Donagi-Witten construction and a graded covering of a supermanifold

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Graded covering of a supermanifold I. The case of a Lie supergroup, arXiv:2103.00665.

Graded covering of a supermanifold II. The case of any supermanifold. Result announced, in preparation.

Joint work with Mikolaj Rotkiewicz

In the paper

“Super Atiyah classes and obstructions to splitting of supermoduli space”,

Donagi and Witten gave a description of the first obstruction class ω_2 for a supermanifold to be split via differential operators.

Let $\mathcal{M}$ be any supermanifold.

Consider two super-charts $\mathcal{U}_1$ and $\mathcal{U}_2$ on $\mathcal{M}$ with local coordinates

$$(x_i, \xi_a), \quad \text{and} \quad (y_j, \eta_b).$$

In $\mathcal{U}_1 \cap \mathcal{U}_2$ we have the following transition functions

$$\begin{aligned} y_j &= F_j + \frac{1}{2} G_j^{a_1 a_2} \xi_{a_1} \xi_{a_2} + \cdots ; \\ \eta_b &= H_b^i \xi_i + \frac{1}{3} R_b^{i_1 i_2 i_3} \xi_{i_1} \xi_{i_2} \xi_{i_3} + \cdots , \end{aligned}$$

where $F_j, G_j^{a_1 a_2}, H_i^b$ are (holomorphic) functions depending only on even coordinates (x_i).

A (modified) Donagi-Witten construction:

$$\mathcal{M} \longmapsto \mathrm{F}_2(\mathcal{M})$$

In $\mathrm{F}_2(\mathcal{U}_1) \cap \mathrm{F}_2(\mathcal{U}_2)$ we have

$$\begin{aligned} y_j &= F(x)_j; \\ \theta_a &= H(x)_a^i \zeta_i; \\ t_j &= (F(x)_j)_{x_b} z_b + 2G(x)_j^{a_1 a_2} \zeta_{a_1} \zeta_{a_2}. \end{aligned}$$

x_i, y_j have weight 0; ζ_i, θ_j have weight 1; z_i, t_j have weight 2.

$\mathrm{F}_2(\mathcal{M})$ is a **graded manifold of degree** 2.

Definition

A **Lie superalgebra** is a $\mathbb{Z}_2$-graded vector superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a $\mathbb{Z}_2$-graded bilinear operation $[\,,]$ satisfying the following conditions:

$$[x, y] = -(-1)^{|x||y|}[y, x],$$

$$[x, [y, z]] - (-1)^{|x||y|}[y, [x, z]] = [[x, y], z],$$

where x, y, and z are homogeneous elements.

A $\mathbb{Z}$-**graded Lie superalgebra** is a $\mathbb{Z}$-graded vector space $\mathfrak{g} = \bigoplus_{a\in\mathbb{Z}} \mathfrak{g}_a$ with a $\mathbb{Z}$-graded bracket $[\mathfrak{g}_a, \mathfrak{g}_b] \subset \mathfrak{g}_{a+b}$, such that

$$[x, y] = -(-1)^{|x||y|}[y, x],$$

$$[x, [y, z]] - (-1)^{|x||y|}[y, [x, z]] = [[x, y], z],$$

where $|t|$ is the parity of a and $t \in \mathfrak{g}_a$.

If $\mathfrak{g} = \bigoplus_{a\in\mathbb{Z}} \mathfrak{g}_a$, we put

$$\mathfrak{g}_{\bar 0} = \bigoplus_{a=2q} \mathfrak{g}_a, \quad \mathfrak{g}_{\bar 1} = \bigoplus_{a=2q+1} \mathfrak{g}_a$$

Definition

Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie superalgebra. Then

$$supp(\mathfrak{g}) = \{a \in \mathbb{Z} \mid \mathfrak{g}_a \neq \{0\}\}.$$

Example. Let $\mathfrak{g}$ be a Lie algebra and M be an $\mathfrak{g}$-module. Then we can define a $\mathbb{Z}$-graded Lie superalgebra structure on

$$\mathfrak{h} := \mathfrak{g} \inplus M, \quad \mathfrak{h}_{\bar{0}} = \mathfrak{h}_0 = \mathfrak{g}, \quad \mathfrak{h}_{\bar{1}} = \mathfrak{h}_1 = M$$

in a natural way.

$$supp(\mathfrak{h}) = \{0, 1\}.$$

The Donagi–Witten construction for Lie superalgebras is a functor

$$\mathrm{F}_2 : \mathrm{SLieAlg} \to \mathrm{grLieAlg}_2$$

from the **category of Lie superalgebras** to the **category of graded Lie superalgebras** with support $\{0, 1, 2\}$.

