Gradient Gibbs measures

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Abstract

In the talk we define (gradient) Gibbs measures of physical models with a countable set of spin values. For SOS (solid-on-solid) model, with spin values from the set of all integers, on a Cayley tree we give some gradient Gibbs measures (GGMs) of the model. Such a measure corresponds to a boundary law (a function defined on vertices of Cayley tree) satisfying a functional equation. In the ferromagnetic SOS case we give several concrete GGMs which correspond to periodic boundary laws.

Preliminaries

σ-algebra, Hamiltonian. The study of random functions ξ_x from a lattice $\mathbb{L}$ (usually $\mathbb{Z}^d$ or Γ^k) to a measure space $(E, \mathcal{E})$ is a central component of ergodic theory and statistical physics.
In many classical models from physics (e.g., the Ising model, the Potts model), E is a finite set (i.e., with a finite underlying measure λ), and ξ_x has a physical interpretation as the spin of a particle at location x in a crystal lattice.

Since 2018 we interested to the models, where $(E, \mathcal{E})$ is a space with an infinite underlying measure λ (i.e. $\mathbb{L}$ with counting measure) where $\mathcal{E}$ is the Borel σ-algebra of E and ξ_x usually has a physical interpretation as the spatial position of a particle at location x in a lattice.

First such models were considered in[1].

The prime examples of unbounded spin systems are harmonic oscillators. Another example is the Ginzburg-Landau interface model; which is obtained from the anharmonic oscillators[2]

[1] Funaki, T., Spohn, H. **Comm. Math. Phys**. 185 (1997), no. 1, 1–36.

[2] H.O. Georgii, *Gibbs Measures and Phase Transitions*, Berlin, 2011.

Denote by Ω the set of functions from $\mathbb{L}$ to E, such a function also is called a configuration.
Assume random field $(\xi_x)_{x\in\mathbb{L}}$ on Ω given as the projection onto the coordinate $x \in \mathbb{L}$:

$$\xi_x(\omega) = \omega(x) = \omega_x, \quad \omega \in \Omega.$$

If $\Lambda \subset \mathbb{L}$, we denote by $\mathcal{F}_\Lambda$ the smallest σ-algebra with respect to which ξ_x is measurable for all $x \in \Lambda$. We write $\mathcal{T}_\Lambda = \mathcal{F}_{\mathbb{L}\setminus\Lambda}$.
A subset of Ω, is called a cylinder set if it belongs to $\mathcal{F}_\Lambda$ for some finite set $\Lambda \subset \mathbb{L}$.
Let $\mathcal{F}$ be the smallest σ-algebra on Ω containing the cylinder sets.
We write $\mathcal{T}$ for the tail-σ-algebra, i.e., intersection of $\mathcal{T}_\Lambda$ over all finite subsets Λ of $\mathbb{L}$ the sets in $\mathcal{T}$ are called tail-measurable sets.

Assume that we are given a family of measurable potential functions $\Phi_\Lambda : \Omega \to \mathbb{R} \cup \{\infty\}$ (one for each finite subset Λ of $\mathbb{L}$) each Φ_Λ is $\mathcal{F}_\Lambda$ measurable.
For each finite subset Λ of $\mathbb{L}$ we also define a Hamiltonian:

$$H_\Lambda(\sigma) = \sum_{\substack{\Delta \subset \mathbb{L}: \\ \Delta \cup \Lambda \neq \emptyset}} \Phi_\Delta(\sigma),$$

where the sum is taken over finite subsets Δ.

Gibbs Measures.

To define Gibbs measures and gradient Gibbs measures, we will need some additional notation[3].

Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be general measure spaces.

A function $\pi : \mathcal{X} \times Y \to [0, \infty]$ is called a probability kernel from $(Y, \mathcal{Y})$ to $(X, \mathcal{X})$ if

1. $\pi(\cdot|y)$ is a probability measure on $(X, \mathcal{X})$ for each fixed $y \in Y$, and
2. $\pi(A|\cdot)$ is $\mathcal{Y}$-measurable for each fixed $A \in \mathcal{X}$.

