Projective flatness and application to uniformization

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Moscow, march 9 2021

X_n complex projective manifold, $\mathcal{E}$ locally free sheaf of rank r on X, H ample divisor.

Suppose $\mathcal{E}$ H-semistable. Then the Bogomolov-Gieseker inequality holds:

$$\boxed{(BG) \quad \left(\frac{r-1}{2r}\right) c_1^2(\mathcal{E}) \cdot H^{n-2} \leq c_2(\mathcal{E}) \cdot H^{n-2}.}$$

If X is Kähler-Einstein, $H = K_X$ or $H = -K_X$, more is true: Miyaoka-Yau inequality

$$\boxed{(MY) \quad \left(\frac{n}{2(n+1)}\right) c_1^2(\Omega^1_X) H^{n-2} \leq c_2(\Omega^1_X) \cdot H^{n-2}.}$$

In case of equality!

- if K_X is ample: X ball quotient
- if $-K_X$ is ample: $X \simeq \mathbb{P}_n$.
  ("X Fano")

Recall : if K_X is ample $\Rightarrow$ X is Kähler - Einstein.

If X Fano $\not\Rightarrow$ X Kähler - Einstein.

- Kanemitsu ; X Fano, $\rho = 1$ $\Rightarrow$ T_X need not even be $(-K_X)$-semistable (few examples).

2a $\rightsquigarrow$

Question What is going on with Fanos which are not KE ?

Further in view of Minimal Model Program study singular Fanos, "$\mathbb{Q}$-Fanos", i.e. X normal projective, K_X is $\mathbb{Q}$-Cartier, $-K_X$ is ample and X has, say, canonical singularities or klt.

jw : D. Greb, S. Kebekus

The examples are two orbit varieties X:

$\mathrm{Aut}^{\circ}(X)$ acts with an open orbit X° and

there is a closed orbit Z such that $X = X^{\circ} \cup Z$.

Picture in Dimension 3:

$\rho = 1$: T_X is $(-K_X)$-stable

$\rho \geq 2$: if T_X is not $(-K_X)$ s.s.,

then $\exists$ Mori fibre space $\varphi : X \longrightarrow Y$

such that $T_{X/Y}$ is maximal destabilizing.

(A. Steffens).

Examples

(1) $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}_3} \oplus \mathcal{O}_{\mathbb{P}_3}(3))$.

Then $(-K_X)^4 = 800$, $(-K_X)^2 c_2(X) = 296$

$\rightsquigarrow$ (BG) violated, and T_X not $(-K_X)$ s.s.

[We write $c_j(X) = c_j(T_X)$].

(2) $X = \mathbb{P}(1, 1, 1, 3)$ weighted projective space. (Prokhorov).

X Gorenstein, i.e. K_X Cartier, with one singularity.

(maximal for can. Gorenstein)

$(-K_X)^3 = 72$, $(-K_X) c_2(X) = 24$

$\rightsquigarrow$ <u>equality</u> in (BG), <u>violates</u> (MY).

<u>Fact</u>: T_X not $(-K_X)$ s.s.

(σ: $Y := \mathbb{P}(\mathcal{O}_{\mathbb{P}_2} \oplus \mathcal{O}_{\mathbb{P}_2}(3)) \to \cancel{X}$ blow down of neg. section; $\sigma_*(T_{Y/\mathbb{P}_2})$ destabilizes).

Principle

(MY) in dimension n looks like (BG) for a sheaf of rank $n+1$.

In case K_X ample, the sheaf is simple $\Omega^1_X \oplus \mathcal{O}_X$, considered as Higgs bundle.

In case $-K_X$ ample:

consider non-split extension

$$0 \to \mathcal{O}_X \to \mathcal{E}_X \to T_X \to 0,$$

given by $\zeta \in \mathrm{Ext}^1(T_X, \mathcal{O}_X) \simeq H^1(X, \Omega^1_X)$, $\zeta := c_1(-K_X)$.

Observation If $\mathcal{E}_X$ is $(-K_X)$-s.s, then (MY) holds on X. (Fano).