The functor $\mathrm{F}_2 : \mathrm{SLieAlg} \to \mathrm{grLieAlg}_2$ is a composition of four functors:

1. T (*tangent*);
2. gr (*split*);
3. π (*parity change*);
4. ι (*inverse*).

The **functor inverse** ι is the inversion of a functor defined independently by

- Bruce, Grabowski, Rotkiewicz (SIGMA 12 (2016), 106.)
- E.V. (Letters in Mathematical Physics 109 (2), 2018, 243-293.)

for the category of **graded manifolds**. (And in a particular case independently by del Carpio-Marek and by Jotz Lean.)

Tangent functor T. If $\mathfrak{g}$ is a Lie superalgebra, its antitangent bundle (odd tangent) is the following linear superspace

$$\mathrm{T}(\mathfrak{g}) = \mathfrak{g} \oplus \mathrm{d}(\mathfrak{g}).$$

Here $\mathrm{d}(\mathfrak{g})$ denote a copy of $\mathfrak{g}$ with reversed parity.

The multiplication in $\mathrm{T}(\mathfrak{g})$:

- The Lie bracket in $\mathfrak{g} \oplus \{0\}$ coincides with the Lie bracket of $\mathfrak{g}$;
- $[X, \mathrm{d}Y] := (-1)^{|X|}\mathrm{d}([X, Y])$;
- the product $[\mathrm{d}(\mathfrak{g}), \mathrm{d}(\mathfrak{g})]$ is trivial.

The Lie superalgebra $\mathrm{T}(\mathfrak{g})$ sometimes is called in the literature a **Takiff superalgebra**:

$$\mathrm{T}(\mathfrak{g}) = \mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[\tau],$$

where τ is an odd element and $\mathbb{K}[\tau]$ are all polynomials in τ.

Functor split gr. For a Lie superalgebra $\mathfrak{g}$ we put $\mathrm{gr}(\mathfrak{g}) := \mathfrak{g}'$, where $\mathfrak{g}'$ is obtained from the Lie superalgebra $\mathfrak{g}$ putting

$$[\mathfrak{g}'_{\bar{1}}, \mathfrak{g}'_{\bar{1}}] = 0.$$

Observation. The Lie superalgebra $\mathfrak{a} := \mathrm{gr}(\mathrm{T}(\mathfrak{g}))$ is a $\mathbb{Z}\alpha \times \mathbb{Z}\beta$-graded Lie superalgebra with support

$$\{0, \alpha, \beta, \alpha + \beta\},$$

where α is odd and β is even. Indeed, we put

$$\mathfrak{a}_0 := \mathfrak{g}_{\bar{0}}, \quad \mathfrak{a}_\alpha := \mathfrak{g}_{\bar{1}}, \quad \mathfrak{a}_\beta := \mathrm{d}\mathfrak{g}_{\bar{1}}, \quad \mathfrak{a}_{\alpha+\beta} := \mathrm{d}\mathfrak{g}_{\bar{0}}.$$

For example

$$[\mathfrak{a}_\alpha, \mathfrak{a}_\alpha] = [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]_{\mathrm{gr}(\mathrm{T}(\mathfrak{g}))} = \{0\}.$$

Observation. The superalgebra $\mathfrak{a} := \mathrm{gr}(\mathrm{T}(\mathfrak{g}))$ contains the whole information about the Lie superalgebra $\mathfrak{g}$.

Indeed, $\mathfrak{g}$ is defined by three bilinear maps:

$$\mathfrak{g}_{\bar{0}} \times \mathfrak{g}_{\bar{0}} \to \mathfrak{g}_{\bar{0}}; \qquad \mathfrak{g}_{\bar{0}} \times \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{1}}; \qquad \mathfrak{g}_{\bar{1}} \times \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}},$$

$$\mathfrak{a}_{0} \times \mathfrak{a}_{0} \to \mathfrak{a}_{0}; \qquad \mathfrak{a}_{0} \times \mathfrak{a}_{\alpha} \to \mathfrak{a}_{\alpha}; \qquad \mathfrak{a}_{\alpha} \times \mathfrak{a}_{\beta} \to \mathfrak{a}_{\alpha+\beta}.$$