Such a kernel maps each measure μ, on $(Y, \mathcal{Y})$ to a measure $\mu\pi$ on $(X, \mathcal{X})$ by

$$\mu\pi(A) = \int \pi(A|\cdot) d\mu$$

[3] S. Sheffield, Random surfaces: Large deviations principles and gradient Gibbs measure classifications. Thesis (Ph.D.) Stanford University. 2003.

The following is a probability kernel from $(\Omega, \mathcal{T}_\Lambda)$ to $(\Omega, \mathcal{F})$:

$$\gamma_\Lambda(A, \omega) = Z_\Lambda(\omega)^{-1} \int \exp(-H_\Lambda(\sigma_\Lambda \omega_{\Lambda^c})) \mathbf{1}_A(\sigma_\Lambda \omega_{\Lambda^c}) \nu^{\otimes \Lambda}(d\sigma_\Lambda),$$

where $\nu = \{\nu(i) > 0, i \in E\}$ is a counting measure.
We say σ has finite energy if $\Phi_\Lambda(\sigma) < \infty$ for all finite Λ. We say σ is Φ-admissible if each $Z_\Lambda(\sigma)$ is finite and non-zero.
Given a measure μ on $(\Omega, \mathcal{F})$, we define a new measure $\mu\gamma_\Lambda$ by

$$\mu\gamma_\Lambda(A) = \int \gamma_\Lambda(A, \cdot) d\mu$$

Definition 1. A probability measure μ on $(\Omega, \mathcal{F})$ is called a Gibbs measure if μ is supported on the set of Φ-admissible configurations in Ω and for all finite subset Λ we have

$$\mu\gamma_\Lambda = \mu.$$

Gradient Gibbs measure

For any configuration $\omega = (\omega(x))_{x\in\mathbb{L}} \in E^{\mathbb{L}}$ and edge $b = \langle x, y\rangle$ of $\mathbb{L}$ the *difference* along the edge b is given by $\nabla\omega_b = \omega_y - \omega_x$ and we also call $\nabla\omega$ the *gradient field* of ω.
The gradient spin variables are now defined by $\eta_{\langle x,y\rangle} = \omega_y - \omega_x$ for each $\langle x, y\rangle$.
The space of *gradient configurations* denoted by Ω^∇. The measurable structure on the space Ω^∇ is given by σ-algebra

$$\mathcal{F}^\nabla := \sigma(\{\eta_b \,|\, b \in \mathbb{L}\}).$$

Note that $\mathcal{F}^\nabla$ is the subset of $\mathcal{F}$ containing those sets that are invariant under translations $\omega \to \omega + c$ for $c \in E$.
Similarly, we define

$$\mathcal{T}^\nabla_\Lambda = \mathcal{T}_\Lambda \cap \mathcal{F}^\nabla, \ \ \mathcal{F}^\nabla_\Lambda = \mathcal{F}_\Lambda \cap \mathcal{F}^\nabla.$$

Let Φ be an translation invariant gradient potential. Since, given any $A \in \mathcal{F}^\nabla$, the kernels $\gamma_\Lambda^\Phi(A, \omega)$ are $\mathcal{F}^\nabla$-measurable functions of ω, it follows that the kernel sends a given measure μ on $(\Omega, \mathcal{F}^\nabla)$ to another measure $\mu\gamma_\Lambda^\Phi$ on $(\Omega, \mathcal{F}^\nabla)$.

Definition 2. A measure μ on $(\Omega, \mathcal{F}^\nabla)$ is called a gradient Gibbs measure if for all finite subset Λ we have

$$\mu\gamma_\Lambda^\Phi = \mu.$$

Note that, if μ is a Gibbs measure on $(\Omega, \mathcal{F})$, then its restriction to $\mathcal{F}^\nabla$ is a gradient Gibbs measure.

A gradient Gibbs measure is said to be localized or smooth if it arises as the restriction of a Gibbs measure in this way. Otherwise, it is non-localized or rough.

It is known[4] that many natural Gibbs measures are rough when $d \in \{1, 2\}$.

[4] H.O. Georgii, *Gibbs Measures and Phase Transitions*, Berlin, 2011.

