Pair correlation of real sequences: metric results

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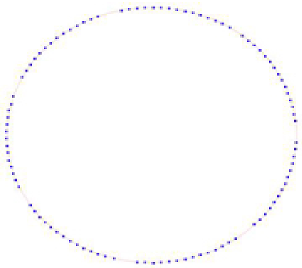
Uniform distribution modulo 1

A sequence of real numbers $(x_n)_{n\geq 1}$ is *uniformly distributed modulo 1* (or shortly u. d.) if

$$\frac{\#\{n \leq N,\ \mathbf{1}_A(x_n) = 1\}}{N} \underset{N\to+\infty}{\longrightarrow} \text{meas(A)}$$

for every interval $A \subset [0,1)$ where $\mathbf{1}_A$ is the indicator function of A, extended periodically with period 1.



Distribution of points $\{n\pi\}, n = 1..100$

A well-known criterion and examples

Theorem (Weyl's criterion)

A sequence is u.d modulo 1 *if and only if*

$$\frac{1}{N}\sum_{n=1}^{N} e^{2i\pi h x_n} \underset{N\to+\infty}{\longrightarrow} 0$$

for all $h \in \mathbb{Z}\backslash\{0\}$.

- Kronecker sequences $x_n = \{n\alpha\}$ for α irrational are u. d.
- Generalization $x_n = \{n^k\alpha\}$ for any $k \geq 2$ and α irrational.
- The sequence $x_n = \{n^\theta\}$ for $\theta > 0$ non-integer is u. d.
- Metric result (Weyl): $x_n = \{a_n\alpha\}$ with $(a_n)_{n\geq 1}$ integers is u. d for almost all α (L^2 argument with Weyl sums).

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Gap distribution

Given a sequence of N elements $x_1, \ldots, x_N$ in $[0,1)$, reorder them modulo 1 in the following way

$$0 \leq \theta_{1,N} \leq \theta_{2,N} \leq \cdots \leq \theta_{N,N} < 1.$$

The gap distribution is then defined for any interval $I \subset [0, +\infty)$ by

$$P_N(I; x_n) := \frac{\#\{j \leq N, \theta_{j,N} - \theta_{j-1,N} \in N^{-1} I\}}{N}.$$

We say that the gap distribution of $\{x_n\}_{n \geq 1}$ is Poissonian if for any I we have

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A more local statistic: pair correlation

A sequence $(x_n)_{n\geq 1}$ is said to have Poissonian pair correlation if

$$R_2([-s,s],N) := \frac{1}{N} \sum_{\substack{1\leq m,n\leq N,\\ m\neq n}} \mathbf{1}_{[-s/N,s/N]}(x_n - x_m) \underset{N\to+\infty}{\longrightarrow} 2s$$

for all real numbers $s \geq 0$.

For a "random" sequence, any interval $(x - s/N, x + s/N)$ should contain $2s$ points on average.

If $X_n : [0,1) \to [0,1)$ are uniformly distributed, independent random variables, then $X_n(\alpha)$ has almost surely Poissonian pair correlation (or shortly PPC).

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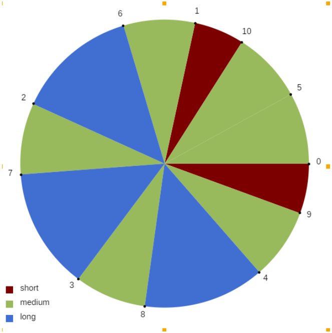
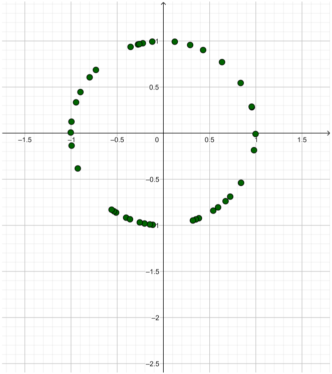
“Poissonian” versus “non Poissonian”





Kronecker sequences $\{n\alpha\}$ have at most 3 gaps

Polynomial sequence $\{n^6\sqrt{3}\}$

The beginning of the story

Conjecture (Rudnick-Sarnak-Zaharescu): For any Diophantine α and any integer $k \geq 2$, the sequence $\{\alpha n^k\}$ has Poissonian pair correlations.

- "Diophantine" = not too well approximated by rationals, true for almost all α.
- Under some extra conditions on α, they prove PPC along subsequences.
- Further partial results by Heath-Brown and Truelsen: link with questions on average of divisor functions in arithmetic progressions.

