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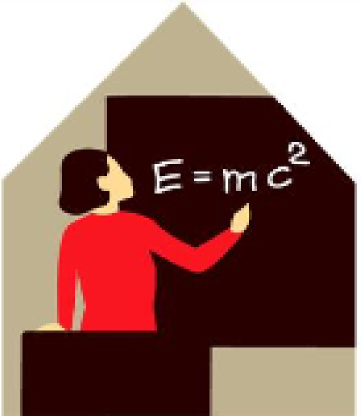

On integrability of some Riemann type hydrodynamical systems and Dubrovin's integrability scheme classification of perturbed Korteweg-de Vries type equations

Devoted to the bright memory of untimely passed away brilliant mathematician Boris Dubrovin (†19 March 2019)

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1. Introduction: Dubrovin's integrability scheme

We will recall from the very beginning some very interesting works by B. Dubrovin with collaborators [1, 2, 3] (Dubrovin B., Liu S.-Q., Zhang Y. On Hamiltonian perturbations of hyperbolic systems of conservation laws I: quasi-triviality of bi-Hamiltonian perturbations. Commun. Pure Appl. Math. 59 (2006) 559–615; Dubrovin B., Zhang Y. Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov–Witten invariants. arxiv: math.DG/0108160.), in which there was posed the following classification problem:

Consider a general evolution equation

$$\begin{aligned} u_t &= f(u)u_x + \varepsilon[f_{21}(u)u_{xx} + f_{22}(u)u_x^2] + \\ &\quad +\varepsilon^2[f_{31}(u)u_{xxx} + f_{32}(u)u_x u_{xx} + f_{33}u_x^3] + \ldots + \\ &\quad +\varepsilon^{N-1}[f_{N,1}(u)u_{Nx} + \ldots], \end{aligned} \tag{1.1}$$

with graded homogeneous polynomials of the jest-variables $\{u_x, u_{xx}, \ldots, u_{kx} \ldots\}$ with $deg\ u_{kx} := k \in \mathbb{N}$, where $f'(u) \neq 0$ for arbitrary $u \in C^\infty(\mathbb{R};\mathbb{R})$, and describe **the set** $\mathcal{F}$ of smooth functions $f_{jk}(u)$, $k = \overline{1,j}$, $j = \overline{1,N}$, with fixed $N \in \mathbb{N}$, the equation (1.1) reduces by means of the following transformation

$$u \to v + \sum_{k \in \mathbb{N}} \varepsilon^k F_k(v, v_x, v_{xx}, \ldots, v^{(m_k)}) \tag{1.2}$$

to the Riemann type equation

$$v_t = f(v)v_x, \tag{1.3}$$

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where numbers $m_k \in \mathbb{Z}_+$, $k \in \mathbb{N}$, are finite and ε is a formal parameter. The transformation (1.2) is often called a quasi-Miura transformation and naturally acts as *an automorphism* ot the ring $\mathcal{A}_\varepsilon := C^\infty(u)\ [u_1, u_2, ..., u_k, ...][[\varepsilon]]$ of formal functional series with respect to the parameter ε. It is worth to mention that this ring is a topological ring $\mathcal{A}_\varepsilon$ with respect to the natural metric, within which adding and multiplication of series is continuous. Moreover, the related group of Miura-type automorphisms, being the semidirect product of the local diffeomorphism group $Diff_{loc}(\mathbb{R})$ of the real axis $\mathbb{R}$ and the quasi-identical authomorphism subgroup of self-mappings

$$u \to u + \sum_{k \in \mathbb{N}} \varepsilon^k F_k(u, u_x, u_{xx}, ..., u^{(m_k)}) \tag{1.4}$$

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$$u \to u + \sum_{k \in \mathbb{N}} \varepsilon^k F_k(u, u_x, u_{xx}, ..., u^{(m_k)}) \tag{1.4}$$

with finite $m_k \in \mathbb{N}, k \in \mathbb{N}$, naturally generates the Lie subalgbra $\mathcal{D}(\mathcal{A}_\varepsilon)$ of the natural derivations of the ring $\mathcal{A}_\varepsilon$, whose representatives exactly coincide with the considered above equations (1.1).

In the Dubrovin's works there was formulated the following *integrability criterium:*

Definition 1.1. The evolution equation (1.1) **is defined to be formally integrable,** iff the corresponding inverse to (1.2) transformation

$$v \to u + \sum_{k\in\mathbb{N}} \varepsilon^k G_k(u, u_x, u_{xx}, \dots, u^{(m_k)}) \in \mathcal{A}_\varepsilon \tag{1.5}$$

with finite orders $m_k \in \mathbb{N}, k \in \mathbb{N}$, being applied to an arbitrary Riemann **type symmetry flow**

$$v_s = h(v)v_x \tag{1.6}$$

with respect to an evolution parameter $s \in \mathbb{R}$, reduces to the form

$$u_s = h(u)u_x + \sum_{k\in\mathbb{N}} \varepsilon^k W_k(u, x, u_{xx}, \dots, u^{(k)}) \in \mathcal{A}_\varepsilon \tag{1.7}$$

with uniform homogeneous differential polynomials $W_k(u, x, u_{xx}, \dots, u^{(k)})$ of the order $k \in \mathbb{N}$.

In their works B. Dubrovin and its collaborators applied this scheme to the equation

(1.7) $$u_t = uu_x + \varepsilon^2[f_{31}(u)u_{xxx} + f_{32}(u)u_xu_{xx} + f_{33}(u)u_x^3]$$

and presented a list of 9 (!) equations [3]:

(1.8)
$$1)\ u_t = uu_x + \varepsilon u_{xxx} \quad (KdV)$$
$$2)\ w_t = w^2w_x + \varepsilon^2 w_{xxx} \quad (w^2 := u)$$
$$3)\ u_t = uu_x + \varepsilon^2(u_{xxx} - \frac{3u_xu_{xx}}{u} + \frac{15u_x^2}{8u^2})$$
$$4)\ w_t = w^3w_x + \varepsilon^2 w_{xxx} \quad (w^3 := u)$$
$$5)\ w_t = w^3w_x + \varepsilon^2(3w^2w_xw_{xx} + w^3w_{xxx}), \quad (w^3 := u),$$
$$6)\ w_t = w^2w_x + \varepsilon^2(3w^2w_xw_{xx} + w^3w_{xxx}), \quad (w^2 := u)$$
$$7)\ w_t = w^2w_x + \varepsilon^2(\frac{3}{2}w^2w_xw_{xx} + w^3w_{xxx}), \quad (w^2 := u)$$
$$8)\ u_t = uu_x + \varepsilon^2(u^2u_{xxx} + \frac{1}{9}u_x^3)$$
$$9)\ w_t = \frac{w_x}{w} + \varepsilon^2 w^3w_{xxx}, \quad (\frac{1}{w} := u).$$

The first two equations are the KdV ad mKdV, the third equation is equivalent via the Miura transformation $u \to u + \varepsilon^2[3u_{xx}/(2u) - 15u_x^2/(8u^2)]$ to the KdV equation 1). The last ones 4)-9) are reduced by means of suitable reciprocity transformations

$$dy = \alpha(u)dx + \rho dt, \qquad ds = dt,$$

parametrized by a smooth function $\alpha(u)$, to the equations 1)-3) from this listing.