Construction of gradient Gibbs measure on Cayley tree.
Following[5] we consider models where spin-configuration ω is a function from the vertices of the Cayley tree $\Gamma^k = (V, \vec{L})$ to the set $E = \mathbb{Z}$.
For nearest-neighboring interaction potential $\Phi = (\Phi_b)_b$, where bonds are denoted $b = \langle x, y \rangle$, define symmetric transfer matrices Q_b by

$$Q_b(\omega_b) = e^{-\left(\Phi_b(\omega_b) + |\partial x|^{-1}\Phi_{\{x\}}(\omega_x) + |\partial y|^{-1}\Phi_{\{y\}}(\omega_y)\right)}.$$

Define the Markov (Gibbsian) specification as

$$\gamma_\Lambda^\Phi(\sigma_\Lambda = \omega_\Lambda | \omega) = (Z_\Lambda^\Phi)(\omega)^{-1} \prod_{b \cap \Lambda \neq \emptyset} Q_b(\omega_b).$$

[5] C. Külske, P. Schriever. *Gradient Gibbs measures and fuzzy transformations on trees*, **Markov Process. Relat. Fields**, 23, (2017), 553-590.

If for any bond $b = \langle x, y \rangle$ the transfer operator $Q_b(\omega_b)$ is a function of gradient spin variable $\zeta_b = \omega_y - \omega_x$ then the underlying potential Φ is called a *gradient interaction potential*.

Introduce a notion of *boundary laws* that allows to describe the Gibbs measures that are Markov chains on trees.

Definition 3. A family of vectors $\{l_{xy}\}_{\langle x,y \rangle \in \vec{L}}$ with $l_{xy} = \{l_{xy}(i) : i \in \mathbb{Z}\} \in (0, \infty)^{\mathbb{Z}}$ is called a *boundary law for the transfer operators* $\{Q_b\}_{b \in \vec{L}}$ if for each $\langle x, y \rangle \in \vec{L}$ there exists a constant $c_{xy} > 0$ such that the consistency equation

$$l_{xy}(i) = c_{xy} \prod_{z \in \partial x \setminus \{y\}} \sum_{j \in \mathbb{Z}} Q_{zx}(i, j) l_{zx}(j) \tag{1}$$

holds for every $i \in \mathbb{Z}$.

A boundary law is called *q-periodic* if $l_{xy}(i+q) = l_{xy}(i)$ for every oriented edge $\langle x, y \rangle \in \vec{L}$ and each $i \in \mathbb{Z}$.

It is known that there is a one-to-one correspondence between boundary laws and tree-indexed Markov chains if the boundary laws are *normalisable* in the sense of Zachary[6]:

Definition 4. A boundary law l is said to be *normalisable* if and only if

$$\sum_{i \in \mathbb{Z}} \Big(\prod_{z \in \partial x} \sum_{j \in \mathbb{Z}} Q_{zx}(i,j) l_{zx}(j) \Big) < \infty \tag{2}$$

at any $x \in V$.

[6] S. Zachary, *Countable state space Markov random fields and Markov chains on trees*, **Ann. Probab.** 11(4) (1983), 894–903.

The correspondence now reads the following:

Theorem 1 (Zachary) For any Markov specification γ with associated family of transfer matrices $(Q_b)_{b \in L}$ we have

1. Each *normalisable* boundary law $(l_{xy})_{x,y}$ for $(Q_b)_{b \in L}$ defines a unique tree-indexed Markov chain $\mu \in \mathcal{G}(\gamma)$ via the equation given for any connected set $\Lambda \in \mathcal{S}$

$$\mu(\sigma_{\Lambda \cup \partial \Lambda} = \omega_{\Lambda \cup \partial \Lambda}) = (Z_\Lambda)^{-1} \prod_{y \in \partial \Lambda} l_{y y_\Lambda}(\omega_y) \prod_{b \cap \Lambda \neq \emptyset} Q_b(\omega_b), \quad (3)$$

   where for any $y \in \partial \Lambda$, y_Λ denotes the unique *n.n.* of y in Λ.

2. Conversely, every tree-indexed Markov chain $\mu \in \mathcal{G}(\gamma)$ admits a representation of the form (3) in terms of a *normalisable* boundary law (unique up to a constant positive factor).