From now on

X_n normal projective, K_X $\mathbb{Q}$-Cartier, canonical or klt singularities.

$\mathcal{T}_X$ tangent sheaf, $= (\Omega^1_X)^*$. $\rightsquigarrow$ $\mathcal{E}_X$ again reflexive: $0 \to \mathcal{O}_X \to \mathcal{E}_X \to \mathcal{T}_X \to 0$, resp

$0 \to \Omega^{[1]}_X \to \mathcal{E}^*_X \to \mathcal{O}_X \to 0$.

Now (BG) holds in singular setting:

$\mathcal{E}$ general reflexive sheaf of rank r, $\mathcal{E}$ H-s.s.

$\Rightarrow$

Thm 1 $\quad \left(\frac{r-1}{2r}\right) c_1^2(\mathcal{E}).H^{n-2} \leq \hat{c}_2(\mathcal{E}).H^{n-2}$.

Here: $\hat{c}_2$ "orbifold Chern class".

Key: X has quotient singularities in codim. 2

$\rightsquigarrow \hat{c}_2(\mathcal{E}).H^{n-2}$ is defined (although $\hat{c}_2(\mathcal{E})$ does not exist on all of X):

$\hat{c}_2(\mathcal{E}).H^{n-2} = \hat{c}_2(\mathcal{E}|S)$, S cut out by

$m_1 H_1, \dots m_{n-2} H$.

<u>Corollary</u> Assume $\mathcal{E}_X$ H-s.s for some H ample,

then

$$(*) \quad \frac{n}{2(n+1)} \, \hat{c}_1^2(\mathcal{J}_X) . H^{n-2} \le \hat{c}_2(X) . H^{n-2}.$$

$$\hat{c}_1^2(\mathcal{J}_X) . H^{n-2} = K_X^2 . H^{n-2}$$

<u>Remark</u> Suppose X Fano Kähler–Einstein (smooth)

Tian: $\mathcal{E}_X$ has Hermitian–Yang–Mills metric w.r.t. ω (KE metric), hence $\mathcal{E}_X$ is $(-K_X)$–s.s. Remains true in "singular case", below.

<u>Theorem 2</u> Assume $(-K_X)$ nef, X klt ~~Fano~~, ~~$\mathcal{E}_X$ H-s.s.~~

Then there exists H ample with $\mathcal{E}_X$ H s.s such that equality holds in $(*)$ if and only if

$$X \simeq \begin{cases} \mathbb{P}_n / G \\ A / G \quad (A \text{ abelian}) \end{cases},$$

G finite group without fixed points in codimension one.

(" $\mathbb{P}_n \to X$ resp $A \to X$ is quasi-étale").

Tian's result generalized by Druel-Guenancia-Păun (2020) to $\mathbb{Q}$-Fano Kähler Einstein (klt).

Hence

~~Theorem~~ Corollary X $\mathbb{Q}$-Fano, Kähler-Einstein. If equality holds in $(*)$, then $X = \mathbb{P}_n / G$.

(and $\mathcal{E}_X$ always $(-K_X)$-s.s.).

When is $\mathcal{E}_X$ s.s. ?

Notation $\mu_H(\mathcal{F}) = \dfrac{c_1(\mathcal{F}) H^{n-1}}{\mathrm{rk}\, \mathcal{F}}$ (slope)

Definition X normal projective variety, H ample, $\delta \in \mathbb{R}_+$. $\mathcal{E}$ tor free sheaf.

$\mathcal{E}$ is δ-stable (w.r.t H) $\Longleftrightarrow$

$\forall \mathcal{S} \subset \mathcal{E}$, $0 < \mathrm{rk}\, \mathcal{S} < \mathrm{rk}\, \mathcal{E}$:

$$\mu_H(\mathcal{S}) < \delta\, \mu_H(\mathcal{E})$$

(s.s : " $\leq$ ").