Keeping in mind this result, we have decided to reanalyze the integrability of the evolution equation 3), having rewritten it in the following generalized form:

$$u_t = uu_x + \varepsilon^2 \left(u_{xxx} + c_1 \frac{u_x u_{xx}}{u} + c_2 \frac{u_x^3}{u^2} \right), \tag{1.9}$$

where $c_1, c_2 \in \mathbb{R}$ are constants. As the right-hand side of the flow (1.9) defines a vector field $u_t = K[u]$ on a suitably chosen smooth functional manifold $M \subset C^\infty(\mathbb{R};\mathbb{R})$, (being here locally diffeomorphic to the jet-manifold $J^\infty(\mathbb{R};\mathbb{R})$), we have applied to it our gradient-holonomic integrability scheme [21] and checked the existence of a suitable infinite hierarchy of conservation laws for the flow (1.9) and related Hamiltonian structures on M.

In particular, it means that this hierarchy is suitably ordered and satisfies the well known Noether-Lax equation $d\varphi/dt + K'^{*}[u] \cdot \varphi = 0$ on M, where $\varphi := \varphi[u;\lambda] = \text{grad}\ \xi[u;\lambda] \in T^{*}(M)$ – the functional gradient of a functional $\xi(\lambda)$ on M, chosen to be a **generating function of conservation laws** to the vector field $K : M \to T(M)$ and depending on the constant parameter $\lambda \in \mathbb{C}$ as $|\lambda| \to \infty$.

As a result of simple enough calculations we obtained the following two cases:

$$(1.10) \qquad i) \quad c_1 = -3, c_2 = \frac{15}{8}; \qquad ii) \quad c_1 = -\frac{3}{2}, c_2 = \frac{3}{4}.$$

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The first case gives rise to the equation 3) from the listing above, and the second one gives rise to the new evolution equation

$$(1.11) \qquad u_t = uu_x + \varepsilon^2 \left[u_{xxx} - \frac{3}{4} \left(\frac{u_x^2}{u} \right)_x \right],$$

which *a priori* possesses infinite hierarchy of suitably ordered conservation laws:

$$(1.12) \qquad H_1 = \int dxu, \qquad H_2 = \int dx \left(\frac{3}{2} \frac{u_x^2}{u} - u^2 \right),$$

$$H_3 = \int dx \left(\frac{9u_x u_{xx}}{2u} + \frac{15u_{xx}^2}{4u} - \frac{21u_x^2 u_{xx}}{2u^2} - \right.$$

$$\left. -3uu_{xx} + \frac{69u_x^4}{16u^3} - \frac{7u_x^2}{4} - \frac{u^3}{6}, \right) \quad \ldots \quad \text{and so on,}$$

where we put fo brevity $\varepsilon^2 = 1$.

It is easy to check that this new evolution equation (1.11) is missed in the above listing and can be represented in the following Hamiltonian form

$$u_t = -\theta \operatorname{grad} H_3[u] = -\eta \operatorname{grad} H_2[u], \tag{1.13}$$

where the Poisson operators $\theta, \eta : T^*(M) \to T(M)$ are given, respectively, by the expressions

$$\theta^{-1} = D_x^{-1} + \frac{3}{4}\left(\frac{1}{u}D_x + D_x\frac{1}{u}\right), \qquad D_x := \frac{d}{dx}, \tag{1.14}$$

and

$$\eta = \sqrt{u}D_x\sqrt{u}, \tag{1.15}$$

being compatible on the functional manifold M, that is the operator $(\theta + \lambda\eta)^{-1} : T(M) \to T^*(M)$ is Hamiltonian for any $\lambda \in \mathbb{R}$.

Proposition 1.2. *The above result simply means that the dynamical system*

$$u_t = uu_x + \varepsilon^2 \left[u_{xxx} - \frac{3}{4} \left(\frac{u_x^2}{u} \right)_x \right], \tag{1.16}$$

is a new completely integrable bi-Hamiltonian system on the functional manifold M.

Remark 1.3. Concerning the Dubrovin-Zhang equation 3) from the listing above, we have stated, as a by-product, that it is also a bi-Hamiltonian system and representable in the form

$$u_t = -\vartheta \, \mathrm{grad}\, H_1[u] = -\eta \, \mathrm{grad}\, H_2[u], \tag{1.17}$$

where the compatible Poisson operators $\theta, \eta : T^*(M) \rightarrow T(M)$ are given, respectively, by the expressions

(1.17) $$u_t = -\vartheta \,\mathrm{grad}\, H_1[u] = -\eta \,\mathrm{grad}\, H_2[u],$$

where the compatible Poisson operators $\theta, \eta : T^*(M) \rightarrow T(M)$ are given, respectively, by the expressions

(1.18) $$\vartheta = D_x \sqrt{u} D_x^{-1} \sqrt{u} D_x$$

and

(1.19) $$\eta^{-1} = \frac{3}{u} D_x \frac{1}{u} - \frac{1}{\sqrt{u}} D_x^{-1} \frac{1}{\sqrt{u}} \ ,$$

jointly with the Hamiltonian operators

(1.20) $$H_1 = \int dx \left(\frac{u_x^2}{2u^2} + \frac{4u}{3} \right),$$

$$H_2 = \int dx \left(\frac{17u_x^4}{16u^4} + \frac{40u_x^2}{u} + \frac{u_{xx}^2}{u^2} + \frac{4u^2}{9} - \frac{2u_x^2 u_{xx}}{u^3} \right).$$

2. Reduction integrability properties

Now we proceed to the following reduction of the new equation (1.11) putting $u \to \mu v$ as $\mu \to 0$:

$$(2.1)\quad u_t = uu_x + u_{xxx} - \frac{3}{4}\left(\frac{u_x^2}{u}\right)_x \quad \Rightarrow \quad v_t = v_{xxx} - \frac{3}{4}\left(\frac{v_x^2}{v}\right)_x,$$

called here by KN-3/4 and *a priory* integrable and possessing an infinite hierarchy of conservation laws, which can be easily written down from the hierarchy (1.12) via the limiting procedure

$$\tilde{H}_j := \lim_{\mu \to 0} \mu^{-1} H_j \big|_{u=\mu v}, \quad j \in \mathbb{N}.$$

$$(2.1)\quad u_t = uu_x + u_{xxx} - \frac{3}{4}\left(\frac{u_x^2}{u}\right)_x \quad \Rightarrow \quad v_t = v_{xxx} - \frac{3}{4}\left(\frac{v_x^2}{v}\right)_x ,$$

Remark 2.1. Here we would like to remark that the equation (2.1) above is very similar to the well known Krichever-Novikov (KN-3/2) equation

$$v_t = v_{xxx} - \frac{3}{2}\left(\frac{v_x^2}{v}\right)_x , \tag{2.2}$$

which differs from (2.1) only by the coefficient 3/2 instead of the rational number 3/4 and was studied before by V. Sokolov [4] by means of the well known Mikhailov-Shabat recursion symmetry analysis technique and by G. Wilson [5], using differential-algebraic Galois group solvability reasonings.