Remark

The Markov chain μ defined in (3) has the transition probabilities

$$P_{xy}(i,j) = \mu(\sigma_y = j \mid \sigma_x = i) = \frac{l_{yx}(j)Q_{yx}(j,i)}{\sum_s l_{yx}(s)Q_{yx}(s,i)}. \tag{4}$$

The expressions (4) may exist even in situations where the underlying boundary law $(l_{xy})_{x,y}$ is not normalisable. However, the Markov chain given by 4, in general, does not have an invariant probability measure. Thus, there is no obvious extension of Theorem 1 to non-normalisable boundary laws.
Therefore in[7] the non-normalisable boundary laws are used to give gradient Gibbs measures.

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- C. Külske, P. Schriever. **Markov Process. Relat. Fields**, 23, (2017), 553-590.
- F. Henning, C. Külske, A. LeNy, U. Rozikov. **Elect. Jour. Probab.** 24, (2019), 1–23.

SOS model.

Now we give some results of above mentioned papers. Consider a model on Cayley tree $\Gamma^k = (V, \vec{L})$, where the spin takes values in the set of all integer numbers $\mathbb{Z}$. The set of all configurations is $\Omega := \mathbb{Z}^V$.

The (formal) Hamiltonian of the SOS model is

$$H(\omega) = -J \sum_{\{x,y\} \in L} |\omega_x - \omega_y|, \quad \omega \in \Omega, \tag{5}$$

where $J \in \mathbb{R}$ is a constant and $\{x, y\}$ denotes nearest neighbor vertices.

Then fix a site $w \in \Lambda$. If the boundary law l is assumed to be q-periodic, then take $s \in \mathbb{Z}_q$ and define probability measure $\nu_{w,s}$ on $\mathbb{Z}^{\{b \in \vec{L} | b \subset \Lambda\}}$ by

$$\nu_{w,s}(\eta_{\Lambda \cup \partial\Lambda} = \zeta_{\Lambda \cup \partial\Lambda}) =$$

$$Z_{w,s}^{\Lambda} \prod_{y \in \partial\Lambda} l_{y y_\Lambda}\Big(T_q(s + \sum_{b \in \Gamma(w,y)} \zeta_b)\Big) \prod_{b \cap \Lambda \neq \emptyset} Q_b(\zeta_b),$$

where $Z_{w,s}^{\Lambda}$ is a normalization constant, $\Gamma(w, y)$ is the unique path from w to y and $T_q : \mathbb{Z} \to \mathbb{Z}_q$ denotes the coset projection. In[8] the following theorem is proved:

[8] C. Külske, P. Schriever. **Markov Process. Relat. Fields**, 23, (2017), 553-590.

Theorem 2. (Külske, Schriever) Let $(l_{<xy>})_{<x,y>\in\vec{L}}$ be any q - periodic boundary law to some gradient interaction potential. Fix any site $w \in V$ and any class label $s \in \mathbb{Z}_q$. Then

$$\nu_{w,s}(\eta_{\Lambda\cup\partial\Lambda} = \zeta_{\Lambda\cup\partial\Lambda}) =$$

$$Z_{w,s}^{\Lambda} \prod_{y\in\partial\Lambda} l_{yy_\Lambda}\Big(T_q(s + \sum_{b\in\Gamma(w,y)} \zeta_b)\Big) \prod_{b\cap\Lambda\neq\emptyset} Q_b(\zeta_b), \qquad (6)$$

gives a consistent family of probability measures on the gradient space Ω^∇. Here Λ with $w \in \Lambda \subset V$ is any finite connected set, $\zeta_{\Lambda\cup\partial\Lambda} \in \mathbb{Z}^{\{b\in\vec{L}|b\subset(\Lambda\cup\partial\Lambda)\}}$ and $Z_{w,s}^{\Lambda}$ is a normalization constant. The measures $\nu_{w,s}$ will be called pinned gradient measures.

If q-periodic boundary law and the underlying potential are translation invariant then it is possible to obtain probability measure ν on the the gradient space by mixing the pinned gradient measures:

Theorem 3.(Külske, Schriever) Let a q-periodic boundary law l and its gradient interaction potential are translation invariant. Let $\Lambda \subset V$ be any finite connected set and let $w \in \Lambda$ be any vertex. Then the measure ν with marginals given by

$$\nu(\eta_{\Lambda\cup\partial\Lambda} = \zeta_{\Lambda\cup\partial\Lambda}) = Z_\Lambda \left(\sum_{s\in\mathbb{Z}_q} \prod_{y\in\partial\Lambda} l(s + \sum_{b\in\Gamma(w,y)} \zeta_b) \right) \prod_{b\cap\Lambda\neq\emptyset} Q(\zeta_b), \tag{7}$$

where Z_Λ is a normalisation constant, defines a translation invariant gradient Gibbs measure on Ω^∇.

