Non-Abelian generalizations of integrable PDEs and ODEs

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Moscow, 24.02.2021

[1] **Based on joint papers with V.Adler**

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Introduction

I will speak of integrable matrix differential equations.
Example. Consider the following matrix ODE system

$$\begin{cases} u_t = v^2 + cu + a\,\mathrm{I}, \\ v_t = u^2 - cv + b\,\mathrm{I} \end{cases} . \qquad (1)$$

Here u and v are $m \times m$-matrices, I is the identity matrix, a, b and c are arbitrary scalar constants. For any m the system has first integrals and infinitesimal symmetries.

In the case $m = 1$, we have a system of two equations which can be written in the Hamiltonian form

$$u_t = -\frac{\partial H}{\partial v}, \qquad v_t = \frac{\partial H}{\partial u}$$

with Hamiltonian

$$H = \frac{1}{3}u^3 - \frac{1}{3}v^3 - cuv + bu - av.$$

For generic a, b, c, the relation $H = const$ defines an elliptic curve, and the system describes the motion of a point along this curve.

In the matrix case the system is still Hamiltonial with

$$H = \text{trace}\left(\frac{1}{3}u^3 - \frac{1}{3}v^3 - cuv + bu - av\right).$$

Matrix systems admit various nontrivial reductions. For example, if $a = b = c = 0$ in (1) then the functions $z_i = \lambda_i^{1/2}$, where λ_i are eigenvalues of $u - v$, satisfy the integrable system

$$z_i'' = -z_i^5 + \sum_{j \neq i} \left[(z_i - z_j)^{-3} + (z_i + z_j)^{-3}\right], \qquad i = 1, \dots, m.$$

A simple formalization of matrix equations leads to the so-called Non-Abelean equations. In this case u and v are regarded as generators of free associative algebra $\mathbb{C}[u, v]$.

All main concepts related to integrable scalar equations such as higher symmetries, first integrals, conservation laws, Lax representations, Painlevé approach and so on, can be generalized to the matrix case. Notice that in the case $m = 1$ these generalizations coincide with the corresponding scalar notions.

In particular, integrals and conserved densities become traces of matrix polynomials. A local conservation law is given by

$$(\operatorname{trace} \rho)_t = (\operatorname{trace} \sigma)_x$$

or, the same

$$(\rho)_t = (\sigma)_x + \sum [p_i, q_i].$$

Suppose we have a given scalar differential equation $E = 0$ taken together with additional structures which provide its integrability. We call a matrix equation $\bar{E} = 0$ a non-abelinization of the equation $E = 0$ if

1. the equation $\bar{E} = 0$ possessess matrix generalizations of all structures that provide the integrability of the equation $E = 0$.
2. if the size of matrices is equal to one, then the matrix equation and its additional structures from item 1 coincide with the equation $E = 0$ and corresponding scalar structures.

Main observation. If E is a homogeneous differential polynomial then this definition is very efficient for finding non-abelinizations.

Example. The mKdV equation

$$u_t = u_{xxx} + 6u^2 u_x$$

has higher conservation laws. The first and second order densities are:

$$\rho_1 = u_x^2 - u^4, \qquad \rho_2 = u_{xx}^2 - 10u^2 u_x^2 + 2u^6.$$

Suppose that the non-abelinization is also homogeneous differential polynomial. Then it has the form

$$u_t = u_{xxx} + k_1[u, u_{xx}] + k_2 u^2 u_x + k_3 u u_x u + k_4 u_x u^2, \quad k_2 + k_3 + k_4 = 6.$$

Calculations show that this equation has the conserved density $\mathrm{trace}(u_x^2 - u^4)$ iff $k_3 + 2k_2 = 6$.

For the next matrix density we use the anzats

$$\mathrm{trace}\left(u_{xx}^2 + z u^2 u_x^2 - (z+10) u u_x u u_x + 2u^6 + y(u u_x u_{xx} - u u_{xx} u_x)\right).$$

This is a density iff

$$2k_1 = 3y, \quad 2z = -16 - y^2, \quad 4k_2 = 12 + 3y, \quad y^3 + 4y = 0.$$

Solving this system, we arrive at two different non-abelinizations:

- $k_1 = k_3 = 0, \quad k_2 = k_4 = 3;$
- $k_1 = 3i, \quad k_2 = k_4 = 0, \quad k_3 = 6.$

These matrix mKdV equations

$$\mathbf{u}_t = \mathbf{u}_{xxx} + 3\mathbf{u}^2\mathbf{u}_x + 3\mathbf{u}_x\mathbf{u}^2,$$

and

$$\mathbf{u}_t = \mathbf{u}_{xxx} + 3\mathbf{u}\mathbf{u}_{xx} - 3\mathbf{u}_{xx}\mathbf{u} - 6\mathbf{u}\mathbf{u}_x\mathbf{u}$$

are well-known (Khalilov, Khruslov 1990).

