Intro Let $f : X \to \mathbb{A}^1 / \mathbb{C}$ flat proper rel dim n

X smooth $f^{-1}(t) = X_t$ smooth $t \neq \{0\}$

$f^{-1}(0) = X_0$ has 1 sing pt x_0

$X_0 \setminus x_0$ is smooth

Milnor $\chi^{top}(X_t) - \chi^{top}(X_0) = (-1)^n \dim_{\mathbb{C}} J(f)$

$$J(f) = \mathcal{O}_{X,x_0} / \left(\ldots, \frac{\partial f}{\partial t_i}, \ldots, f \right)$$

$t_0 \ldots t_n$ reg sys of parameters at x_0

Real analytic setting: Eisenbud-Levine
Khimshiashvili

$$\chi^{top}(X_t(\mathbb{R})) - \chi^{top}(X_0(\mathbb{R})) = (-1)^n \operatorname{sig} q_f$$

$\leftarrow$ Scheja-Storch

$e_f \in J(f)$ $\quad Tr : J(f) \to \mathbb{R}$ $\quad \mathbb{R}$-linear

$$\mathrm{Tr}(e_f) = 1$$

$$q_f(x) = \mathrm{Tr}(x^2).$$

Question does this make an identity in "GW(k)" after making the "known" quadratic replacements of χ^{top} + $(J(f), q_f)$?

Ans: Not quite: in cases we know

$$\Delta_t(X, f) = (\langle e \rangle - \langle 1 \rangle) + (1 - \langle e \rangle)^n q_f$$

"cone deformation" with multiplicity $= e$

$\langle u \rangle$ = rank 1 form $x \mapsto ax^2$

§ Motivation / categorical Euler characteristics

If $(\mathcal{C}, \otimes, 1)$ sym. monoidal cat.

$x \in \mathcal{C}$ has <u>dual</u> $(x^\vee, \delta, ev)$

$$\begin{tikzcd} 1 \arrow[r, "\delta"] \arrow[rrr, bend right, "\chi_{\mathcal{C}}(x)"'] & x \otimes x^\vee \arrow[r, "\tau_{x,x^\vee}"] & x^\vee \otimes x \arrow[r, "ev"] & 1 \end{tikzcd}$$

$\chi_{\mathcal{C}}(x) \leftarrow$ categorical Euler characteristic $\in \mathrm{End}_{\mathcal{C}}(1)$

<u>Ex</u>

$\mathcal{C} = k\text{-vs}$

$\chi_{\mathcal{C}}(V) = \dim_k V \cdot 1 \in \mathrm{End}(k) \cong k$

$\mathcal{C} = \mathbb{Z}$ graded vs. $V = \oplus V_n$

$\chi_{\mathcal{C}}(V) = \Sigma (-1)^n \dim_k V_n \in k$

$\mathcal{C} = SH \qquad \mathrm{End}_{SH}(1) = \pi_0^{st}(\mathbb{S}) = \mathbb{Z}$

get word top Euler characteristic

$\mathcal{C} = DM(k)$ get something $\mathrm{End}_{DM(k)}(1) = \mathbb{Z}$

Take $\mathcal{C} = SH(k)$ $\mathrm{End}_{SH(k)}(1) \cong \underline{\underline{GW(k)}}$

Some background on $SH(-)$

$SH(-) : Sch/_k^{op} \to Tr$ Ayoub / Cisinski-Déglise

This has a Groth. 6-functor formalism

$f : X \to Y$ $\leadsto$ $f^* : SH(Y) \to SH(X)$

$f^* \dashv f_*$, $f_! \dashv f^!$, $\wedge$, $\underline{Hom}$

auxiliary gadgets for f smooth

have $f_\# \dashv f^*$

Also for $V \to X$ v. bundle $\Sigma^V, \Sigma^{-V} : SH(X) \circlearrowleft$

extends to $\Sigma^- : K(X) \to \mathrm{Aut}(SH(X))$

$\Sigma^V(\alpha) = Th_X(V)$ & $Th(V) = \Sigma^\infty_T V/V\setminus\{0\}$

$\mathcal{O}_X^N - [V] = [N]$ X affine

$[-V] = [N] - [\mathcal{O}_X^N]$

$Th_X(\mathcal{O}_X) = \mathbb{P}^1_X/\infty_X$ SH(X) invert $[\mathbb{P}^1/\infty]$