We have reanalized these Novikov-Krichever type equations (2.1) and (2.2), having performed the following manipulation:

$$(2.3)\qquad v_t = v_{xxx} - \frac{3}{2}\left(\frac{v_x^2}{v}\right)_x = v_{xxx} - \frac{3}{2}\left(\frac{2v_x v_{xx}}{v} - \frac{v_x^3}{v^2}\right)_x =$$

$$= \quad v_{xxx} - 3\frac{v_x v_{xx}}{v} + \frac{3}{2}\frac{v_x^3}{v^2} \Rightarrow v_{xxx} + c_1\frac{v_x v_{xx}}{v} + c_2\frac{v_x^3}{v^2},$$

where $c_1, c_2 \in \mathbb{R}$ are now arbitrary coefficients and checked the latter equation subject to the existence of an infinite hierarchy of suitably ordered conservation laws. The corresponding calculations gave rise right away that the coefficients $c_1, c_2 \in \mathbb{R}$ should satisfy two related algebraic relationships:

(2.4)

$$(c_1(c_1-3)-9c_2)=0 \qquad \text{and} \qquad (c_1+3)(c_1(c_1-3)-9c_2)=0,$$

whose solutions are the following two cases:

$$(2.5)\quad \begin{array}{lrcl} i) & c_1 & = & -3k, c_2 = k(k+1) \text{ for arbitrary } k \in \mathbb{R}\backslash\{1\}, \\ ii) & c_1 & = & -3, \; c_2 \in \mathbb{R} \text{ is arbitrary.} \end{array}$$

The first case $i)$ gives rise to the new integrable bi-Hamiltonian system on the functional manifold M in the form

$$v_t = v_{xxx} - 3k\frac{v_x v_{xx}}{v} + k(k+1)\frac{v_x^3}{v^2}, \tag{2.6}$$

where $k \in \mathbb{R}$ is an arbitrary parameter, yet $k \neq 1$. For the case $ii)$ when $k = 1$, the equation (2.3) reduces to the modified integrable Krichever-Novikov type system

$$v_t = v_{xxx} - 3\frac{v_x v_{xx}}{v} + c_2\frac{v_x^3}{v^2}, \tag{2.7}$$

integrable Krichever-Novikov type system

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possessing an infinite hierarchy of conservation laws and giving rise at $c_2 = 1/2$ to the well known Krichever-Novikov bi-Hamiltonian system (2.2):

$$v_t = v_{xxx} - \frac{3}{2}\left(\frac{v_x^2}{v}\right)_x . \tag{2.8}$$

The derived above modified Krichever-Novikov type equation (2.7) is also an integrable bi-Hamiltonian flow on the functional manifold M for arbitrary $c_2 \in \mathbb{R}$:

$$v_t = v_{xxx} - 3\frac{v_x v_{xx}}{v} + c_2 \frac{v_x^3}{v^2} = -\theta \operatorname{grad} \tilde{H}^{(c_2)}[v], \tag{2.9}$$

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$$v_t = v_{xxx} - 3\frac{v_x v_{xx}}{v} + c_2\frac{v_x^3}{v^2} = -\theta \operatorname{grad} \tilde{H}^{(c_2)}[v],$$

where the Poisson operator

(2.10)
$$\theta = vD_x^{-1}v$$

forms a compatible pair to the operator

(2.11)
$$\eta = D_x v D_x^{-1} v D_x,$$

and the corresponding Hamiltonian function equals

(2.12)
$$\tilde{H}_2^{(c_2)} = \int dx \left(\frac{v_{xxxx}}{v} - \frac{2v_x v_{xxx}}{v^2} - \frac{3v_{xx}^2}{2v^2} + c_2\frac{v_x^2 v_{xx}}{v^3} + \frac{2v_x^2 v_{xx}}{v^3} - \frac{3c_2 v_x^4}{4v^4} \right).$$

Moreover, one can also easily check that the next slightly modified Krichever-Novikov type equation

(2.13)
$$v_t = v_{xxx} - 3\frac{v_x v_{xx}}{v} + c_2 \frac{v_x^3}{v^2} - \frac{k_0 v_x}{v^2}$$

for arbitrary $c_2, k_0 \in \mathbb{R}$ is also an integrable bi-Hamiltonian flow, possesses an infinite hierarchy of functionally independent conservation laws, which can be generated recursively:

(2.14)
$$H_1^{(k_0,c_2)} = \int dx \left(\frac{v_x^2}{v^2} + \frac{2k_0}{v^2} \right) \rightarrow \tilde{H}_2^{(k_0,c_2)} \rightarrow \ldots \rightarrow \tilde{H}_n^{(k_0,c_2)} \rightarrow \ldots$$

via the Magri gradient recursion scheme:

(2.15)
$$\eta \operatorname{grad} \tilde{H}_n^{(k_0,c_2)} = \theta \operatorname{grad} \tilde{H}_{n+1}^{(k_0,c_2)},$$

for arbitrary $n \in \mathbb{Z}$, using the above mentioned compatible Poisson θ-η pair (2.10) and (2.11).

The same one can state about the new integrable Krichever-Novikov type equation

$$v_t = v_{xxx} - \frac{3}{4}\left(\frac{v_x^2}{v}\right)_x, \tag{2.16}$$

which is also a bi-Hamiltonian flow with respect to a compatible pair of the Poisson operators

$$\theta = vD_x^{-1}v, \qquad \eta = D_x v D_x^{-1} v D_x\ . \tag{2.17}$$

Remark 2.2. It appears interesting to observe that the generalized Krichever-Novikov type equation (2.3)

$$v_t = v_{xxx} + c_1\frac{v_x v_{xx}}{v} + c_2\frac{v_x^3}{v^2} \tag{2.18}$$

transforms via the change of variables $w := v_x/v$ to the following modified Korteweg de Vries type equation:

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transforms via the change of variables $w := v_x / v$ to the following modified Korteweg de Vries type equation:

$$w_t = w_{3x} + \frac{(c_1 + 3)}{2} (w^2)_x + (c_1 + c_2 + 1)(w^3)_x \ , \tag{2.19}$$

whose is, obviously, also integrable for two cases (2.5), mentioned before:

$$\begin{array}{rrcl} i) & c_1 & = & -3k, c_2 = k(k+1) \ \text{ for arbitrary } \ k \in \mathbb{R} \backslash \{1\}, \\ ii) & c_1 & = & -3, \ c_2 \in \mathbb{R} \text{ is arbitrary.} \end{array}$$

$$i) \quad c_1 = -3k, c_2 = k(k+1) \text{ for arbitrary } k \in \mathbb{R}\backslash\{1\},$$

$$ii) \quad c_1 = -3, \; c_2 \in \mathbb{R} \text{ is arbitrary.}$$

The case $i)$ reduces to the well known integrable modified Korteweg de Vries equation

$$w_t = w_{3x} - 3(k-1)ww_x + 3(k-1)^2 w^2 w_x, \tag{2.20}$$

Respectively, at $k = 1/2$, or equivalently at $c_1 = -3/2, c_2 = 3/4$, the modified Korteweg-de Vries equation (2.20) reduces to the classical modified Korteweg de Vries equation

$$w_t = w_{3x} + \frac{3}{2}\, ww_x + \frac{3}{4} w^2 w_x, \tag{2.21}$$

classical modified Korteweg de Vries equation

$$w_t = w_{3x} + \frac{3}{2}\, ww_x + \frac{3}{4} w^2 w_x, \tag{2.21}$$

which, evidently, is also integrable and bi-Hamiltonian on the functional manifold M. The second case ii) of the equation (2.19) also reduces to the classical integrable modified Korteweg-de Vries equation

$$w_t = w_{3x} + 3(c_2 - 2) w^2 w_x \ . \tag{2.22}$$

The special case $k = 1$, equivalent to the choice $c_1 = -3$, corresponds at $c_2 = 2$ exactly to the strictly linear equation, whose exact integrability is trivial.