Tian: X_n smooth Fano, Kähler – Einstein, $\rho = 1$

$\Rightarrow T_X$ is $\left(\frac{n}{n+1}\right)$ s.s.

Also true in the singular case.

Elementary calculation shows

Lemma X_n $\mathbb{Q}$–Fano, T_X $\left(\frac{n}{n+1}\right)$ s.s. $\Rightarrow$

$\mathcal{E}_X$ is s.s. $\left(\text{for } (-K_X)\right)$.

Cases when T_X is $\left(\frac{n}{n+1}\right)$ s.s. (X smooth Fano)

(A) complete weighted intersections X of index r with the following property:

$H^0(X, \Omega_X^p(p)) = 0$, $1 \le p < n - \frac{\delta}{\delta+1}(n+1)$,

$\delta := n - r$.

(B) X is del Pezzo with $\rho = 1$

(C) X has index $r = 1$.

(D) X 3-fold ($\rho = 1$).

Remark (B) follows from (A) by direct calculation and further classification.

<u>Definition</u> X normal irreducible complex space, $\mathbb{P} \to X$ a locally trivial $\mathbb{P}_r$-bundle. $\mathbb{P}$ is projectively flat $:\Longleftrightarrow$

$\exists$ representation $\rho : \pi_1(X) \to PGl(r+1, \mathbb{C})$,

such that

$$\mathbb{P} \simeq_X \mathbb{P}_\rho := \tilde{X} \times \mathbb{P}_r \big/ \pi_1(X)$$

($\tilde{X}$ universal cover).

<u>Definition</u> X normal complex space.

X is <u>maximally quasi-étale</u> $\Longleftrightarrow$

$\hat{\pi}_1(X_{reg}) \longrightarrow \hat{\pi}_1(X)$ is isomorphism

$\Downarrow$

étale fundamental group $=$

profinite completion of fundamental group.

<u>GKP</u> Maximally quasi-étale covers exist:

(jw with Greb, Kebekus) X klt $\Rightarrow \exists\ \tilde{X} \to X$

quasi-étale, s.that $\tilde{X}$ is maximally quasi-étale.

Theorem 3 X normal projective klt variety, maximally quasi-étale, H ample, $\mathcal{F}$ reflexive sheaf of rank r; $\mathcal{F}|X_{reg}$ locally free. Suppose $\mathcal{F}$ H-s.s and that

$$(+) \quad \left(\frac{r-1}{2r}\right) c_1^2(\mathcal{F}) . H^{n-2} = c_2(\mathcal{F}) . H^{n-2} .$$

Then $\mathcal{F}|X_{reg}$ is projectively flat, i.e., $\mathbb{P}(\mathcal{F}|X_{reg})$ is projectively flat.

How to derive Theorem 2 from Theorem 3 ("$\Longrightarrow$").

- By passing to quasi-étale cover: wlog X maximally quasi-étale. Want: $X = \mathbb{P}_n$ or abelian. (étale cover).
- $\mathcal{E}_X$ s.s, loc. free on X_{reg} $\overset{\text{Thm 3}}{\Longrightarrow}$ $\mathcal{E}_X|X_{reg}$ projectively flat.

  (!) $\Longrightarrow$ $\mathcal{E}nd(\mathcal{E}_X)$ is locally free and flat.

  $\mathcal{E}nd(\mathcal{E}_X)$ contains locally T_X as direct summand $\Longrightarrow$

T_X locally free.

„Lipman–Zariski" (see below) $\Rightarrow$ X smooth.

$\mathcal{E}_X$ H–s.s., equality in (BG) $\Rightarrow$

$\mathcal{E}_X \otimes \frac{\det \mathcal{E}_X^*}{n+1}$ is nef. (flat).

$\quad \|$

$\mathcal{E}_X \otimes \frac{K_X}{n+1}$.

$\mathcal{E}_X \to T_X \to 0 \Rightarrow T_X \otimes \frac{K_X}{n+1}$ nef.

By assumption, $-K_X$ is nef $\Rightarrow T_X$ nef.