$\cancel{\mathbb{P}^1/\mathbb{P}^1\setminus\{0\}} = \mathbb{A}^1/\mathbb{A}^1\setminus\{0\} \simeq \mathbb{P}^1/\mathbb{P}^1\setminus\{0\}^{\infty}$

$= \mathbb{P}^1/\infty \simeq \mathbb{P}^1/\mathbb{A}^1$

$Y \xrightarrow{f} X$ smooth

$f_\#(1_Y) \in SH(X)$

$= \Sigma^\infty_T Y/X$

$m(X)$ $\quad X \in \mathrm{Sm}/k$ $\quad$ These are

$\| \quad \pi_{X\#}(1_X) \in SH(k)$ $\quad$ dualizable

(after inverting char $k > 0$)

$Z \in \mathrm{Sch}/k$

$\pi_{Z!}(1_Z) \in SH(k)$ also dualizable

$Z \hookrightarrow BM(Z)$

<u>Note</u> $X \xrightarrow{f} Y$ smooth

$$f_! = f_\# \circ \Sigma^{-T_f} \qquad f^! = \Sigma^{T_f} \circ f^*$$

$\hookrightarrow$ $\chi_{SH(k)}(\pi_{X\#}(1_X)) = \chi(X/k) \in GW(k)$

$\chi_{SH(k)}(\pi_{Z!}(1_Z)) = \chi_c(Z/k)$

Local duality (Jin-?, ...)

$$\mathbb{D}(\alpha) = \underline{\mathrm{Hom}}(\alpha, \pi_Z^! 1_k)$$

$\alpha \in SH(Z)$

$X \to k$ smooth $\quad \pi_X^! 1_k = Th_X(T_X)$

$$Sm/X \to SH(X)$$

$\to \quad Y \xrightarrow{f} X \quad \mapsto \quad f_\#(1_Y)$

$h \uparrow \quad \nearrow g \qquad \qquad \uparrow h$

$Y' \qquad \qquad g_\#(1_{Y'})$

$f_! \xrightarrow{\eta_f} f_*$

iso if f is proper

$$\left(Sch/X\right)^{prop, op} \to SH(X)$$

$\to Z \xrightarrow{f} X \quad \mapsto \quad f_!(1_Z)$

<u>Yes</u>

Some properties of $X(-/k)$, $X(-/1)$

[illegible]

- for X smooth & proper $/k$ $\chi(X/k) = \chi_c(X/k)$
- for U smooth of dim d/k $\chi(U/k) = \langle -1\rangle^d \cdot \chi_c(U/k)$

(X smooth & proper of odd dim $/k$

$\chi(X/k) = m \cdot H \qquad H(x,y) = x^2 - y^2$

$\langle a\rangle \cdot H = H$)

- $Z \hookrightarrow X \quad U = X \setminus Z$ then

$$\chi_c(X/k) = \chi_c(Z/k) + \chi_c(U/k)$$

- $\chi(\mathbb{A}^n/k) = 1 \quad , \quad \chi_c(\mathbb{A}^n/k) = \langle -1\rangle^n$
- $k \hookrightarrow \mathbb{C}$ rank $\chi_c(Z/k) = \chi_c^{top}(Z(\mathbb{C}))$
- $k \hookrightarrow \mathbb{R}$ sig $\sigma_* \chi_c(Z/k) = \chi_c^{top}(Z(\mathbb{R}))$

$\sigma_*^\alpha : GW(k) \to GW(\mathbb{R})$

- $\chi(-/k)$ mult. (for $N_{L/k}$ locally trivial bundles)

§ Gauss-Bonnet Formulas

Take $\mathcal{E} \in SH(k)$ "cohomology theory"

v-Twisted $\mathcal{E}$-cohomology of X $v \in K(X)$

$$\mathcal{E}^{a,b}(X, v) = \mathrm{Hom}_{SH(X)}(1_X, \Sigma^{a,b} \Sigma^{v} \pi_X^* \mathcal{E})$$

$\Sigma_{\mathbb{G}_m} = \Sigma^{1,1}$ $\Sigma_{S^1} = \Sigma^{1,0}$ both invertible