Recently the following integrable generalization of the latter equation was found:

$$\mathbf{u}_t = \mathbf{u}_{xxx} + 3[\mathbf{u}, \mathbf{u}_{xx}] - 6\mathbf{u}\mathbf{u}_x\mathbf{u} + (\mathbf{u}_x + \mathbf{u}^2)b + b(\mathbf{u}_x - \mathbf{u}^2), \qquad b \in \mathrm{Mat}_m$$

(Adler, Sokolov 2020).

Euler top

Consider the system of ODEs

$$u' = z_1\, vw, \qquad v' = z_2\, uw, \qquad w' = z_3\, uv, \qquad z_i \in \mathbb{C}, \qquad (2)$$

where $'$ means the derivative with respect to t. System (2) possesses the first integrals

$$I_1 = z_3 u^2 - z_1 w^2, \qquad I_2 = z_3 v^2 - z_2 w^2.$$

Using scalings of dependent and independent variables, one can reduce the parameters z_i to one. Everywhere below, we assume that $z_1 = z_2 = z_3 = 1$.

Our goal is to generalize system (2) to the case when the unknown variables become matrices of arbitrary size $m \times m$.

Assuming that the right hand sides are still homogeneous quadratic polynomials and that the system coincides with (2) for $m = 1$, we arrive at the following ansatz:

$$\begin{cases} u' = k_1 vw + (1 - k_1)wv + c^1_{12}[v, w] + c^1_{23}[w, u] + c^1_{31}[u, v], \\ v' = k_2 wu + (1 - k_2)uw + c^2_{12}[w, u] + c^2_{23}[u, v] + c^2_{31}[v, w] \\ w' = k_3 uv + (1 - k_3)vu + c^3_{12}[u, v] + c^3_{23}[v, w] + c^3_{31}[w, u]. \end{cases}$$

Proposition. If each integral $I_1^i I_2^j$ of system (2) admits a non-abelinization, then the non-abelian system has the form

$$\begin{cases} u' = \frac{1}{2}(vw + wv) + X[u, v] + Z[u, w], \\ v' = \frac{1}{2}(wu + uw) + Y[v, u] + Z[v, w], \\ w' = \frac{1}{2}(uv + vu) + X[w, v] + Y[w, u], \end{cases} \tag{3}$$

where X, Y, Z are arbitrary constant parameters.

For any parameters X, Y, Z system (3) possesses a Lax representation $L_t = [A,\, L]$ in the Lie algebra of matrices over the skew-field of quaternions. The matrices L and A have the form

$$L = \begin{pmatrix} L_1 & L_2 \\ -L_2 & L_1 \end{pmatrix}, \qquad A = \begin{pmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{pmatrix},$$

where

$$L_1 = 2(\nu - \mu)\mathrm{Det}(S)\, u, \qquad L_2 = \langle P, \Omega \rangle\, v + \langle Q, \Omega \rangle\, w,$$

$$A_1 = (-Y\,u - X\,v - Z\,w)\,\mathbf{1} + \sigma L_1, \qquad A_2 = \mu \langle P, \Omega \rangle\, v + \nu\, \langle Q, \Omega \rangle\, w.$$

Here σ, μ, ν are arbitrary pairwise distinct parameters, $\Omega = (\mathbf{i}, \mathbf{j}, \mathbf{k})$, P and Q are 3–dimension vectors such that $\langle P, Q \rangle = 0$,

$$\langle P, P \rangle = \frac{1}{4(\sigma - \mu)(\nu - \mu)}, \qquad \langle Q, Q \rangle = \frac{1}{4(\sigma - \nu)(\mu - \nu)},$$

and S is the matrix with rows P, Q, Ω. The Lax pair depends on one essential parameter $\kappa = \dfrac{(\sigma - \mu)(\nu - \mu)}{(\sigma - \nu)(\mu - \nu)}$.