Functor parity change π. The Lie superalgebra $\mathfrak{a} = \mathrm{gr}(\mathrm{T}(\mathfrak{g}))$ possesses a **parity reversion** of the weight β. We have

$$\mathfrak{h} := \pi(\mathrm{gr}(\mathrm{T}(\mathfrak{g}))) = \mathrm{gr}(\mathrm{T}(\mathfrak{g})) =: \mathfrak{a}$$

as $\mathfrak{g}_{\bar{0}}$-modules, however now we assume that elements X and $\mathrm{d}(X)$ have the same parities.

The **Lie superalgebra structure** on $\mathfrak{h}$ is defines by

- We set $[X, Y]_{\mathfrak{h}} = [X, Y]_{\mathfrak{a}}$ for all homogeneous X, Y, except for the case $X \in \mathfrak{h}_\alpha$ and $Y \in \mathfrak{h}_\beta$.
- For $X \in \mathfrak{h}_\alpha$ and $Y \in \mathfrak{h}_\beta$ we set $[X, Y]_{\mathfrak{h}} := [X, Y]_{\mathfrak{a}} = -[Y, X]_{\mathfrak{a}} =: [Y, X]_{\mathfrak{h}}$.

Functor inverse ι. Let $\mathfrak{g}$ be a Lie superalgebra. Denote by

$$\mathfrak{p} := \iota \circ \pi \circ \mathrm{gr} \circ \mathrm{T}(\mathfrak{g})$$

a $\mathbb{Z}$-graded subsuperspace of $\mathfrak{h} := \pi(\mathrm{gr}(\mathrm{T}(\mathfrak{g})))$ with support $\{0, 1, 2\}$ given by

$$\begin{aligned} \mathfrak{p}_0 &:= \mathfrak{h}_{\bar{0}}, \\ \mathfrak{p}_1 &= \{Y + \mathrm{d}Y \mid Y \in \mathfrak{g}_{\bar{1}}\} \subset \mathfrak{h}_\alpha \oplus \mathfrak{h}_\beta, \\ \mathfrak{p}_2 &:= \mathfrak{h}_{\alpha+\beta}. \end{aligned}$$

$$\mathrm{F}_2 := \iota \circ \pi \circ \mathrm{gr} \circ \mathrm{T}.$$

A generalized Donagi–Witten construction for Lie superalgebras

The functor F is a composition of four functors:

1. the iterated odd tangent functor T^{∞},
2. the functor split gr,
3. the functor parity change π
4. the functor inverse ι.

Functor T^∞.

As vector spaces

$$\mathrm{T}^\infty(\mathfrak{g}) = \bigoplus_{p\geq 0} \bigoplus_{i_1<\cdots<i_p} \mathrm{d}_{i_1}\cdots\mathrm{d}_{i_p}(\mathfrak{g}).$$

As Lie superalgebras

$$\mathrm{T}^\infty(\mathfrak{g}) \simeq \bigwedge(\mathrm{d}_1, \mathrm{d}_2, \ldots) \otimes \mathfrak{g}.$$

The multiplication is defined by the following formula

$$[\mathrm{d}_i X, \mathrm{d}_j Y] = (-1)^{|X|}\mathrm{d}_i\mathrm{d}_j([X, Y])$$

for any $X, Y \in \mathrm{T}^\infty(\mathfrak{g})$.

Observation. The Lie superalgebra

$$\mathrm{T}^{\infty}(\mathfrak{g}) = \bigoplus_{p\geq 0} \bigoplus_{i_1<\cdots<i_p} \mathrm{d}_{i_1}\cdots \mathrm{d}_{i_p}(\mathfrak{g})$$

is called in the literature a **current superalgebra**. Very often one studies its even part

$$\left(\bigoplus_{p\geq 0} \bigoplus_{i_1<\cdots<i_p} \mathrm{d}_{i_1}\cdots \mathrm{d}_{i_p}(\mathfrak{g})\right)_{\bar{0}}.$$

This Lie algebra contains the whole information about $\mathfrak{g}$.

The **functor** gr is defined as above:

we replace any product of two odd elements by 0.