3. Gradient-holonomic integrability scheme: the integrability of the Riemann type hydrodynamical systems

At the end of our presentation we will we will dwell for a short time on the integrability theory aspects, based on the gradient-holonomic integrability scheme, devised and applied by me jointly with Maxim Pavlov and collaborators to a virtually new important Riemann type hierarchy

$$D_t^{N-1} u = \bar{z}_x^s, \qquad D_t \bar{z} = 0, \tag{3.1}$$

where $s, N \in \mathbb{N}$ are arbitrary natural numbers, before proposed in our work (M. Pavlov, A. Prykarpatsky and others: J. Math. Phys. 53, 2012, 103521) as a nontrivial generalization of the infinite hierarchy of the Riemann type flows, suggested recently by M. Pavlov and D. Holm in the form of dynamical systems

$$D_t^N u = 0, \tag{3.2}$$

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defined on a 2π-periodic functional manifold $\bar{M}^N \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^N)$, the vector $(u, D_t u, D_t^2 u, ..., D_t^{N-1} u, \bar{z})^\intercal \in \bar{M}^N$, the differentiations

$$D_x := \partial/\partial x, D_t := \partial/\partial t + u\partial/\partial x \tag{3.3}$$

satisfy as above the Lie-algebraic commutator relationship

$$[D_x, D_t] = u_x D_x \tag{3.4}$$

and $t \in \mathbb{R}$ is an evolution parameter.

The mentioned above dynamical systems

$$D_t^{N-1} u = \bar{z}_x^s, \qquad D_t \bar{z} = 0, \tag{3.5}$$

where $s, N \in \mathbb{N}$, were considered at $s = \overline{1,2}$ and $N = \overline{2,3}$, respectively, appeared to be related to nontrivial generalizations of the well known Camassa-Holm and Degasperis-Procesi equations,

$$D_t^{N-1} u = \bar{z}_x^s, \qquad D_t \bar{z} = 0, \tag{3.5}$$

where $s, N \in \mathbb{N}$, were considered at $s = \overline{1,2}$ and $N = \overline{2,3}$, respectively, appeared to be related to nontrivial generalizations of the well known Camassa-Holm and Degasperis-Procesi equations, which were extensively studied by many researchers. The case $s = 2$ and $N = 2$ is a generalization of the known Gurevich-Zybin dynamical system in cosmology, whose integrability was analyzed by M. Pavlov in [18] and later in our works [21, 14, 8] within the gradient-holonomic scheme. There was shown that this system, namely,

$$D_t u = \bar{z}_x^2, \qquad D_t \bar{z} = 0, \tag{3.6}$$

is a smooth integrable bi-Hamiltonian flow on the 2π-periodic functional maniifold $\bar{M}_2$, whose Lax type representation is given by the compatible linear Lax type system

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is a smooth integrable bi-Hamiltonian flow on the 2π-periodic functional maniifold $\bar{M}_2$, whose Lax type representation is given by the compatible linear Lax type system

(3.7)

$$D_x f = \begin{pmatrix} \bar{z}_x & 0 \\ -\lambda(u + u_x/\bar{z}_x) & -\bar{z}_{xx}/\bar{z}_x \end{pmatrix} f, \qquad D_t f = \begin{pmatrix} 0 & 0 \\ -\lambda \bar{z}_x & u_x \end{pmatrix} f,$$

where $f \in C^\infty(\mathbb{R}^2; \mathbb{R}^2)$ and $\lambda \in \mathbb{R}$ is an arbitrary spectral parameter.

At $s = 2$ and $N = 3$ dynamical system (3.5) is equivalent to the evolution flow

$$D_t u = v, \qquad D_t v = \bar{z}_x^2, \qquad D_t \bar{z} = 0, \tag{3.8}$$

considered here on a 2π-periodic functional manifold $\bar{M}_3 \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}^3)$ for a point $(u, v, \bar{z})^\intercal \in \bar{M}_3$.

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considered here on a 2π-periodic functional manifold $\bar{M}_3 \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^3)$ for a point $(u,v,\bar{z})^\intercal \in \bar{M}_3$.

Below we will analyze this dynamical system (3.8) by means of the symplectic gradient-holonomic integrability scheme, devised and developed in collaboration with Maxim Pavlov (M. Pavlov, A. Prykarpatsky and others. J. Phys. A: Math. Theor. 43, 2010, 295205; J. Math. Phys. 53, 2012, 103521) and which takes us a way towards the desired result.

3.1. **Poissonian structure analysis on $\bar{M}_3$.** In what follows, we first construct the Poissonian structures for dynamical system (3.8) on the manifold $\bar{M}_3$, rewritten in the equivalent componentwise form

$$
(3.9) \qquad \frac{d}{dt}\begin{pmatrix} u \\ v \\ \bar{z} \end{pmatrix} = \bar{K}[u, v, \bar{z}] := \begin{pmatrix} v - uu_x \\ \bar{z}_x^2 - uv_x \\ 0 \end{pmatrix},
$$

where $\bar{K} : \bar{M}_3 \to (\bar{M}_3)$ is the corresponding vector field on $\bar{M}_3$. To proceed, we need to obtain additional solutions to the basic Noether-Lax gradient equation

$$
(3.10) \qquad D_t\bar{\psi} + \bar{K}'^{,*}[u, v, z]\bar{\psi} = \text{grad } \bar{\mathcal{L}},
$$

on the functional manifold $\bar{M}_3$, where the matrix operator

$$
(3.11) \qquad \bar{K}'^{,*}[u, v, \bar{z}] := \begin{pmatrix} 0 & -v_x & -\bar{z}_x \\ 1 & u_x & 0 \\ 0 & -2\partial\ \bar{z}_x & u_x \end{pmatrix}
$$

