, 2, 3, 4$, we extend the locally nilpotent derivation $D_i$ of $B$ to a locally nilpotent derivation $\widetilde{D_i}$ of $\widetilde{B}$, by defining $\widetilde{D_i}u = 1$. Let

$$\widetilde{D_5} = \frac{\partial}{\partial u} \text{ and } Ker\ \widetilde{D_i} = \widetilde{A_i}.$$

# Example 5 (contd)

**Proof (contd) :** By the Slice Theorem, we have

$$\widetilde{A_1} = k[x, z, y - zu, t - xu],$$

$$\widetilde{A_2} = k[x, t, z - xu, y - tu],$$

$$\widetilde{A_3} = k[y, z, x - zu, t - yu],$$

$$\widetilde{A_4} = k[y, t, x - tu, z - yu] \text{ and}$$

$$\widetilde{A_5} = k[x, y, z, t].$$

Clearly $k[x, z + t - xu] \subseteq \widetilde{A_1} \cap \widetilde{A_2}$. Since $k[x, z + t - xu]$ and $\widetilde{A_1} \cap \widetilde{A_2}$ are algebraically closed in $B[u]$ and they have the same transcendence degree over $k$, we have

$$\widetilde{A_1} \cap \widetilde{A_2} = k[x, z + t - xu].$$

# Example 5 (contd)

**Proof (contd) :** Similarly,

$$\widetilde{A_3} \cap \widetilde{A_4} = k[y, z + t - yu].$$

Again,

$$\widetilde{A_5} \cap k[x, z + t - xu] = k[x] \text{ and } \widetilde{A_5} \cap k[y, z + t - yu] = k[y].$$

Hence $\bigcap_i \widetilde{A}_i = k$. Thus $ML^*(\widetilde{B}) = ML(\widetilde{B}) = k$. But $\widetilde{B} \neq k^{[4]}$ (for instance, $K_1(\widetilde{B}) = K_1(B) \neq k^*$).

# Example 6 (Lemma 1 does not hold in *dim* 4)

Let $k$ be an algebraically closed field of characteristic zero and $R = k[X, Y, Z]/(X^2 + Y^3 + Z^7) = k[x, y, z]$, where $x$, $y$ and $z$ denote the images of $X$, $Y$ and $Z$ in $R$. Let $C = R[U, V]/(xU - yV - 1) = R[u, v]$, where $u$ and $v$ denote the images of $U$ and $V$ in $C$ and $B = C[T] = C^{[1]}$. Then the following hold.

(i) $B$ is a UFD of dimension 4.

(ii) $ML(B) = ML^*(B) = R$.

(iii) $B \neq R^{[2]}$.

(iv) $B \neq S^{[2]}$ for any $k$-subalgebra $S$ of $B$.

# Example 6 (contd)

**Proof :** (i) By Nagata's criterion, $R$ and $C$ are UFDs. Hence $B$ is a UFD. Clearly dim $B = 4$.
(ii) By [3, Lemma 2], $R \subseteq ML(B) \subseteq ML^*(B)$. Consider the $R$-linear derivations $\delta_1$ and $\delta_2$ on $B$ as follows:

$$\delta_1(u) = y, \quad \delta_1(v) = x \quad \text{and} \quad \delta_1(T) = 1$$

and

$$\delta_2(u) = yT, \quad \delta_2(v) = xT \quad \text{and} \quad \delta_1(T) = 1.$$

Clearly they are locally nilpotent derivations with slices $T$. By Slice Theorem, $A_1 := Ker\ \delta_1 = R[u - yT, v - xT]$ and $A_2 := Ker\ \delta_2 = R[2u - yT^2, 2v - xT^2]$. Then $A_{1x} = R_x[v - xT]$ and $A_{2x} = R_x[2v - xT^2]$ and the two rings $A_1$ and $A_2$ are clearly different. Therefore $A_1 \cap A_2 \subsetneqq A_2$. As $A_1 \cap A_2$ is an inert subring of $B$ containing $R$, we have $A_1 \cap A_2 = R$ by comparing the dimensions.

# Example 6 (contd)

**Proof (contd) :** (iii) Since $(x, y)B = B$, it follows that $B \neq R^{[2]}$.
(iv) Suppose there exists a $k$-subalgebra $S$ of $B$ such that $B = S^{[2]}$. Then $R = ML(B) \subseteq S$. Since $\operatorname{tr.deg}_k R = \operatorname{tr.deg}_k S$ and both $R$ and $S$ are algebraically closed in $B$, it follows that $R = S$, contradicting (iii). Hence the result.

# A Question

Given an $n$-dimensional affine domain $B$, $ML(B) = ML^*(B)$ holds when

- $\operatorname{tr.deg}_k ML^*(B) = n - 1$ for $n \geq 2$ [2, Lemma 3.2]
- $dim\ B = 2$ and $ML^*B \neq B$ [2, Lemma 3.4]
- $dim\ B = 3$ and $ML^*B \neq B$ and $B$ is a UFD [Lemma 2]

**Question** : Does there exist a three-dimesnional affine $k$-domain such that $ML(B) = k$ but $\operatorname{tr.deg}_k ML^*B = 1$ ?

# Russell-Koras threefold

The well-known Russell-Koras threefold (denoted by $B$) is defined as

$$B := \frac{\mathbb{C}[X, Y, Z, T]}{(X + X^2Y + Z^2 + T^3)}$$

L.G. Makar-Limanov showed that $B$, which was a candidate for a counterexample to the Linearization Problem, is not isomorphic to $\mathbb{C}^{[3]}$, by proving $ML(B) \neq \mathbb{C}$.([6]).

An important problem in Affine Algebraic Geometry asks whether

$$B^{[1]} = \mathbb{C}^{[4]}$$

. An affirmative answer will give a negative solution to the Zariski Cancellation Problem for the three-space in characteristic zero.

# Russell-Koras threefold (cont'd)

**Some results and remarks:**

- A. Dubouloz showed that $ML(B^{[1]}) = \mathbb{C}$(2009, [11]).
- More recently, A. Dubouloz and J. Fasel have shown that $X = Spec(B)$ is $\mathbb{A}^1$-contractible (2018, [12]).
- It follows from our result (Lemma 3.1.1, [2]) that $ML^*(B^{[2]}) \subseteq ML(B^{[1]}) = \mathbb{C}$. As a consequence, we have $ML^*(B^{[n]}) = \mathbb{C}$ for any $n \geqslant 2$.

This leads to the following open question:

**Question :** Is $ML^*(B^{[1]}) = \mathbb{C}$ ?

A negative answer to this problem will confirm that $B^{[1]} \neq \mathbb{C}^{[4]}$.

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