Observation. The information about the bracket in $\mathfrak{g}$ is not lost. For example the information about the product $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ is contained in

$$[\mathrm{d}_1\mathfrak{g}_{\bar{1}}, \mathrm{d}_1\mathfrak{g}_{\bar{1}}] = -\mathrm{d}_1\mathrm{d}_2[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}],$$

since $\mathrm{d}_i\mathfrak{g}_{\bar{1}}$ is an even subspace in $\mathrm{gr}(\mathrm{T}^\infty(\mathfrak{g}))$.

Properties of $\mathrm{gr} \circ \mathrm{T}^{\infty}(\mathfrak{g})$.

- $\mathrm{gr} \circ \mathrm{T}^{\infty}(\mathfrak{g})$ is a $\mathbb{Z}\alpha \times \mathbb{Z}\beta_1 \times \cdots$-graded Lie superalgebra with support Δ, where Δ is the maximal **multiplicity free** weight system generated by $\alpha, \beta_1, \ldots, \beta_k, \ldots$.
- α is odd and $\beta_1, \ldots, \beta_k, \ldots$ are even.
- (1) We assign the weight $\alpha + \beta_{i_i} + \cdots + \beta_{i_p}$ to $\mathrm{d}_{i_1} \cdots \mathrm{d}_{i_p}(Z)$, if $\mathrm{d}_{i_1} \cdots \mathrm{d}_{i_p}(Z)$ is odd.

  (2) We assign the weight $\beta_{i_i} + \cdots + \beta_{i_p}$ to $\mathrm{d}_{i_1} \cdots \mathrm{d}_{i_p}(Z)$ if $\mathrm{d}_{i_1} \cdots \mathrm{d}_{i_p}(Z)$ is even.

Example. Elements of the subspace $\mathfrak{g}_{\bar{0}}$ have weight 0 and elements of $\mathfrak{g}_{\bar{1}}$ have weight α. The product $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ has weight 2α, which is not multiplicity free. Hence it is 0 in $\mathrm{gr} \circ \mathrm{T}^{\infty}(\mathfrak{g})$.

Parity change π

Theorem (M. Rotkiewicz, E.V.)

The Lie superalgebra $\mathrm{gr} \circ \mathrm{T}^{\infty}(\mathfrak{g})$ *possesses a parity reversion:*

all β_i change their parities from even to odd.

The resulting Lie superalgebra we denote by $\pi(\mathrm{gr}(\mathrm{T}^{\infty}(\mathfrak{g})))$.

Functor inverse ι. (M. Rotkiewicz, E.V.) A $\mathbb{Z}$-graded subsuperalgebra $\mathrm{F}(\mathfrak{g}) := \iota \circ \pi \circ \mathrm{gr} \circ \mathrm{T}^{\infty}(\mathfrak{g})$ in $\pi \circ \mathrm{gr} \circ \mathrm{T}^{\infty}(\mathfrak{g})$:

$$\mathrm{F}(\mathfrak{g})_0 := \mathfrak{g}_{\bar{0}};$$

$$\mathrm{F}(\mathfrak{g})_1 := \mathrm{diag}(\mathfrak{g}_{\bar{1}} \oplus \bigoplus_i \mathrm{d}_i(\mathfrak{g}_{\bar{1}}));$$

$$\mathrm{F}(\mathfrak{g})_2 := \mathrm{diag}(\bigoplus_i \mathrm{d}_i(\mathfrak{g}_{\bar{0}}) \oplus \bigoplus_{i<j} \mathrm{d}_i\mathrm{d}_j(\mathfrak{g}_{\bar{0}}));$$

$$\mathrm{F}(\mathfrak{g})_3 := \mathrm{diag}(\bigoplus_{i<j} \mathrm{d}_i\mathrm{d}_j(\mathfrak{g}_{\bar{1}}) \oplus \bigoplus_{i<j<k} \mathrm{d}_i\mathrm{d}_j\mathrm{d}_k(\mathfrak{g}_{\bar{1}}));$$

$$\ldots$$

$$\mathrm{F}(\mathfrak{g})_n := \mathrm{diag}(\bigoplus_{C(I)=n-1} \mathrm{d}_I(\mathfrak{g}_{\bar{n}}) \oplus \bigoplus_{C(J)=n} \mathrm{d}_J(\mathfrak{g}_{\bar{n}}));$$

$$\ldots$$

Here $C(I)$ is the cardinality of I.