Classification problems

1. Matrix systems of NLS type.

Theorem (Mikhailov, Shabat, Yamilov.) i) If a system of the form

$$\begin{cases} u_t = u_{xx} + A_1(u,v)\,u_x + A_2(u,v)\,v_x + A_0(u,v), \\ v_t = -v_{xx} + B_1(u,v)\,v_x + B_2(u,v)\,u_x + B_0(u,v) \end{cases} \tag{4}$$

admits an infinite sequence of conservation laws then it is polynomial.
ii) A homogeneous polynomial non-triangular system (4) admits higher conservation laws if and only if it belongs to one of the following lists, up to the scaling of the variables x, t, u, v and the interchange $(u, v) \mapsto (v, u)$:

1. NLS type systems

$$\begin{cases} u_t = u_{xx} + 2uvu, \\ v_t = -v_{xx} - 2vuv, \end{cases} \qquad S_1$$

$$\begin{cases} u_t = u_{xx} + 2uu_x + 2vu_x + 2uv_x, \\ v_t = -v_{xx} + 2vv_x + 2vu_x + 2uv_x, \end{cases} \qquad S_2$$

$$\begin{cases} u_t = u_{xx} + 2u_xv + 2uv_x, \\ v_t = -v_{xx} + 2vv_x + 2u_x, \end{cases} \qquad S_3$$

$$\begin{cases} u_t = u_{xx} + 2(u+v)u_x, \\ v_t = -v_{xx} + 2(u+v)v_x, \end{cases} \qquad S_4$$

$$\begin{cases} u_t = u_{xx} + 2\alpha u^2v_x + 2\beta uvu_x + \alpha(\beta - 2\alpha)u^3v^2, \\ v_t = -v_{xx} + 2\alpha v^2u_x + 2\beta uvv_x - \alpha(\beta - 2\alpha)u^2v^3; \end{cases} \qquad S_5$$

2. Boussinesq type systems

$$\begin{cases} u_t = u_{xx} + (u+v)^2, \\ v_t = -v_{xx} - (u+v)^2, \end{cases} \qquad B_1$$

$$\begin{cases} u_t = u_{xx} + 2vv_x, \\ v_t = -v_{xx} + u_x, \end{cases} \qquad B_2$$

$$\begin{cases} u_t = u_{xx} + 6(u+v)v_x - 6(u+v)^3, \\ v_t = -v_{xx} + 6(u+v)u_x + 6(u+v)^3, \end{cases} \qquad B_3$$

$$\begin{cases} u_t = u_{xx} + 2vv_x, \\ v_t = -v_{xx} + 2uu_x. \end{cases} \qquad B_4$$

Our goal is to find all non-commutative generalizations with conservation laws for the systems from the above lists.

We postulate that:

1. a non-commutative generalization is polynomial, homogeneous and admits the scaling group

$$(x,\ t,\ \mathbf{u},\ \mathbf{v}) \longrightarrow (\tau^{-1}x,\ \tau^{-\mu}t,\ \tau^{\nu_1}\mathbf{u}, \tau^{\nu_2}\mathbf{v})$$

   with μ, ν_1 and ν_2 which are the same as for the original system. If the weights of the scalar system contain an arbitrary parameter, like in the case of the NLS system, then we assume that this parameter is preserved also for the non-commutative generalization;

2. the generalization turns into the original system in the case of 1×1 matrices;

3. for any homogeneous conserved density of the original system, there exists a homogeneous conserved density of its non-Abelian analog, which turns into the scalar one if $m = 1$.

Theorem. If a non-Abelian analog of one of the scalar systems S_1–S_5, B_1–B_4 satisfies the conditions **1–3** then it is equivalent to one of the systems listed in the second column of the Table.

		N	weights of u, v
S_1	$\mathbf{S'1}$	1	$(\nu, 1-\nu)$
S_2	$\mathbf{S'2}$, $\mathbf{S''2}$	4	$(1,1)$
S_3	$\mathbf{S'3}$	2	$(2,1)$
S_4	$\mathbf{S'4}$, $\mathbf{S''4}$	4	$(1,1)$
$S_5(\alpha,\beta)$		0	$(\nu, 1-\nu)$
$S_5(1,2)$	$\mathbf{S'512}$, $\mathbf{S''512}$	3	
$S_5(0,1)$	$\mathbf{S'501}$, $\mathbf{S''501}$, $\mathbf{S'''501}$	6	
$S_5(1,0)$	$\mathbf{S'510}$, $\mathbf{S''510}$	3	
B_1		0	$(2,2)$
B_2	$\mathbf{B'2}$	2	$(3,2)$
B_3	$\mathbf{B'3}$, $\mathbf{B''3}$	4	$(1,1)$
B_4	$\mathbf{B'4}$, $\mathbf{B''4}$	8	$(1,1)$

In order to make sure that the found non-commutative systems are integrable indeed, we find for each system a zero curvature representation

$$U_t = V_x + [V,\, U], \tag{5}$$

where U and V are homogeneous polynomial matrices depending on the spectral parameter λ.

