Hyperbolic CR singularities

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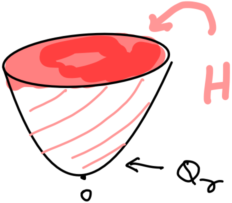
Holomorphic extension

Holomorphic Hull $(K) = K$: $\nexists$ open set $\supset K$ to which all holom. fct on K can be extended to

Q_γ: $z_2 = |z_1|^2 + \gamma(z_1^2 + \bar{z}_1^2)$, $0 < \gamma < 1/2$, real surface in $\mathbb{C}^2$.



Bishop ('65)

Hull(Q_γ): fills the "interior".

Surfaces with CR singularity

Surface with CR singularity : real analytic surface $M \subset (\mathbb{C}^2, 0)$:

$$M : z_2 = z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2) + O^3(z_1, \bar{z}_1), \quad \gamma \geq 0.$$

r.a. perturbation of the *Bishop quadric* $Q_\gamma : z_2 = z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2)$
$\gamma \in \mathbb{R}^+$ — *Bishop invariant*

If $\gamma \neq \frac{1}{2}$, the origin is an isolated *Cauchy-Riemann singularity* :

- $\forall\ p \neq 0$, "$\mathbb{C} \not\subset T_pM$" (ie. totally real at $p \neq 0$)
- $T_0M = \{z_2 = 0\}$

M is said to be :

- *elliptic* si $0 \leq \gamma < \frac{1}{2}$
- *hyperbolic* if $\gamma > \frac{1}{2}$
- *parabolic* if $\gamma = \frac{1}{2}$

Geometry near an elliptic CR singularity

Questions

- *Holomorphic Flattening* : is $\phi(M) \subset \mathrm{Im}(z_2) = 0$?
- *What is the local hull of holomorphy ?*

Answers through :

- Normal form of M with respect to holomorphic change of coordinates near the origin.

Normalization near an elliptic CR singularity

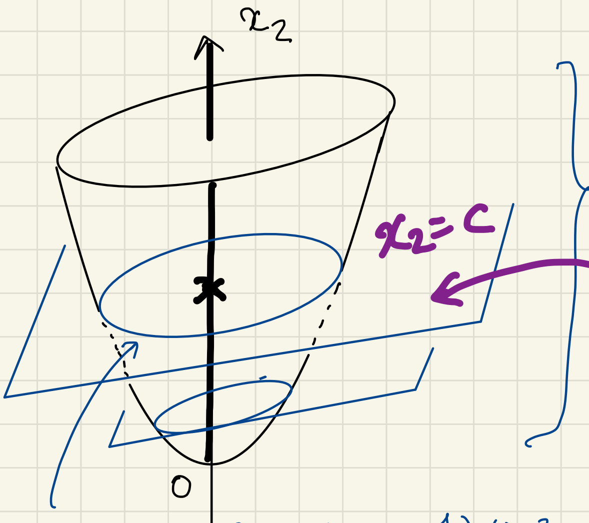
Theorem (Moser-Webster 1983)

If $0 < \gamma < \frac{1}{2}$*, there exists a holomorphic change of variables near the origin such that* M *reads*

$$x_2 = z_1\bar{z}_1 + (\gamma + \delta x_2^s)(z_1^2 + \bar{z}_1^2), \quad y_2 = 0, \quad z_2 = x_2 + \mathrm{i}y_2$$

avec $\delta = \pm 1$ *si* $s \in \mathbb{N}^*$ *ou* $\delta = 0$ *si* $s = \infty$.

Elliptic case



real analytic family (in c) of holomorphic curve intersecting M along an ellipsis.

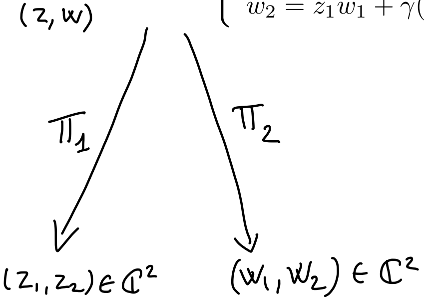
$$\underbrace{c = |z_1|^2 + (\gamma + c^1)(\bar{z}_1^2 + z_1^2)}_{\mathcal{E}_c \,=\, \text{ellipsis}}$$

This follows from Moser-Webster thm, using holomorphic coordinates in which M is a normal form.