Noether-Lax gradient equation

$$D_t\bar\psi + \bar K'^{,*}[u,v,z]\bar\psi = \mathrm{grad}\ \bar{\mathcal{L}}, \tag{3.10}$$

on the functional manifold $\bar M_3$, where the matrix operator

$$\bar K'^{,*}[u,v,\bar z] := \begin{pmatrix} 0 & -v_x & -\bar z_x \\ 1 & u_x & 0 \\ 0 & -2\partial\ \bar z_x & u_x \end{pmatrix} \tag{3.11}$$

is an endomorphism of the cotangent space $T^*(\bar M_3)$, adjoint to the corresponding Frechet derivative $K'[u,v,\bar z] : T(\bar M_3) \to T(\bar M_3)$ at $(u,v,\bar z) \in \bar M_3$ subject to the natural bi-linear form $(\cdot|\cdot) : T^*(\bar M_3)\times T(\bar M_3) \to \mathbb{R}$, and which we may rewrite in the component wise form

$$\begin{aligned} D_t\bar\psi^{(1)} &= v_x\bar\psi^{(2)} + \bar z_x\bar\psi^{(3)} + \delta\bar{\mathcal{L}}/\delta u, \\ D_t\bar\psi^{(2)} &= -\bar\psi^{(1)} - u_x\bar\psi^{(2)} + \delta\bar{\mathcal{L}}/\delta v, \\ D_t\bar\psi^{(3)} &= 2(\bar z_x\bar\psi^{(2)})_x - u_x\bar\psi^{(3)} + \delta\bar{\mathcal{L}}/\delta\bar z, \end{aligned} \tag{3.12}$$

$$(3.12)\qquad \begin{aligned} D_t\bar{\psi}^{(1)} &= v_x\bar{\psi}^{(2)} + \bar{z}_x\bar{\psi}^{(3)} + \delta\bar{\mathcal{L}}/\delta u,\\ D_t\bar{\psi}^{(2)} &= -\bar{\psi}^{(1)} - u_x\bar{\psi}^{(2)} + \delta\bar{\mathcal{L}}/\delta v,\\ D_t\bar{\psi}^{(3)} &= 2(\bar{z}_x\bar{\psi}^{(2)})_x - u_x\bar{\psi}^{(3)} + \delta\bar{\mathcal{L}}/\delta\bar{z}, \end{aligned}$$

where the vector $\bar{\psi} := (\bar{\psi}^{(1)}, \bar{\psi}^{(2)}, \bar{\psi}^{(3)})^{\intercal} \in T^*(\bar{M}_3)$. As a simple consequence of (3.12), one obtains the following system of linear differential relationships:

$$(3.13)\qquad \begin{aligned} D_t^3\tilde{\psi}^{(2)} &= -2\bar{z}_x^2\tilde{\psi}_x^{(2)} + D_t^2\partial^{-1}(\delta\bar{\mathcal{L}}/\delta v) - \\ &\quad -\partial^{-1}\langle \text{grad } \bar{\mathcal{L}} | (u_x, v_x, \bar{z}_x)^{\intercal}\rangle,\\ D_t\tilde{\psi}^{(2)} &= -\tilde{\psi}^{(1)} + \partial^{-1}(\delta\bar{\mathcal{L}}/\delta v),\\ D_t\tilde{\psi}^{(3)} &= 2\bar{z}_x\tilde{\psi}_x^{(2)} + \partial^{-1}(\delta\bar{\mathcal{L}}/\delta\bar{z}). \end{aligned}$$

Here we have defined $(\bar{\psi}^{(1)}, \bar{\psi}^{(2)}, \bar{\psi}^{(3)})^{\intercal} := (\tilde{\psi}_x^{(1)}, \tilde{\psi}_x^{(2)}, \tilde{\psi}_x^{(3)})^{\intercal}$ and made use of the commutator relationship $[D_x, D_t] = u_x D_x$ to derive the following important operator property

$$[D_t, (\alpha D_x)^j] = 0, \tag{3.14}$$

which holds for the function $\alpha := 1/\bar{z}_x$, where $D_t \bar{z} = 0$, and any $j \in \mathbb{N}$.

Let us now construct a differential ring $\mathcal{K}\{u\} \subset \mathcal{K} := \mathbb{R}\{\{x,t\}\}$, generated by a fixed functional variable $u \in \mathbb{R}\{\{x,t\}\}$ and invariant with respect to two differentiations $D_x := \partial/\partial x$ and $D_t := \partial/\partial t + u\partial/\partial x$, satisfying the Lie-algebraic commutator relationship (3.4)

$$[D_x, D_t] = u_x D_x \tag{3.15}$$

together with the constraint (3.6)

$$D_t u = \bar{z}_x^2, \qquad D_t \bar{z} = 0. \tag{3.16}$$

$$
(3.13)\quad
\begin{aligned}
D_t^3\tilde{\psi}^{(2)} &= -2\bar{z}_x^2\tilde{\psi}_x^{(2)} + D_t^2\partial^{-1}(\delta\bar{\mathcal{L}}/\delta v) - \\
&\quad -\partial^{-1}\langle \text{grad } \bar{\mathcal{L}} | (u_x, v_x, \bar{z}_x)^{\intercal}\rangle, \\
D_t\tilde{\psi}^{(2)} &= -\tilde{\psi}^{(1)} + \partial^{-1}(\delta\bar{\mathcal{L}}/\delta v), \\
D_t\tilde{\psi}^{(3)} &= 2\bar{z}_x\tilde{\psi}_x^{(2)} + \partial^{-1}(\delta\bar{\mathcal{L}}/\delta\bar{z}).
\end{aligned}
$$

Based now on the property that for any $m \in \mathbb{N}$ any additive set $I_m := \{\sum_{j=\overline{1,m}} a_j\alpha^j : a_j \in \mathcal{K}\{u\},\ j = \overline{1,m}\} \subset \mathcal{K}\{u\}$ is an ideal in the functional ring $\mathcal{K}\{u\} \subset \mathcal{K} := \mathbb{R}\{\{x,t\}\}$, one can easily solve the first equation of the linear system system (3.13) above and next solve recursively the remaining two equations. In particular, one simply gets that the three vector elements

(3.17)

$$
\begin{aligned}
&\tilde{\psi}_0 = (-v, u, -2\bar{z}_x)^{\intercal}, \quad \bar{\mathcal{L}}_0 = 0; \\
&\tilde{\psi}_\theta = (-u_x/\bar{z}_x, 1/\bar{z}_x, (u_x^2 - 2v_x)/(2\bar{z}_x^2))^{\intercal}, \bar{\mathcal{L}}_\theta = 0; \\
&\tilde{\psi}_\eta = (u/2, -x/2, \partial^{-1}[(2v_x - u_x^2)/(2\bar{z}_x)]), \bar{\mathcal{L}}_\eta = (D_x\tilde{\psi}_\eta, \bar{K}) - H_\vartheta,
\end{aligned}
$$