For example, the system S_4 admits two nonequivalent non-commutative analogs:

$$\begin{cases} \mathbf{u}_t = \quad \mathbf{u}_{xx} + 2\mathbf{u}_x(\mathbf{u}+\mathbf{v}), \\ \mathbf{v}_t = -\mathbf{v}_{xx} + 2(\mathbf{u}+\mathbf{v})\mathbf{v}_x, \end{cases}$$

$$\begin{cases} \mathbf{u}_t = \quad \mathbf{u}_{xx} + 2(\mathbf{u}+\mathbf{v})\mathbf{u}_x + 2[\mathbf{v}_x, \mathbf{u}] - 2\mathbf{u}^2\mathbf{v} + 4\mathbf{uvu} - 2\mathbf{vu}^2, \\ \mathbf{v}_t = -\mathbf{v}_{xx} + 2\mathbf{v}_x(\mathbf{u}+\mathbf{v}) + 2[\mathbf{v}, \mathbf{u}_x] + 2\mathbf{uv}^2 - 4\mathbf{vuv} + 2\mathbf{v}^2\mathbf{u}. \end{cases}$$

The zero curvature representation matrices for the first one are of the form

$$U = \begin{pmatrix} \mathbf{u} & (\lambda\mathbf{I} - \mathbf{u})(\lambda\mathbf{I} - \mathbf{v}) \\ \mathbf{I} & \mathbf{v} \end{pmatrix},$$

$$V = U^2 + 2\lambda U - \lambda^2 I + \begin{pmatrix} \mathbf{u}_x & \lambda(\mathbf{v}_x - \mathbf{u}_x) + \mathbf{u}_x\mathbf{v} - \mathbf{u}\mathbf{v}_x \\ 0 & -\mathbf{v}_x \end{pmatrix};$$

the matrices for the second system read

$$U = \begin{pmatrix} -\mathbf{v} & (\lambda\mathbf{I} - \mathbf{u})(\lambda\mathbf{I} - \mathbf{v}) \\ \mathbf{I} & -\mathbf{u} \end{pmatrix},$$

$$V = -U^2 + 2\lambda U + (\lambda^2\mathbf{I} + 2[\mathbf{u}, \mathbf{v}])I + \begin{pmatrix} \mathbf{v}_x & \lambda(\mathbf{v} - \mathbf{u})_x + \mathbf{u}_x\mathbf{v} - \mathbf{u}\mathbf{v}_x \\ 0 & -\mathbf{u}_x \end{pmatrix}.$$

2. Systems of two quadratic ODEs.

Consider the simplest class of matrix ODEs of the form

$$\begin{cases} u_t = \alpha_1 u\, u + \alpha_2 u\, v + \alpha_3 v\, u + \alpha_4 v\, v, \\ v_t = \beta_1 v\, v + \beta_2 v\, u + \beta_3 u\, v + \beta_4 u\, u. \end{cases} \tag{6}$$

Some examples of systems (6) that have higher symmetries were found by A. Mikhailov and VS.

Let us assume that:

1. The system (6) is integrable in a common sense if the size of matrices is equal to 1.

In the case $m = 1$ we have the system of ODEs of the form

$$\begin{cases} u_t = a_1 u^2 + a_2 uv + a_3 v^2, \\ v_t = b_1 v^2 + b_2 uv + b_3 u^2, \end{cases} \tag{7}$$

where

$$a_1 = \alpha_1, \qquad a_2 = \alpha_2 + \alpha_3, \qquad a_3 = \alpha_4,$$

$$b_1 = \beta_1, \qquad b_2 = \beta_2 + \beta_3, \qquad b_3 = \beta_4.$$

In addition to Item **1** we assume that

2. The commutative limit of the symmetry hierarchy of system (6) coincides with the whole symmetry hierarchy of the system (7).

Theorem. Suppose that a system (7) has a polynomial first integral I and satisfies the Kowalevski-Lyapunov test. Then the integral of minimal degree is equivalent to one of the following:

1. $I = u(u - v)v;$
2. $I = u(u - v)^2 v;$
3. $I = u(u - v)^2 v^3;$
4. $I = uv;$
5. $I = u.$

Lemma. Let I be a first integral of a system (7). Then for any N

$$\begin{cases} u_\tau = I^N (a_1 u^2 + a_2 uv + a_3 v^2), \\ v_\tau = I^N (b_1 v^2 + b_2 uv + b_3 u^2) \end{cases}$$

is a symmetry of (7).