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A first metric result

Theorem (Rudnick-Sarnak)

For $k \geq 2$, the sequence of fractional parts $\{\alpha n^k\}$ has Poissonian pair correlation for almost every α.

- Not a single explicit α known.
- For $d = 2$, this implies a result for the energy levels of the "boxed oscillator", a special case of Berry and Tabor conjecture about integrable systems.
- The proof establishes a convergence in L^2 of the pair correlation function R_2 and bounds on Weyl's exponential sums $\sum_{n \leq N} e(\alpha h n^k)$.

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Towards a general criterion

Rudnick/Zaharescu criterion: Let $(u_n)_{n\geq 1}$ be a sequence of integers such that there are at most $O(MN^{2+\varepsilon})$ solutions to the equation

$$j_1(u_{n_1} - u_{m_1}) = j_2(u_{n_2} - u_{m_2})$$

with $1 \leq n_i \neq m_i \leq N$ and $1 \leq |j_i| \leq M$, $M \ll N^k$ for some $k > 0$.
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A more ready-to-use criterion

For any set S of real numbers, the additive energy of S is defined by

$$E(S) = \sum_{a+b=c+d} 1$$

where the sum is over quadruples $(a, b, c, d) \in S^4$. For a sequence $X = (x_n)_{n\geq 1}$, we denote by X_N the first N elements of X.

Theorem (Aistleitner-Larcher-Lewko)

Assume that $(x_n)_{n\geq 1}$ is a sequence of integers and $E(X_N) \ll N^{3-\varepsilon}$ for some $\varepsilon > 0$. Then for almost all α, the sequence $\{\alpha x_n\}$ has PPC.

Remarks: Apply to integer polynomial sequences, lacunary sequences or convex sequences ($E(X_N) \ll N^{5/2}$).

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Some comments

- The proof relies on estimates by Bondarenko and Seip for GCD sums

$$\sum_{k,l=1}^{M} c_k c_l \frac{gcd(n_k, n_l)}{\sqrt{n_k n_l}}$$

where n_i are any distinct integers and $(c_i)_{i=1..M}$ is such that $||c||_2 \leq 1$.

- Using specific structure of the coefficients c_i, the energy estimate has been refined by Bloom and Walker.
- A converse theorem is true. Large energy $\Longrightarrow$ PPC is not true almost everywhere.

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What about real sequences?

Very little is known when $(x_n)_n$ is a sequence of reals.

Rudnick-Technau's criterion: Let $(x_n)_{n\geq 1}$ be a sequence of reals such that there are at most $O(N^{4-\delta})$ solutions to the inequality

$$|j_1(x_{n_1} - x_{m_1}) - j_2(x_{n_2} - x_{m_2})| < N^{\varepsilon}$$

with $1 \leq n_i \neq m_i \leq N$ and $1 \leq |j_i| \leq N^{1+\varepsilon}$.

Then the sequence $\{\alpha x_n\}$ has PPC for almost all α.

- Diophantine inequality appears in the variance computation while before only equality mattered for integers.
- Apply to any real lacunary sequence.
- Not easy to verify in general, can we do better?

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An additive energy adapted for the real case

Let E_N^* denote the number of solutions (n_1, n_2, n_3, n_4) of the inequality

$$|x_{n_1} - x_{n_2} + x_{n_3} - x_{n_4}| < 1,$$

subject to $n_i \leq N, \; i = 1, 2, 3, 4.$

Question: Can we prove that $E_N^* \ll N^{3-\varepsilon}$ implies that $\{\alpha x_n\}$ has PPC for almost all α?

Answer: Not completely but we tried our best!

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An easier criterion as in the case of real sequences?

Theorem (Aistleitner, El-Baz, Munsch (2020))

Assume that there exists some $\delta > 0$ such that $E_N^* \ll N^{220/91-\delta}$ as $N \to \infty$. Then the sequence $(x_n\alpha)_{n\geq 1}$ has Poissonian pair correlation for almost all $\alpha \in \mathbb{R}$.

- The exponent $220/91 \approx 2.417$ comes from the order of magnitude of the Riemann zeta function $\zeta(1/2 + it)$ for large t as well as estimates on the moments of ζ.
- Under the Lindelöf hypothesis, our bound for E_N^* can be relaxed to $E_N^* \ll N^{3-\varepsilon}$.

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An easier criterion as in the case of real sequences?