(3.17)

$$\tilde{\psi}_0 = (-v, u, -2\bar{z}_x)^{\intercal}, \quad \bar{\mathcal{L}}_0 = 0;$$
$$\tilde{\psi}_\theta = (-u_x/\bar{z}_x, 1/\bar{z}_x, (u_x^2 - 2v_x)/(2\bar{z}_x^2))^{\intercal}, \bar{\mathcal{L}}_\theta = 0;$$
$$\tilde{\psi}_\eta = (u/2, -x/2, \partial^{-1}[(2v_x - u_x^2)/(2\bar{z}_x)]), \bar{\mathcal{L}}_\eta = (D_x\tilde{\psi}_\eta, \bar{K}) - H_\vartheta,$$

are solutions to our linear system (3.13). The first two elements of (3.17) lead to the Volterra symmetric vectors $\bar{\psi}_0 = D_x\tilde{\psi}_0, \bar{\psi}_\theta = D_x\tilde{\psi}_\theta \in T^*(\bar{M}_3) : \bar{\psi}_0' = \bar{\psi}_0'^{,*}, \bar{\psi}_\theta' = \bar{\psi}_\theta'^{,*}$ entailing the trivial conservation laws $(\bar{\psi}_0, \bar{K}) = 0 = (\bar{\psi}_\theta, \bar{K})$. The third element of (3.17) gives rise to the Volterra asymmetric vector $\bar{\psi}_\eta := D_x\tilde{\psi}_\eta : \bar{\psi}_\eta' \neq \bar{\psi}_\eta'^{,*}$, entailing the following inverse co-symplectic differential-integral expression:

(3.18)

$$\bar{\eta}^{-1} := \bar{\psi}_\eta' - \bar{\psi}_\eta'^{,*} = \begin{pmatrix} \partial & 0 & -\partial\frac{u_x}{\bar{z}_x} \\ 0 & 0 & \partial\frac{1}{\bar{z}_x} \\ -\frac{u_x}{\bar{z}_x}\partial & \frac{1}{\bar{z}_x}\partial & \begin{array}{c}\frac{u_x}{2\bar{z}_x}\partial\frac{u_x}{\bar{z}_x} - \\ -\frac{v_x}{\bar{z}_x}\partial\frac{1}{\bar{z}_x} - \frac{1}{\bar{z}_x}\partial\frac{v_x}{\bar{z}_x}\end{array} \end{pmatrix}.$$

(3.18)

$$\bar{\eta}^{-1} := \bar{\psi}'_\eta - \bar{\psi}'^{,*}_\eta = \left(\begin{array}{ccc} \partial & 0 & -\partial \frac{u_x}{\bar{z}_x} \\ 0 & 0 & \partial \frac{1}{\bar{z}_x} \\ -\frac{u_x}{\bar{z}_x}\partial & \frac{1}{\bar{z}_x}\partial & \begin{array}{c} \frac{u_x}{2\bar{z}_x}\partial \frac{u_x}{\bar{z}_x} - \\ -\frac{v_x}{\bar{z}_x}\partial \frac{1}{\bar{z}_x} - \frac{1}{\bar{z}_x}\partial \frac{v_x}{\bar{z}_x} \end{array} \end{array} \right).$$

Having inverted the expression (3.18), one easily obtains the Poisson operator $\bar{\eta} : T^*(\bar{M}_3) \rightarrow T(\bar{M}_3)$ on the manifold $\bar{M}_3$

$$\bar{\eta} = \left(\begin{array}{ccc} \partial^{-1} & u_x\partial^{-1} & 0 \\ \partial^{-1}u_x & v_x\partial^{-1} + \partial^{-1}v_x & \partial^{-1}\bar{z}_x \\ 0 & \bar{z}_x\partial^{-1} & 0 \end{array} \right), \tag{3.19}$$

subject to which the following Hamiltonian representation

$$\bar{K}[u, v, \bar{z}] = -\bar{\eta} \text{ grad } \bar{H}_\eta \tag{3.20}$$

holds, where the Hamiltonian function $\bar{H}_\eta : \bar{M}_3 \rightarrow \mathbb{R}$ is given by the polynomial functional

$$\bar{H}_\eta = \frac{1}{2}\int_0^{2\pi} dx(2u\bar{z}_x^2 - v^2 - u^2 v_x). \tag{3.21}$$

The same way makes it possible to derive easily enough the second Poisson operator $\bar{\vartheta} : T^*(\bar{M}_3) \rightarrow T(\bar{M}_3)$, suitably compatible with the above Poisson operator $\bar{\eta} : T^*(\bar{M}_3) \rightarrow T(\bar{M}_3)$ (3.19) and equal to

$$\bar{\vartheta} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1/2D^{-1} \end{pmatrix} \tag{3.22}$$

on the manifold $\bar{M}_3$.

3.2. **Lax type integrability analysis.** Next, we return to the Lax type integrability analysis of the dynamical system (3.9)

$$\frac{d}{dt}\begin{pmatrix} u \\ v \\ \bar{z} \end{pmatrix} = \bar{K}[u,v,\bar{z}] := \begin{pmatrix} v - uu_x \\ \bar{z}_x^2 - uv_x \\ 0 \end{pmatrix} \tag{3.23}$$

on the functional manifold $\bar{M}_3$. As the Poissonian operators (3.22) and (3.19) are compatible [12, 21, 8, 10] on the manifold $\bar{M}_3$, that is, the operator pencil $(\bar{\vartheta} + \lambda\bar{\eta}) : T^*(\bar{M}_3) \rightarrow T(\bar{M}_3)$ is also Poissonian for arbitrary $\lambda \in \mathbb{R}$, all operators of the form

$$\bar{\vartheta}_n := \bar{\vartheta}(\bar{\vartheta}^{-1}\bar{\eta})^n \tag{3.24}$$

$$\bar{\vartheta}_n := \bar{\vartheta}(\bar{\vartheta}^{-1}\bar{\eta})^n \tag{3.24}$$

for arbitrary $n \in \mathbb{Z}$ are then Poissonian too on the functional manifold $\bar{M}_3$. Using now the classical homotopy formula [11, 8, 21] and the recursion property of the Poissonian pair (3.22) and (3.19), it is easy to reconstruct the related infinite hierarchy of mutually commuting conservation laws

$$\begin{gathered} \bar{\gamma}_j = \int_0^1 d\mu(\text{grad } \bar{\gamma}_j[\mu u, \mu v, \mu \bar{z}], (u, v, \bar{z})^\intercal), \\ \text{grad } \bar{\gamma}_j[u, v, \bar{z}] := \Lambda^j \text{grad } \bar{H}_\eta, \end{gathered} \tag{3.25}$$

for our dynamical system (3.23), where $j \in \mathbb{Z}$ and $\bar{\Lambda} := \bar{\vartheta}^{-1}\bar{\eta}$: $T^*(\bar{M}_3) \to T^*(\bar{M}_3)$ is the corresponding recursion operator, which satisfies the so called associated Lax type commutator relationship

$$d\Lambda/dt = [\bar{\Lambda}, \bar{K}^{\prime,*}]. \tag{3.26}$$

Remark 3.1. It is evident that the trace-functionals

$$\bar{\gamma}_n := \int_0^{2\pi} \mathrm{Tr}(\bar{\Lambda}^n)dx, \tag{3.27}$$

where Tr:$= res_{D-1}$ tr : $End\ (T^*(\bar{M}_3) \to C^\infty(\bar{M}_3; C^\infty([0,2\pi];\mathbb{R}))$ is the usual Adler type trace operation on the algebra of periodic pseudo-differential operators $PDO(\mathbb{R}/\{2\pi\mathbb{Z}\})$, are for any $n \in \mathbb{Z}$ conservation laws to our dynamical system (3.23). In particular, this property was put into the background of the Shabat-Mikhailov integrability classification scheme.