Consider Case 1. The system (7) is uniquely defined by the existence of the integral I:

$$\begin{cases} u_t = -u^2 + 2\,uv \\ v_t = -v^2 + 2\,uv. \end{cases}$$

This system has symmetries of degrees $2 + 3N$.

Any its non-commutative generalization has the form

$$\begin{cases} u_t = -u^2 + 2\,uv + \alpha(uv - vu) \\ v_t = -v^2 + 2\,vu + \beta(vu - uv). \end{cases} \tag{8}$$

Theorem 1. In Case 1 there exist the following 5 non-equivalent integrable non-commutative systems (8):

- **1.** $\alpha = -1, \quad \beta = -1;$
- **2.** $\alpha = 0, \quad \beta = -1;$
- **3.** $\alpha = 0, \quad \beta = -2;$
- **4.** $\alpha = 0, \quad \beta = 0;$
- **5.** $\alpha = 0, \quad \beta = -3.$

Similar theorems were proved by T.Wolf and VS for each of the five cases.

Integrable inhomogeneous generalizations.

Consider non-homogeneous matrix generalizations of the form

$$\begin{cases} u_t = \alpha_1 u^2 + \alpha_2 u\, v + \alpha_3 v\, u + \alpha_4 v^2 + \gamma_1 u + \gamma_2 v + \gamma_3\, \mathrm{I}, \\ v_t = \beta_1 v^2 + \beta_2 v\, u + \beta_3 u\, v + \beta_4 u^2 + \gamma_4 u + \gamma_5 v + \gamma_6\, \mathrm{I} \end{cases} \tag{9}$$

of non-commutative homogeneous integrable systems (6) found above. The system (9) coincides with (6) up to linear terms.

Definition. We call two inhomogeneous generalizations *equivalent* if they are related by a shift

$$u \to u + c_1\, \mathrm{I}, \qquad v \to v + c_2\, \mathrm{I},$$

where c_i are constants.

Consider the non-commutative systems from Theorem 1. By a shift we reduce γ_2 and γ_4 in (9) to zero.

Proposition. For each of cases **1 - 5** of Theorem 1 there exists an inhomogeneous generalization iff $\gamma_5 = -\gamma_1$.

The commutative limits of all these generalizations have the same polynomial inhomogeneous cubic integral

$$I = v(v-u)u + \gamma_1 uv + \gamma_3 v - \gamma_6 u.$$

Matrix Painlevé-2 equations

The second of the six famous Painlevé equations reads

$$y'' = 2y^3 + zy + a.$$

It is clear that in the non-commutative case one can change the principal differential-homogeneous part $y'' = 2y^3$ of this equation by adding the term of the same weight $\kappa[y, y']$, where $\kappa \in \mathbb{C}$.

Consider matrix generalizations of the P_2 equation of the general form

$$y'' = \kappa[y, y'] + 2y^3 + zy + b_1 y + y b_2 + a, \tag{10}$$

where a, b_1 and b_2 are matrix constants and κ is a scalar constant.

Theorem. The equation (10) satisfies the matrix Painlevé –Kovalevskaya test only in the following cases:

$$y'' = 2y^3 + zy + by + yb + \alpha I, \qquad \alpha \in \mathbb{C}, \quad b \in \mathrm{Mat}_m,$$

$$y'' = \pm[y, y'] + 2y^3 + zy + a, \qquad a \in \mathrm{Mat}_m,$$

$$y'' = \pm 2[y, y'] + 2y^3 + zy + by + yb + a, \qquad a, b \in \mathrm{Mat}_m, \quad [b, a] = \pm 2b.$$

The Painlevé–Kovalevskaya matrix test for equation (10) is based on counting of arbitrary scalar constants in a formal solution of the form

$$y = \frac{p}{z - z_0} + c_0 + c_1(z - z_0) + \cdots, \qquad p, c_j \in \mathrm{Mat}_m, \quad z_0 \in \mathbb{C}. \tag{11}$$

Here z_0 is one of these arbitrary constants. In order for the series y to represent a generic solution, it is necessary that the matrices p and c_j contain additionally $2m^2 - 1$ arbitrary constants. We assume that the Painlevé–Kovalevskaya test is fulfilled if such matrices exist.

Substituting the series into the equation and collecting the coefficients at powers of $z - z_0$, we obtain relations of the form

$$p^3 = p, \tag{12}$$

$$L_{\frac{j+1}{2}\kappa}(c_j) - \frac{j(j-1)}{2} c_j = f_j(z_0, p, c_0, \ldots, c_{j-1}), \quad j \geqslant 0, \tag{13}$$

where

$$L_\sigma(c) \stackrel{def}{=} p^2 c + pcp + cp^2 + \sigma(pc - cp).$$