$\Sigma_{\mathbb{P}^1} = \Sigma_{\mathbb{G}_m} \Sigma_{S^1} = \Sigma^{2,1}$

Euler class (Degliso-Jin-Khan)

$v = [V]$ $V \to X$ rank n v. bundle

$$X \xrightarrow{s_0} V \to V/_{V \setminus \{0\}} = Th_X(V)$$

stable

$$1_X \to \Sigma^V 1_X$$

$$e(V) \in Hom_{SH(X)}(1_X, \Sigma^V \pi_X^* 1_k)$$

$$\in 1_k^{0,0}(X, V)$$

$$1_k \xrightarrow{u} \mathcal{E}$$

$$e^{\mathcal{E}}(V) \in \mathcal{E}^{0,0}(X, V)$$

$\pi_{X\#} = \pi_{X!} \Sigma^{T_X}$

B-M homology

$$\mathcal{E}^{BM}_{a,b}(X,\nu) = \mathrm{Hom}\left(\Sigma^{a,b}\Sigma^{\pi_X^!\nu} 1_X, \pi_X^!\mathcal{E}\right)$$

X smooth over k "points iso"

$$\mathcal{E}^{a,b}(X,\nu) = \mathcal{E}^{BM}_{-a,-b}(X, T_X - \nu)$$

<u>Ex</u> $\nu = T_X$

$$e(T_X) \in \mathcal{E}^{0,0}(X, T_X) = \mathcal{E}^{BM}_{0,0}(X, 0)$$

$\pi_X : X \to \mathrm{Spec}\, k$ ~~smooth~~ proper

$$\downarrow \pi_{X*}$$

$$\mathcal{E}_{0,0}(\mathrm{Spec}\, k) \cong \mathcal{E}^{BM}_{0,0}(\mathrm{Spec}\, k, 0)$$

$\pi_{X*}(e^{\mathcal{E}}(T_X)) \in \mathcal{E}_{0,0}(\operatorname{Spec} k)$

<u>Thm</u> (motivic Gauss-Bonnet - DJK)

$u : 1_k \to \mathcal{E}$ unit. Then

$u(\chi(X/k)) = \pi_{X*}(e^{\mathcal{E}}(T_X))$

$\in \mathcal{E}_{0,0}(\operatorname{Spec} k)$

<u>Thm</u> (concrete GB - L.-Raksit)

Take X smooth + proper over k ($n = \dim_k X$)

$$\chi(X/k) = \left(\bigoplus_{i,j} H^j(X, \Omega^i_{X/k})[i-j], \operatorname{Tr}(-\wedge-)\right)$$

$\in GW(k)$ $\qquad v^r$

$$\begin{tikzcd}
H^i(X, \Omega^i)[i-j] \times H^{n-j}(X, \Omega^{n-i}_{X/k})[n-i] \arrow[r, "- \wedge -"] \arrow[d, "\mathrm{Tr}(- \wedge -)"'] & H^n(X, \Omega^n_{X/k}) \arrow[dl, "\mathrm{Tr}"] \\
k &
\end{tikzcd}$$

Case $\underline{n \text{ odd}}$ then $\chi(X/k)$ is hyperbolic

n odd $\Rightarrow$ nothing in $\bigoplus H^i(\Omega^i)[i-j]$

gets paired with itself

n even $\chi(X/k) = M \cdot H + q$

$n = 2m$

q Hodge, m

$$H^m(\Omega^m) \to k$$

$$\eta \to \mathrm{Tr}(\eta^2)$$

1 word about pf — use hermitian K^{th} KQ

In SH(k)
Homotopical
Schlichting
Tripathi

• $1_k \xrightarrow{u} KQ$ — index 1 ??