Now using Theorem 3 and following[9] we give some gradient Gibbs measures.

Let $\beta > 0$ be inverse temperature and $\theta := \exp(-\beta) < 1$. The transfer operator Q then reads $Q(i-j) = \theta^{|i-j|}$ for any $i, j \in \mathbb{Z}$, and a translation invariant boundary law, denoted by $\mathbf{z}$, is any positive function on $\mathbb{Z}$ solving the consistency equation, whose values we will denote by z_i instead of $z(i)$. By definition of the boundary law it is only unique up to multiplication with any positive prefactor. Hence we may choose this constant in a way such that we have $z_0 = 1$.

Set $\mathbb{Z}_0 := \mathbb{Z} \setminus \{0\}$. Then the boundary law equation reads

$$z_i = \left(\frac{\theta^{|i|} + \sum_{j \in \mathbb{Z}_0} \theta^{|i-j|} z_j}{1 + \sum_{j \in \mathbb{Z}_0} \theta^{|j|} z_j} \right)^k, \quad i \in \mathbb{Z}_0. \tag{8}$$

[9] F. Henning, C. Külske, A. LeNy, U. Rozikov. **Elect. Jour. Probab.** 24, (2019), 1–23.

Let $\mathbf{z}(\theta) = (z_i = z_i(\theta), i \in \mathbb{Z}_0)$ be a solution to (8). Denote

$$l_i \equiv l_i(\theta) = \sum_{j=-\infty}^{-1} \theta^{|i-j|} z_j, \quad r_i \equiv r_i(\theta) = \sum_{j=1}^{\infty} \theta^{|i-j|} z_j, \quad i \in \mathbb{Z}_0. \tag{9}$$

Note that each l_i and r_i can be a finite positive number or $+\infty$.
Lemma. For each $i \in \mathbb{Z}_0$ we have

- $l_i < +\infty$ if and only if $l_0 < +\infty$;
- $r_i < +\infty$ if and only if $r_0 < +\infty$.

In what follows, we will assume that $l_0 < +\infty$ and $r_0 < +\infty$.

Denote $u_i = \sqrt[k]{z_i}$ and assume $u_0 = 1$.
Then we have the following
Proposition. If $z_0 = 1$ (i.e. $u_0 = 1$) then the equation (8) is equivalent to the following

$$u_i^k = \frac{u_{i-1} + u_{i+1} - \tau u_i}{u_{-1} + u_1 - \tau}, \quad i \in \mathbb{Z}, \tag{10}$$

where $\tau = \theta^{-1} + \theta$.

The following theorem is proved for $k = 2$ and 4-periodic boundary laws:

Theorem 4. For the SOS model (5) on the binary tree (i.e. $k = 2$) with parameter $\tau = \theta + \theta^{-1}$ the following assertions hold

1. If $\tau \leq 4$ then there is precisely one GGM associated to a 4-periodic boundary law.
2. If $4 < \tau \leq 6$ then there are precisely two GGMs.
3. If $6 < \tau < 2 + 2\sqrt{5}$ then there are precisely three GGMs.
4. If $\tau \geq 2 + 2\sqrt{5}$ then there are precisely four such measures.

The following theorem is proved for any $k \geq 2$ and 3-periodic boundary laws.
Denote

$$\tau_0 := \frac{2k+1}{k-1}.$$

Theorem 5. For the SOS-model on the k-regular tree, $k \geq 2$, with parameter τ there is τ_c such that $0 < \tau_c < \tau_0$ and the following holds:

1. If $\tau < \tau_c$ then there are no GGM corresponding to nontrivial 3-periodic boundary.
2. At $\tau = \tau_c$ there is a unique GGM corresponding to a nontrivial 3-periodic boundary law.
3. For $\tau > \tau_c$, $\tau \neq \tau_0$ (resp. $\tau = \tau_0$) there are exactly two such (resp. one) GGMs.

The GGMs described above are all different from the GGMs mentioned in Theorem 4.

Thank you!