Summing up, we constructed the following functor

$$\mathrm{F} := \iota \circ \pi \circ \mathrm{gr} \circ \mathrm{T}^{\infty} : \mathrm{SLieAlg} \to \mathrm{grLieAlg}.$$

from the category of Lie superalgebras to the category of **non-negatively** graded Lie superalgebras.

Example.

$$\mathfrak{gl}_{m|n}(\mathbb{K}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\},$$

$\mathrm{F}(\mathfrak{gl}_{m|n}(\mathbb{K}))$ contains all matrices in the following form

$$\begin{pmatrix} A_1 & 0 & 0 & 0 & \cdots \\ C_1 & D_1 & 0 & 0 & \cdots \\ A_2 & B_1 & A_1 & 0 & \cdots \\ C_2 & D_2 & C_1 & D_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

We can define the functors

$$\mathrm{F}_n = \iota \circ \pi \circ \mathrm{gr} \circ \mathrm{T}^{n-1} : \mathrm{SLieAlg} \to \mathrm{grLieAlg}_n$$

as well.

For example $\mathrm{F}_2(\mathfrak{gl}_{m|n}(\mathbb{K}))$ (the Donagi–Witten case) contains all matrices in the following form

$$\begin{pmatrix} A_1 & 0 & 0 & 0 \\ C_1 & D_1 & 0 & 0 \\ A_2 & B_1 & A_1 & 0 \\ C_2 & D_2 & C_1 & D_1 \end{pmatrix}.$$

mod C_2.

We can see $\mathrm{F}(\mathfrak{g})$ as a subalgebra of a loop superalgebra corresponding to a Lie superalgebra $\mathfrak{g}$:

$$\mathfrak{p} := \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_{\bar{i}} \otimes t^i, \quad \mathfrak{p}_i \simeq \mathfrak{g}_{\bar{i}} \otimes t^i,$$

(Allison, Berman, Faulkner, Pianzola)

We have

$$\mathrm{F}(\mathfrak{g}) = \bigoplus_{i\geq 0} \mathfrak{g}_{\bar{i}} \otimes t^i.$$

Definition

A $\mathbb{Z}^{\geq 0}$-**covering of a Lie superalgebra** $\mathfrak{g}$ is a non-negatively $\mathbb{Z}$-graded superalgebra $\mathfrak{p} = \bigoplus_{a\geq 0} \mathfrak{p}_a$ with a surjective homomorphism

$$\Pi : \mathfrak{p} \to \mathfrak{g}$$

such that $\Pi|_{\mathfrak{p}_a} : \mathfrak{p}_a \to \mathfrak{g}_{\bar{a}}$ is a linear bijection for any $a \geq 0$.

Universal properties. Let $\mathfrak{a}$ be an $\mathbb{Z}^{\geq 0}$-graded Lie superalgebra and

$$\psi : \mathfrak{a} \to \mathfrak{g}$$

be a homomorphism of Lie superalgebras. Then there exists a unique $\mathbb{Z}$-graded homomorphism $\Psi : \mathfrak{a} \to \mathfrak{p}$ such that

$$\psi = \Pi \circ \Psi.$$

Supermanifolds

Definition

A **supermanifold** $\mathcal{M} = (\mathcal{M}_0, \mathcal{O}_{\mathcal{M}})$ of dimension $n|m$ is a $\mathbb{Z}_2$-graded ringed space that is locally isomorphic to a superdomain in $\mathbb{C}^{n|m}$. That is to a ringed space

$$(\mathbb{C}^n, \mathcal{F} \otimes \bigwedge(\xi_1, \ldots \xi_m)),$$

where $\mathcal{F}$ is a sheaf of smooth or holomorphisc functions in $\mathbb{C}^n$.

Example 1. [Split supermanifold]

$$\mathcal{M} = (\mathcal{M}_0, \bigwedge \mathcal{E}^*).$$

According to the Batchelor–Gawedzki Theorem any smooth supermanifold is split.

Any Lie supergroup is split.

Example 2. A **superquadric** in $\mathbb{CP}^{2|2}$ is a supermanifold of dimension $1|2$ with the base space $\mathbb{CP}^1$, which has the following transition functions

$$y = \frac{1}{x} + \frac{1}{x^3}\xi_1\xi_2;$$
$$\eta_j = \frac{1}{x^2}\xi_j,\ j = 1, 2.$$

(Berezin, Green.)

Supergrassmannians and flag supermanifolds are in general non-split.

Let $\mathcal{M}$ be any supermanifold.