$GW(k) = 1_k^{0,0}(k) \xrightarrow{u_*} \underline{\underline{KQ}}^{0,0}(k)$

• $e^{KQ}(V) = KO(K_\infty(V^\vee, 0), \sim \wedge \sim)$

rank

$\bigoplus \Lambda^i V^\vee[i]$

$\Lambda^i V^\vee[i] \otimes \Lambda^{n-i} V^\vee[n-i] \to \Lambda^n V^\vee$

$KQ^{2n,n}(X; \det V)$

$\det^{-1} V[n]$

X smooth w proper / k

π_{X*} on KQ with twist — Serre duality

$V \xrightarrow{\varphi} \mathrm{Hom}(V, \omega_{X/k}) \in KO^{2n,n}(X, \omega)$

$$\pi_{\text{fdg}}(V,\varphi) : R\pi_{0}V \to R_{\pi_0}\mathrm{Hom}(\ )$$

$$\pi_{fg}(\varphi) \searrow \overset{sh}{\mathrm{Hom}}(R\pi_0 V, k)$$

$V = T_X$

$$e(T_X) = -n - a \oplus \Omega^{i}_{X/k}(i)$$

$$\pi_{fdg}(e T_X) = T_2(-n-)$$

Conductor formula

Ex $F \in k[X_0 \ldots X_n]_e \rightsquigarrow V(F) \subset \mathbb{P}^n_k$ smooth

$(e, \mathrm{char}\, k) = 1$

$$\tilde{F} = F(X_0 \ldots X_n) - tX^e$$

$$\boxed{F - tX^{e}_{n+1}} \rightsquigarrow ? \quad F - pX^{e}_{n+1} \text{ in mixed char.?}$$

$n+1$

$$X = V(\widetilde{F}) \subset \mathbb{P}^{n+1} \times \mathbb{A}^1 \quad \text{smooth}$$

$$\Big\downarrow f \qquad\qquad \mathbb{A}^1 \ni t$$

$\mathbb{A}^1$ $\quad X_t : V(F_t) \subset \mathbb{P}^{n+1}$ smooth $t \neq 0$

? Conductor formula?

$$X_0 = \mathrm{Cone}(V(F))$$

vertex $(0 \sim 0,1) \in X_0$

$$\chi(X_t/k(t)) \in GW(k(t)) \quad \langle u \rangle \quad \langle t \rangle$$

$$\Big\downarrow sp_t \qquad \Big\downarrow \qquad \Big\downarrow$$

$$GW(k) \quad \langle u(0) \rangle \quad \langle 1 \rangle$$

$$\Delta_t(X) = sp_t\, \chi(X_t/k(t)) - \underset{\shortparallel}{\chi_c}(X_0/k)$$

For $\cancel{X}\ Y = V(F) \subset \mathbb{P}^n_{k}$

Griffiths

Carlson-Griffiths

$$H^{q}(\Omega^{p}_{Y})_{prim} \overset{\sim}{=} J(F)_{(q+1)e-n-2}$$

mult ion

$$e_F \in J(F)_{(e-2)(n+1)} \xrightarrow{T_n} k \qquad g_F(x) = T_n(x^2)$$

Pepin-Lehalleur, Srinivas

$$\Rightarrow \quad \mathcal{K}\left(\bigoplus H^{q}(\Omega^{p}_{Y}), T_n(-\cdot-)\right)$$

$$\cong \qquad \dim = \text{even}$$

$$\langle e \rangle + \langle -e \rangle \cdot g_F$$

use this formula get the expression mentioned at the beginning

$\Delta_f(X)$ ~ lifts to $K_0(\mathrm{Var}_k)$

Denef-Loeser +

Algonb - Turn - Sebaj

$$\boxed{\gamma_f = e_{X_0}(\gamma_f)}$$

$X \xrightarrow{f} \mathbb{A}^1$

$X_0 \quad \partial X_0 \quad \sum x_i \, dx_i^{\wedge}$

$C^{n+1}(\mathcal{U}, \mathbb{P}^{n+1})$

$A \quad \Omega^{n+2}\big/\Pi\frac{\partial f}{\partial x_i}\,(e-1)(n+2)$

$X \subset \mathbb{P}^{n+1}$

$\Omega^{n+1} \qquad (e-2)(n+2) \qquad \mathcal{O}_{\mathbb{P}^{n+1}}(e-n-2)$

$\Omega^{n+1}_{\mathbb{P}^{n+1}}(X)$

$\downarrow \mathrm{res}$

$\Omega^n_X \qquad i_x$

$H^n(X, \Omega^n_X) \xrightarrow{\delta} H^{n+1}(\mathbb{P}, \Omega^{n+1})$

$T_n \backslash \quad / T_n$

k