In the course of above analysis and observations, we have stated the following result.

$$
(3.23) \qquad \frac{d}{dt}\begin{pmatrix} u \\ v \\ \bar{z} \end{pmatrix} = \bar{K}[u, v, \bar{z}] := \begin{pmatrix} v - uu_x \\ \bar{z}_x^2 - uv_x \\ 0 \end{pmatrix}
$$

Proposition 3.2. *The Riemann type hydrodynamic system (3.23) is a bi-Hamiltonian dynamical system on the functional manifold $\bar{M}_3$ with respect to the compatible Poissonian structures $\bar{\vartheta}, \bar{\eta} : T^*(\bar{M}_3) \to T(\bar{M}_3)$*

(3.28)

$$
\bar{\vartheta} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1/2D^{-1} \end{pmatrix}, \bar{\eta} = \begin{pmatrix} \partial^{-1} & u_x \partial^{-1} & 0 \\ \partial^{-1} u_x & v_x \partial^{-1} + \partial^{-1} v_x & \partial^{-1} \bar{z}_x \\ 0 & \bar{z}_x \partial^{-1} & 0 \end{pmatrix}
$$

and possesses an infinite hierarchy of mutually commuting conservation laws (3.25).

Remark 3.3. Concerning the existence of an additional infinite and parametrically $\mathbb{R} \ni \lambda$-ordered hierarchy of conservation laws for the dynamical system (3.23), it is instructive to consider the new dispersive nonlinear dynamical system

(3.29)

$$\begin{pmatrix} du/d\tau \\ dv/d\tau \\ d\bar{z}/d\tau \end{pmatrix} = -\bar{\vartheta}\ \mathrm{grad}\ \bar{H}_0[u,v,\bar{z}] := \begin{pmatrix} -(\bar{z}_x^{-1})_x \\ -(u_x \bar{z}_x^{-1})_x \\ 1/4(u_x^2 - 2v_x)\bar{z}_x^{-2} \end{pmatrix} = \tilde{K}[u,v,\bar{z}]$$

with respect to a new evolution parameter $\tau \in \mathbb{R}$, a priori commuting to the initial flow (3.23). By solving the corresponding Noether-Lax equation

$$d\tilde{\varphi}/dt + \tilde{K}'^{,*}\tilde{\varphi} = 0 \qquad (3.30)$$

Noether-Lax equation

$$d\tilde{\varphi}/dt + \tilde{K}^{\prime,*}\tilde{\varphi} = 0 \tag{3.30}$$

for an element $\tilde{\varphi} \in T^*(\bar{M}_3)$ in a suitably chosen asymptotic form, one can construct an infinite ordered hierarchy of conservation laws for our Riemann type dynamical system

$$\frac{d}{dt}\begin{pmatrix} u \\ v \\ \bar{z} \end{pmatrix} = \bar{K}[u, v, \bar{z}] := \begin{pmatrix} v - uu_x \\ \bar{z}_x^2 - uv_x \\ 0 \end{pmatrix}, \tag{3.31}$$

$$\frac{d}{dt}\begin{pmatrix} u \\ v \\ \bar{z} \end{pmatrix} = \bar{K}[u, v, \bar{z}] := \begin{pmatrix} v - uu_x \\ \bar{z}_x^2 - uv_x \\ 0 \end{pmatrix} \tag{3.23}$$

concerning which we will not delve into here. This hierarchy and the existence of an infinite and parametrically $\mathbb{R} \ni \lambda$-ordered hierarchy of conservation laws for the Riemann type dynamical system (3.23) provides compelling indications that it is completely integrable in the sense of Lax on the functional manifold $\bar{M}_3$. We shall complete our integrability analysis in the next section using rather powerful differential-algebraic tools that were devised recently in [22, 19, 23].

4. Differential-algebraic integrability analysis: $N = 3$

Consider now the above introduced differential ring $\mathcal{K}\{u\} \subset \mathcal{K} := \mathbb{R}\{\{x,t\}\}$, generated by a fixed functional variable $u \in \mathbb{R}\{\{x,t\}\}$ and invariant with respect to two differentiations $D_x := \partial/\partial x$ and $D_t := \partial/\partial t + u\partial/\partial x$ that satisfy the Lie-algebraic commutator relationship (3.4)

$$[D_x, D_t] = u_x D_x \tag{4.1}$$

together with the constraint (3.6) expressed in the following differential-algebraic functional form

$$D_t^3 u = -2D_t^2 u D_x u.$$

Since the Lax representation for the dynamical system (3.23) can be interpreted [8, 22] as the existence of a finite-dimensional invariant ideal $\mathcal{I}\{u\} \subset \mathcal{K}\{u\}$ realizing the corresponding finite-dimensional representation of the the Lie-algebraic commutator relationship (4.1), this ideal can be constructed in the finitely generated form as

(4.2)

$$\mathcal{I}\{u\} := \{\lambda^2 u f_1 + \lambda v f_2 + \bar{z}_x f_3 \in \mathcal{K}\{u\} : f_j \in \mathcal{K}, 1 \leq j \leq 3, \lambda \in \mathbb{R}\},$$

where $v = D_t u, \bar{z}_x^2 = D_t^2 u, D_t \bar{z} = 0$ and $\lambda \in \mathbb{R}$ is an arbitrary real parameter. To construct finite-dimensional representations of

$$\mathcal{I}\{u\} := \{\lambda^2 u f_1 + \lambda v f_2 + \bar{z}_x f_3 \in \mathcal{K}\{u\} : f_j \in \mathcal{K}, 1 \leq j \leq 3, \lambda \in \mathbb{R}\},$$

where $v = D_t u$, $\bar{z}_x^2 = D_t^2 u$, $D_t \bar{z} = 0$ and $\lambda \in \mathbb{R}$ is an arbitrary real parameter. To construct finite-dimensional representations of the D_x- and D_t-differentiations, it is necessary [22] first to find the D_t-invariant kernel $\ker D_t \subset \mathcal{I}\{u\}$ and next to check its invariance with respect to the D_x-differentiation. It is easy to show that

$$\ker D_t = \{f \in \mathcal{K}^3\{u\} : D_t f = q(\lambda) f, \quad \lambda \in \mathbb{R}\}, \tag{4.3}$$

where the matrix $q(\lambda) := q[u, v, \bar{z}; \lambda] \in$End $\mathcal{K}\{u\}^3$ is given as

$$q(\lambda) = \begin{pmatrix} 0 & 0 & 0 \\ -\lambda & 0 & 0 \\ 0 & -2\lambda\ \bar{z}_x \bar{z}_{xx} & u_x \end{pmatrix}. \tag{4.4}$$