Complexification of M

Complexification of M: $(z_1, z_2, \bar{z}_1, \bar{z}_2) \leftarrow (z_1, z_2, w_1, w_2) =: (z, w) \in \mathbb{C}^4$

$$\mathcal{M} \subset \mathbb{C}^4 : \begin{cases} z_2 = z_1 w_1 + \gamma(z_1^2 + w_1^2) + H(z_1, w_1) \\ w_2 = z_1 w_1 + \gamma(z_1^2 + w_1^2) + \bar{H}(w_1, z_1) \end{cases}$$



2-1 branched covering

↓

2 pre-images (a.e)

↓

Deck transformations are involutions

Moser-Webster involutions

$\rightsquigarrow$ pair of holomorphic involutions: pour $\gamma > 0$

$$\tau_1 : \begin{cases} z_1' = -z_1 - \frac{1}{\gamma} w_1 + \underbrace{h_1(z_1, w_1)}_{\mathrm{ord}_0 \geq 2} \\ \\ w_1' = w_1 \end{cases} \quad - - - - - - - \ \tau_1 \circ \tau_1 = Id$$

$$\tau_2 : \begin{cases} z_1' = z_1 \\ w_1' = -\frac{1}{\gamma} z_1 - w_1 + h_2(z_1, w_1) \end{cases} \quad - - - - - - - \ \tau_2 \circ \tau_2 = Id$$

$$\tau_2 = \rho \tau_1 \rho, \quad \rho(z, w) := (\bar{w}, \bar{z})$$

Proposition (Moser-Webster 1983)

Holomorphic classification of surface $\mathcal{M} \in \mathbb{C}^4 \leftrightsquigarrow$ *Holomorphic classification of* (τ_1, τ_2)

Remark. Normal form of $M \subset \mathbb{C}^2 \leftrightsquigarrow$ Normal form of (τ_1, τ_2).

Appropriate coordinates

$$\tau_1 : \begin{cases} \xi' = \lambda\eta + \text{h.o.t.} \\ \eta' = \lambda^{-1}\xi + \text{h.o.t.} \end{cases}, \quad \tau_2 : \begin{cases} \xi' = \lambda^{-1}\eta + \text{h.o.t.} \\ \eta' = \lambda\xi + \text{h.o.t.} \end{cases},$$

$$\sigma := \tau_1 \circ \tau_2 : \begin{cases} \xi' = \lambda^2\xi + \text{h.o.t.} \\ \eta' = \lambda^{-2}\eta + \text{h.o.t.} \end{cases},$$

λ is a root of $\gamma\lambda^2 - \lambda + \gamma = 0$

Remark

- elliptic surface M, $0 < \gamma < \frac{1}{2} \Longrightarrow \lambda = \bar{\lambda}$ and $|\lambda| \neq 1$
  — origin is an hyperbolic fixed point of $\sigma = \tau_1 \circ \tau_2$
- hyperbolic surface M, $\gamma > \frac{1}{2} \Longrightarrow |\lambda| = 1$
  — origin is an elliptic fixed point of $\sigma = \tau_1 \circ \tau_2$

Normal forms of involutions

Theorem (Moser-Webster 1983, formal normal form)

Assume: *λ not a root of unity*
Conclusion : exists a unique formal normalized transformation ψ s.t.

$$\psi^{-1}\circ\tau_1\circ\psi:\begin{cases}\xi'=\Lambda(\xi\eta)\eta\\ \eta'=\Lambda^{-1}(\xi\eta)\xi\end{cases},\quad \psi^{-1}\circ\tau_2\circ\psi:\begin{cases}\xi'=\Lambda^{-1}(\xi\eta)\eta\\ \eta'=\Lambda(\xi\eta)\xi\end{cases},$$

where $\Lambda(t)\in\mathbb{C}[[t]]$. s.t. $\Lambda(t)=\bar\Lambda(t)$ (elliptic case) or $\Lambda(t)\cdot\bar\Lambda(t)=1$ (hyperbolic case).

Theorem (Moser-Webster 1983, Convergence in elliptic case)

If $\lambda=\bar\lambda$ and $|\lambda|\neq 1$, then Λ and ψ are holomorphic on a neighborhood of the origin.

$\Longrightarrow$ Holomorphic equivalence of inital manifold M to NF manifold

Non exceptional hyperbolic CR singularity

$|\lambda| = 1$ not a root of unity (*non exceptional*).
Moser-Webster $\rightsquigarrow$ normalizing transformation ψ might not converge at the origin: no holomorphic equivalence to a normal form and even, no holomorphic flattening.