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Consequences and further comments

- The Diophantine inequality involves only terms of the sequence $\{x_n\}_{n\geq 1}$ and get rid of the coefficients in Rudnick-Technau's result.
- Recover the result of Rudnick-Technau about lacunary sequences.
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A more precise result for the sequence $x_n = n^\theta$ with $\theta > 1$

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Let $\theta > 1$ be a real number. Then the sequence $(n^\theta \alpha)_{n \geq 1}$ has Poissonian pair correlation for almost all $\alpha \in \mathbb{R}$.

- In another direction Technau and Yesha proved that n^α has PPC for almost all $\alpha > 7$.
- If the sequence grows linearly or slower, the variance cannot converge to 0, similarly the energy is too large and the result is in this sense optimal.
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Much finer inequalities and a result of Robert and Sargos

Theorem (Robert-Sargos)

Let $E(M,\gamma)$ denote the number of 4-tuples $(n_1, n_2, n_3, n_4) \in \{M+1, M+2, \dots, 2M\}^4$ for which

$$\left| n_1^\theta - n_2^\theta + n_3^\theta - n_4^\theta \right| \le \gamma.$$

Then for every $\varepsilon > 0$,

$$E(M,\gamma) \ll M^{2+\varepsilon} + \gamma M^{4-\theta+\varepsilon}.$$

This uniformity in γ can be incorporated in our argument to prove our result for $\theta > 1$ whereas our previous requirement on the energy would only allow to prove it for $\theta > 1.59$.

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General strategy of the proof

As before the pair correlation function is

$$R_2([-s,s],\alpha,N) := \frac{1}{N} \sum_{\substack{1\le m,n\le N,\\ m\neq n}} \mathbf{1}_{[-s/N,s/N]}(x_n\alpha - x_m\alpha).$$

We study the expectation and the variance of R_2 and show

$$\mathbb{E}(R_2) := \int_0^1 R_2([-s,s],\alpha,N)d\alpha \sim 2s$$

and

$$Var(R_2) := \int_0^1 (R_2([-s,s],\alpha,N) - 2s)^2 d\alpha \to 0.$$

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From variance to counting problem

More or less bounding the variance amounts to bound

$$\sum_{\substack{1\le n_1,n_2,n_3,n_4\le N,\\ n_1>n_2,\ n_3>n_4}} \sum_{1\le j_1,j_2\le N} \mathbf{1}\left(|j_1(x_{n_1}-x_{n_2})-j_2(x_{n_3}-x_{n_4})|<1\right).$$

Introduce the set of differences $z_M = x_n - x_m$ and the number of representations $r(z) := \sum_{\substack{n\neq m\\ x_n-x_m=z}} 1.$

Integer case: $|j_1z_1 - j_2z_2| < 1$ is the same as $j_1z_1 - j_2z_2 = 0$.
This boils down to bound

$$\sum_{z_1,z_2} r(z_1)r(z_2)\frac{\gcd(z_2,z_1)}{\sqrt{z_1z_2}}.$$

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A way to detect the inequality

We can assume $2^{u-1} \leq j_1, j_2 \leq 2^u$. This forces z_1 and z_2 to be of comparable size. Say we want to count solutions when $z_i \geq N$, then we have

$$\left|\frac{j_1 z_1}{j_2 z_2} - 1\right| < \frac{1}{2^u N}.$$

Set $T = 2^u N^{1-\varepsilon}$. Defining $\Phi(t) = e^{-t^2}$, the solutions can be bounded by

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What it has to do with the Riemann zeta function?

We define a "Dirichlet polynomial" $P(t)$ supported on z_n such that

$$\int_{\mathbb{R}} |P(t)|^2 \Phi(t/T) dt \quad \ll \quad TE_N^*. \tag{1}$$

Recall the approximation
$\zeta(1/2+it) = \sum_{n \le T} n^{-1/2-it} + O(T^{-1/2})$ for $t \approx T$.

Expanding the squares we see that our counting problem reduces to the twisted moment of zeta

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Suitable Dirichlet polynomial and orthogonality

For any integer $k \geq 0$, set

$$b_k = \sum_{m=1}^{M} \mathbf{1}\big(z_m \in [k, k+1)\big). \tag{2}$$

Let $I_h = \left[\left(1+\frac{1}{T}\right)^h, \left(1+\frac{1}{T}\right)^{h+1}\right)$, and set $a_h = \left(\sum_{k \in I_h} b_k^2\right)^{1/2}$.
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Comments on the proof

- One major difficulty is the choice of T.
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Open questions

- Find other interesting examples of applications of our Theorem, meaning sequences such that the $E_N^* \ll N^{2.41}$ ...
- Show that $\{x_n \alpha\}$ has PPC for almost all α under the weaker assumption $E_N^* \ll N^{3-\delta}$ for some $\delta > 0$.
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