To obtain the corresponding representation of the D_x-differentiation in the space $\mathcal{K}\{u\}^3$, it suffices to find a matrix $l(\lambda) := l[u, v, \bar{z}; \lambda] \in \text{End}\ \mathcal{K}\{u\}^3$satisfying the linear relationship

$$D_x f = l(\lambda) f \tag{4.5}$$

for $f \in \mathcal{K}\{u\}^3$ under condition that the related $\{f\}$-generated ideal

$$\mathcal{R}\{u\} := \{\langle g|f\rangle_{\mathcal{K}^3} : f \in \ker D_t \subset \mathcal{K}^3\{u\},\ g \in \mathcal{K}^3\} \tag{4.6}$$

is D_x-invariant with respect to the matrix differentiation representation 4.5). Straightforward calculations using this invariance ideal

$$\mathcal{R}\{u\} := \{\langle g|f\rangle_{\mathcal{K}^3} : f \in \ker D_t \subset \mathcal{K}^3\{u\},\ g \in \mathcal{K}^3\} \tag{4.6}$$

is D_x-invariant with respect to the matrix differentiation representation 4.5). Straightforward calculations using this invariance

condition then yield the following matrix

$$l(\lambda) = \begin{pmatrix} \lambda^2 u\bar{z}_x & \lambda v\bar{z}_x & \bar{z}_x^2 \\ -t\lambda^3 u\bar{z}_x & -t\lambda^2 v\bar{z}_x & -t\lambda\bar{z}_x^2 \\ \lambda^4(tuv-u^2)- & -\lambda v_x\bar{z}_x^{-1}+ & \lambda^2\bar{z}_x(u-tv)- \\ -\lambda^2 u_x\bar{z}_x^{-1} & +\lambda^3(tv^2-uv) & -\bar{z}_{xx}\bar{z}_x^{-1} \end{pmatrix} \tag{4.7}$$

entering the linear equation 4.5). Thus, the following proposition is stated.

Proposition 4.1. *The generalized Riemann type dynamical system (3.23) is a bi-Hamiltonian integrable flow possessing a non-autonomous Lax type representation of the form*

(4.8)

$$D_t f = \begin{pmatrix} 0 & 0 & 0 \\ -\lambda & 0 & 0 \\ 0 & -2\lambda\ \bar{z}_x \bar{z}_{xx} & u_x \end{pmatrix} f,$$

$$D_x f = \begin{pmatrix} \lambda^2 u \bar{z}_x & \lambda v \bar{z}_x & \bar{z}_x^2 \\ -t\lambda^3 u \bar{z}_x & -t\lambda^2 v \bar{z}_x & -t\lambda \bar{z}_x^2 \\ \lambda^4 (tuv - u^2) - & -\lambda v_x \bar{z}_x^{-1} + & \lambda^2 \bar{z}_x (u - tv) - \\ -\lambda^2 u_x \bar{z}_x^{-1} & +\lambda^3 (tv^2 - uv) & -\bar{z}_{xx} \bar{z}_x^{-1} \end{pmatrix} f,$$

where $\lambda \in \mathbb{R}$ *is an arbitrary spectral parameter and* $f \in C^{(\infty)}(\mathbb{R}^2; \mathbb{R}^3)$.

Remark 4.2. Simple analogs of the above differential-algebraic calculations for the case $N = 2$ gives rise readily to the corresponding Riemann type hydrodynamic system

$$D_t u = \bar{z}_x^2, \qquad D_t \bar{z} = 0 \tag{4.9}$$

on the functional manifold $\bar{M}_2$, which possesses the following matrix Lax type representation:

(4.10)

$$D_t f = \begin{pmatrix} 0 & 0 \\ -\lambda \bar{z}_x & u_x \end{pmatrix}, \qquad D_x f = \begin{pmatrix} \bar{z}_x & 0 \\ -\lambda(u + u_x/\bar{z}_x & -\bar{z}_{xx}/\bar{z}_x \end{pmatrix} f,$$

where $\lambda \in \mathbb{R}$ is an arbitrary spectral parameter and $f \in C^{(\infty)}(\mathbb{R}^2; \mathbb{R}^2)$.

The matrices (4.8) are not of standard form since they depend explicitly on the temporal evolution parameter $t \in \mathbb{R}$. Nonetheless, the matrices (4.4) and (4.7):

(4.11)

$$q(\lambda) = \begin{pmatrix} 0 & 0 & 0 \\ -\lambda & 0 & 0 \\ 0 & -2\lambda\ \bar{z}_x \bar{z}_{xx} & u_x \end{pmatrix},$$

$$l(\lambda) = \begin{pmatrix} \begin{matrix} \lambda^2 u \bar{z}_x \\ -t\lambda^3 u \bar{z}_x \end{matrix} & \begin{matrix} \lambda v \bar{z}_x \\ -t\lambda^2 v \bar{z}_x \end{matrix} & \begin{matrix} \bar{z}_x^2 \\ -t\lambda \bar{z}_x^2 \end{matrix} \\ \begin{matrix} \lambda^4 (tuv - u^2) - \\ -\lambda^2 u_x \bar{z}_x^{-1} \end{matrix} & \begin{matrix} -\lambda v_x \bar{z}_x^{-1} + \\ +\lambda^3 (tv^2 - uv) \end{matrix} & \begin{matrix} \lambda^2 \bar{z}_x (u - tv) - \\ -\bar{z}_{xx} \bar{z}_x^{-1} \end{matrix} \end{pmatrix}$$

satisfy for all $\lambda \in \mathbb{R}$ the Zakharov–Shabat type compatibility condition

$$D_t l(\lambda) = [q(\lambda), l(\lambda)] + D_x l(\lambda) - u_x l(\lambda), \tag{4.12}$$

$$D_t l(\lambda) = [q(\lambda), l(\lambda)] + D_x l(\lambda) - u_x l(\lambda), \tag{4.12}$$

which follows from the linear Lax type relationships (4.3) and (4.5)

$$D_t f = q(\lambda) f, \qquad D_x f = l(\lambda) f \tag{4.13}$$

$$[D_x, D_t] = u_x D_x \tag{4.1}$$

and the commutator condition (4.1). In particular, taking into account that the dynamical system (3.23) has a compatible Poissonian pair (3.28), depending only on the variables $(u, v, \bar{z})^{\intercal} \in \bar{M}_3$ and not depending on the temporal variable $t \in \mathbb{R}$, one can certainly assume that it also possesses a standard autonomous Lax type representation, which can possibly be found by means of a suitable gauge transformation of the linear relationships (4.13).

5. Ackowledgements

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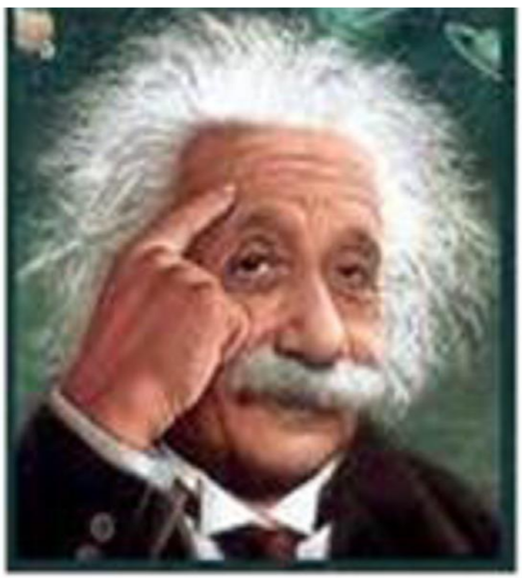
Dziękuję serdecznie za uwagę!
Спасибо за внимание!



Thanks for your attention!

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