Theorem (Gong 1994: non exceptional degenerate case)

Assumptions:

1. $|\lambda| = 1$ *and* λ *satisfies* *diophantine condition:*

$$|\lambda^n - 1| > \frac{c}{n^\delta}$$

2. M *formally equivalent to the quadric (i.e.* $\Lambda(\xi\eta) = \lambda$; τ_1 *et* τ_2 *are* *formally linearizable* *),*

Then, M *is holomorphically equivalent to the quadric (*$\leftrightsquigarrow$ ψ *is holomorphic in a neighborhood of the origin).*

Non degenerate hyperbolic CR singularity surface

$$\tau_1 : \begin{cases} \xi' = \lambda\eta + \text{h.o.t.} \\ \eta' = \lambda^{-1}\xi + \text{h.o.t.} \end{cases}, \quad \tau_2 : \begin{cases} \xi' = \lambda^{-1}\eta + \text{h.o.t.} \\ \eta' = \lambda\xi + \text{h.o.t.} \end{cases}$$

$$\lambda := e^{\frac{i}{2}\alpha}, \frac{\alpha}{\pi} \in \mathbb{R} \setminus \mathbb{Q}, \quad \Lambda(\xi\eta) = \lambda + \sum_{n\geq 1} \tilde{c}_n (\xi\eta)^n.$$

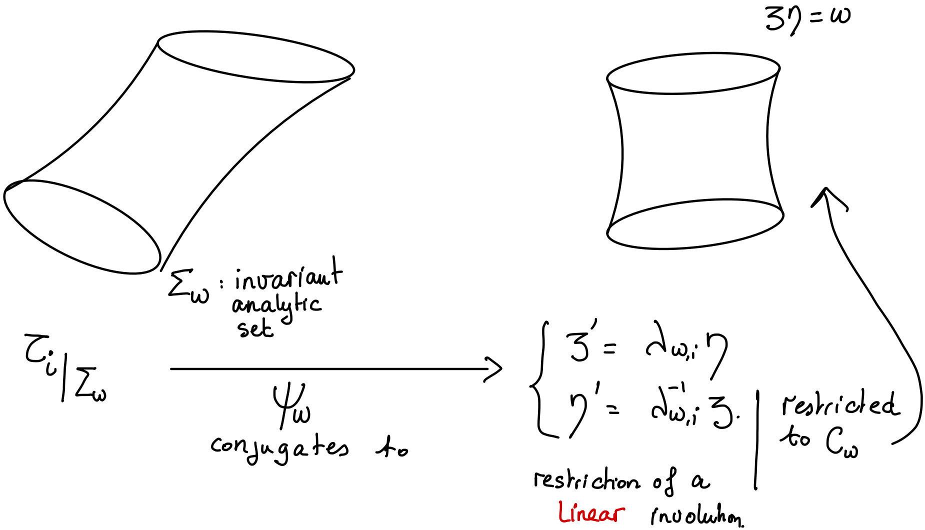
Theorem (S.-Zhao 2020)

Assume $\Lambda(\xi\eta) \neq \lambda$. If $r > 0$ is small enough, there exists a "asymptotic full measure" parameters set $\mathcal{O}_r \subset]-r^2, r^2[$ s.t. $\forall\ \omega \in \mathcal{O}_r$, $\exists\ \mu_\omega \in \mathbb{R}$ and an holomorphic transformation Ψ_ω, Whitney smooth in ω, on $\mathcal{C}^r_\omega := \{\xi\eta = \omega,\ |\xi|, |\eta| < r\}$ with $\Psi_\omega \circ \rho = \rho \circ \Psi_\omega$ and s.t. , on $\mathcal{C}^r_\omega$,

$$\Psi_\omega^{-1} \circ \tau_1 \circ \Psi_\omega : \begin{cases} \xi' = e^{\frac{i}{2}\mu_\omega}\eta \\ \eta' = e^{-\frac{i}{2}\mu_\omega}\xi \end{cases}, \quad \Psi_\omega^{-1} \circ \tau_2 \circ \Psi_\omega : \begin{cases} \xi' = e^{-\frac{i}{2}\mu_\omega}\eta \\ \eta' = e^{\frac{i}{2}\mu_\omega}\xi \end{cases},$$

Remark $\Psi_\omega(\mathcal{C}^r_\omega)$ is a holomorphic invariant set of τ_i's and their restriction is conjugated to a linear map . "Asymptotic full measure"$= \frac{|\mathcal{O}_r|}{2r^2} \xrightarrow{r\to 0} 1$.