Example 3. Consider two super-charts $\mathcal{U}_1$ and $\mathcal{U}_2$ on $\mathcal{M}$ with local coordinates (x_i, ξ_a), and (y_j, η_b).

In $\mathcal{U}_1 \cap \mathcal{U}_2$ we have the following transition functions

$$
\begin{aligned}
y_j &= F_j + \frac{1}{2} G_j^{a_1 a_2} \xi_{a_1} \xi_{a_2} + \cdots ; \\
\eta_b &= H_b^i \xi_i + \frac{1}{3} R_b^{i_1 i_2 i_3} \xi_{i_1} \xi_{i_2} \xi_{i_3} + \cdots ,
\end{aligned}
$$

where $F_j, G_j^{a_1 a_2}, H_i^b$ are (holomorphic) functions depending only on even coordinates (x_i).

Now we apply the functors $\mathrm{F}_n = \iota \circ \pi \circ \mathrm{gr} \circ \mathrm{T}^{n-1}$ to $\mathcal{M}$, where $n = 2, 3, \ldots, \infty$.

Let us present the result of application of

$$\mathrm{F}_2 = \iota \circ \pi \circ \mathrm{gr} \circ \mathrm{T}$$

to $\mathcal{M}$ in coordinates. In $\mathrm{F}(\mathcal{U}_1) \cap \mathrm{F}(\mathcal{U}_2)$ we have

$$\begin{aligned} y_j &= F_j(x); \\ \theta_a &= H_a^i \zeta_i; \\ t_j &= \sum_{b=1}^{n} (F_j)_{x_b} z_b + 2 G_j^{a_1 a_2} \zeta_{a_1} \zeta_{a_2}. \end{aligned}$$

x_i, y_j have weight 0;

ζ_i, θ_j have weight 1;

z_i, t_j have weight 2.

$\mathrm{F}_2(\mathcal{M})$ a graded manifold of degree 2.

Definition

A **graded manifold of degree** 2 is a $\mathbb{Z}$-graded ringed space that is locally isomorphic to

$$(\mathbb{K}^n, S^*(V)),$$

where

$$V = V_0 \oplus V_1 \oplus V_2$$

is a graded vector space with support $0, 1, 2$.

This graded manifold of degree 2 is defined by

- the underlying manifold $M = \mathcal{M}_0$;
- the vector bundle E;
- by the following exact sequence

$$0 \to \bigwedge^2 E \to D_\omega \to \mathrm{T}M \to 0$$

The Atiyah class of this exact sequence is called the first obstruction class ω to splitting a supermanifold.

A generalized Donagi–Witten construction

$$\mathcal{M} \mapsto \iota \circ \pi \circ \mathrm{gr} \circ \mathrm{T}^{\infty}(\mathcal{M}) = \mathrm{F}(\mathcal{M})$$

Tangent functor T^{n-1} + inverse limit.

If $\mathcal{M}$ is a supermanifold, we can define its (odd) tangent bundle $\mathrm{T}^{n-1}(\mathcal{M})$ as in classical geometry.

In term of coordinates in $\mathrm{T}(\mathcal{U}_1) \cap \mathrm{T}(\mathcal{U}_2)$

$$(x_i, \xi_a, \mathrm{d}x_i, \mathrm{d}\xi_a) \quad \text{and} \quad (y_j, \eta_b, \mathrm{d}y_j, \mathrm{d}\eta_b),$$

For Lie algebras $T^{\infty}(\mathfrak{g}) = \bigwedge(\mathrm{d}_1, \mathrm{d}_2, \ldots) \otimes \mathfrak{g}$.

Functor split gr.

Let $\mathcal{M} = (\mathcal{M}_0, \mathcal{O})$ be a supermanifold.

$$\mathcal{O} = \mathcal{J}^0 \supset \mathcal{J} \supset \mathcal{J}^2 \supset \cdots \supset \mathcal{J}^p \supset \cdots ,$$

where $\mathcal{J}$ is the sheaf of ideals generated by odd elements in $\mathcal{O}$. We define

$$\mathrm{gr}(\mathcal{M}) := (\mathcal{M}_0, \mathrm{gr}\mathcal{O}),$$

where

$$\mathrm{gr}\mathcal{O} = \bigoplus_{p \geq 0} \mathcal{J}^p / \mathcal{J}^{p+1}.$$

The supermanifold $\mathrm{gr}(\mathcal{M})$ is split, that is its structure sheaf is isomorphic to $\bigwedge \mathcal{E}^*$, where $\mathcal{E}^* = \mathcal{J}/\mathcal{J}^2$ is a locally free sheaf.

n-fold vector bundle

Definition 1. A **double vector bundle** is a bundle in the category of vector bundles, a triple vector bundle is a bundle in the category of double vector bundles and so on.