[DPS 94]: $\alpha : X \to Alb(X)$ submersion, conn. fibres, with Fano fibres F (possibly after étale cover);

T_F nef.

If α iso: done; so assume $\dim F > 0$.

Take a curve $C \subset F$, $\eta : \tilde{C} \to F$ normalization

$\Rightarrow \deg \eta^*(-K_X) = \deg \eta^*(-K_F) > 0$

Since $\eta^*\left(T_X \otimes \frac{K_X}{n+1}\right)$ is nef $\Rightarrow$

$\eta^* T_X$ ample. Then $F = X$;

otherwise $T_X|F \to N_{F/X} \to 0$

$\qquad\qquad\qquad \Vert$

$\qquad\qquad\qquad \mathcal{O}_F^{\oplus}$

leads to $\eta^* T_X \twoheadrightarrow \eta^*(N_F) = \mathcal{O}_{\tilde{C}}$, $\lightning$.

So X is Fano $\rightsquigarrow \pi_1(X) = 0$

$\mathcal{E}_X$ projectively flat $\underset{(+\pi_1(X)=0)}{\rightsquigarrow}$ $\mathcal{E}_X \simeq \mathcal{L}^{\oplus(n+1)}$

for some $\mathcal{L} \in \mathrm{Pic}(X)$

$\Rightarrow \underset{\substack{\| \\ -K_X}}{\det \mathcal{E}_X} = (n+1)\mathcal{L} \xRightarrow[\text{Ochiai}]{\text{Kobayashi-}} X \simeq \mathbb{P}_n$.

Lipman - Zariski

X normal, T_X locally free $\overset{?}{\Longrightarrow}$ X smooth.

known if X is klt (Greb-Kebekus-Kovács-P),

generalized to X lc by Graf, Druel.

unknown in general even in dimension 2.

(older work by Steenbrink, van Straten, Flenner).

Facts on projective flatness

(1) X normal, $\mathcal{E}$ loc. free of rank r, projectively flat $\Rightarrow$ $\mathcal{E}nd(\mathcal{E})$, $S^r\mathcal{E} \otimes \det\mathcal{E}^*$ are flat (given by $\rho : \pi_1(X) \to Gl(\cdot, \mathbb{C})$).

(2) X normal, maximal quasi-étale, $\mathcal{E}$ reflexive, $\mathcal{E}|X_{reg}$ locally free, projectively flat $\Rightarrow$ $\mathcal{E}nd(\mathcal{E})$ and $(S^r\mathcal{E} \otimes \det\mathcal{E}^*)^{**}$ are locally free and flat on X.

Reason: $\hat{\pi}_1(X_{reg}) \xrightarrow{\simeq} \hat{\pi}_1(X)$ implies the following:

given $\rho_0 : \pi_1(X_{reg}) \to Gl(N, \mathbb{C})$ $\rightsquigarrow$

$\rho : \pi_1(X) \to Gl(N, \mathbb{C})$ and factorization

$$\begin{tikzcd} \pi_1(X_{reg}) \arrow[r] \arrow[rr, bend right, "\rho_0"'] & \pi_1(X) \arrow[r, "\rho"] & Gl(N, \mathbb{C}) \end{tikzcd}$$

(basic fact: Mal'cev's theorem: subgroups of linear groups are residually finite).

Assume X $\mathbb{Q}$ – Fano, T_X $(-K_X)$ s.s.,

such that equality in (BG) holds for T_X ($\Rightarrow$ $\mathcal{E}_X$ wrt $(-K_X)$ s.s).

This cannot happen :

<u>Theorem 4</u> X projective klt space, H ample,

T_X H-s.s, with equality in (BG)

$\Rightarrow$ $\exists\, \tilde{X} \to X$ quasi-étale with $\tilde{X}$ abelian.

- shown by Jahnke-Radloff in case X smooth.

<u>References</u> :

- Projective flatness over klt spaces and uniformization of varieties with nef anticanonical class

arXiv: 2006.08769 v1

- Projectively flat klt varieties

arXiv: 2010.06878