Nonstandrd KAM theorem

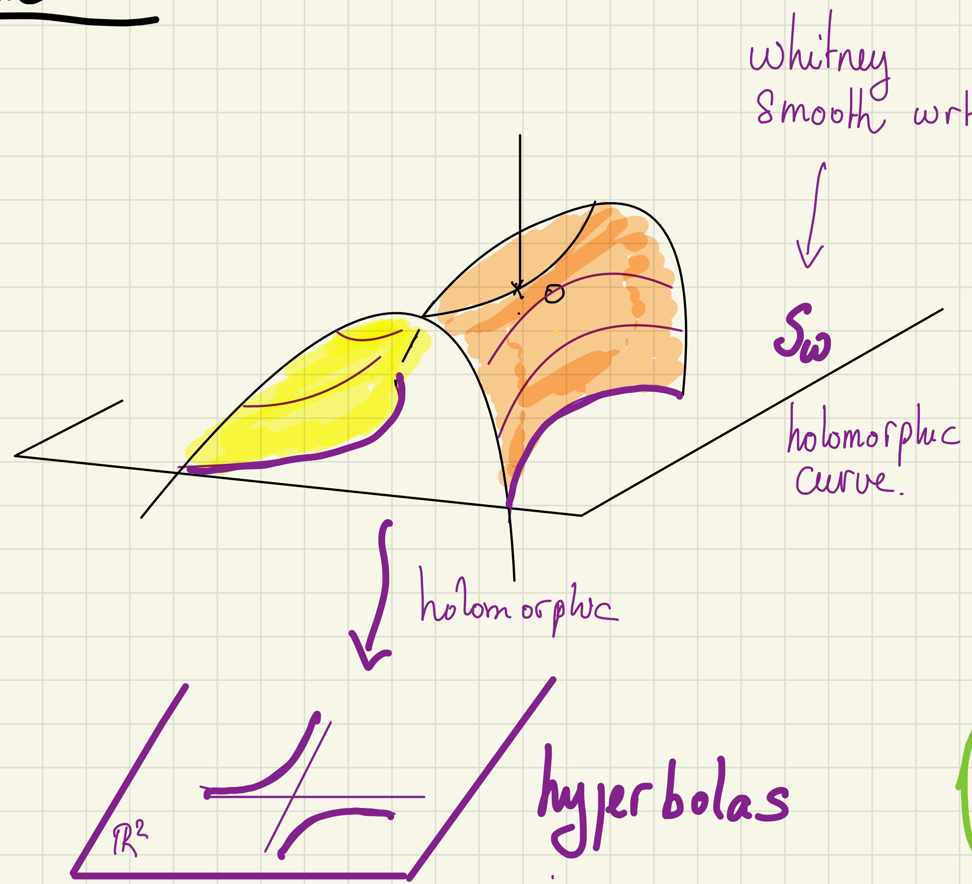


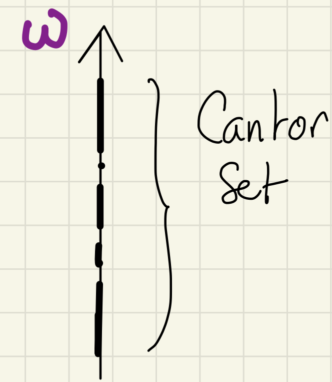
CR-Geometric consequences

Theorem (S.-Zhao 2020)

Let M be a surface with an hyberbolic CR singularity at the origin which is non exceptionnal and not formally equivalent to a quadric. Then: there exist a neighborhood of the origin and a Whitney smooth family of holomorphic curves $\{\mathcal{S}_\omega\}_{\omega\in\mathcal{O}}$ which intersects M along holomorphic hyperbolas : 2 real curves which are simultaneously holomorphically mapped to the two branches of the hyperbolas $\xi\eta=\omega$, $\omega\neq 0$.

Hyperbolic case

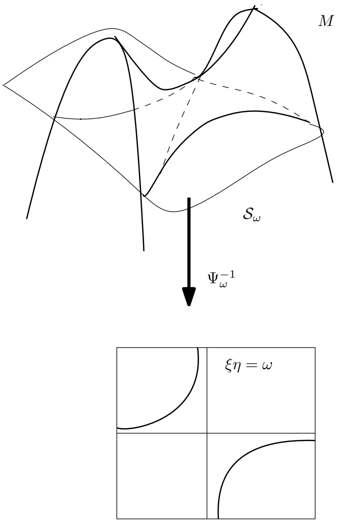




$\omega \to S_\omega$ is a "smooth" fct on Cantor set $\subset \mathbb{R}$

($c \to E_c$ is analytic fct on interval $\subset \mathbb{R}$.
Elliptic)

Intersection of M by holomorphic curves



Idea of the proof : KAM (Kolmogorv-Arnold-Moser) scheme