Definition 2. Let Δ is the maximal multiplicity free weight system generated by $\alpha, \beta_1, \ldots, \beta_r$, where α is odd and all β_i are even. Consider the following vector superspace.

$$V = \bigoplus_{\delta \in \Delta} V_\delta, \quad V_{\bar{0}} = \bigoplus_{\delta \in \Delta_{\bar{0}}} V_\delta, \quad V_{\bar{1}} = \bigoplus_{\delta \in \Delta_{\bar{1}}} V_\delta.$$

A n-**fold vector bundle** is a $\mathbb{Z}^r$-graded manifold, which is locally isomorphic to the ringed space

$$(V_0^*, S^*(V)).$$

Theorem (M. Rotkiewicz, E.V.)

The supermanifold $\mathrm{gr} \circ \mathrm{T}^{(n-1)}(\mathcal{M})$ *is an n-fold vector bundle of type* Δ*, where* Δ *is the maximal multiplicity free system generated by an odd weight* α *and by even weights* $\beta_1, \dots, \beta_{n-1}$*.*

The functor $\mathrm{gr} \circ \mathrm{T}^{(n-1)}$ *is an embedding of the category of supermanifolds of odd dimension n into the category of n-fold vector bundles of type* Δ*.*

Observation. Any n-fold vector bundle possesses a parity reversion in all possible directions. That is we can change a parity of any weight α or β_i.

The **functor parity** change π changes the parity of all β_i from even to odd.

Functor inverse ι.

The functor inverse ι is the inversion of a functor defined independently by Bruce, Grabowski, Rotkiewicz and E.V. (And in the case of double vector bundle independently by del Carpio-Marek and by Jotz Lean.)

It is a functor from the category of **graded manifolds of degree** n to the category of n-**fold vector bundles with additional structures**. It was shown that this functor is an equivalence of categories.

Theorem (M. Rotkiewicz, E.V.)

The image $\pi \circ \mathrm{gr} \circ \mathrm{T}^{(n-1)}(\mathcal{M})$ *possesses these additional structures, therefore we can apply* ι *and get a graded manifold*

$$\mathrm{F}(\mathcal{M}) := \iota \circ \pi \circ \mathrm{gr} \circ \mathrm{T}^{(n-1)}(\mathcal{M})$$

of degree n

The functor F is an embedding of the category of supermanifolds of odd dimension n into the category of graded manifolds if degree n.

Definition (M. Rotkiewicz, E.V.)

A $\mathbb{Z}^{\geq 0}$-**covering of a supermanifold** $\mathcal{M}$ is a graded manifold $\mathcal{P}$ of infinite degree with $\mathcal{P}_0 = \mathcal{G}_0$ together with a morphism

$$\Pi : \mathcal{P} \to \mathcal{G}$$

such that we can choose atlases $\{\mathcal{U}_i\}$ and $\{\mathcal{V}_i\}$ on $\mathcal{M}$ and $\mathcal{P}$, respectively, with the same base space $(\mathcal{U}_i)_0 = (\mathcal{V}_i)_0$, with even and odd coordinates (x_a, ξ_b) in $\mathcal{U}_i$ and with graded coordinates (y^s_a, η^t_b), where s is an even integer and t is an odd integer, in $\mathcal{V}_i$ such that

$$\Pi^*_s(x_a) = y^s_a, \quad \Pi^*_t(\xi_b) = \eta^t_b.$$

A $\mathbb{Z}^{\geq 0}$-covering of a supermanifold $\mathcal{M}$ satisfies universal properties: it is unique up to isomorphism and a morphism of a graded manifold $\mathcal{N}$ to a supermanifold $\mathcal{M}$ can be lifted to the covering.

Theorem (M. Rotkiewicz, E.V.)

For any supermanifold $\mathcal{M}$ the graded manifold $\mathrm{F}(\mathcal{M})$ is a $\mathbb{Z}^{\geq 0}$-covering of $\mathcal{M}$.