Abelian varieties not isogenous to any Jacobian

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Scuola Normale Superiore

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$$j(z) = q^{-1} + 744 + 196884q + \ldots, \qquad q = \exp 2\pi i z.$$

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It may be proved that each abelian variety is isogenous to one which is principally polarised.

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In general, the curve X embeds in J_X and the variety J_X is *birationally isomorphic* to the symmetric power $X^{(g)}$.
Many properties of X transfer to J_X, and it is often of interest to discover whether a given abelian variety is of the shape J_X for some curve X.

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For $g > 1$ we have (essentially due to RIEMANN)

$$\dim \mathcal{T}_g = 3g - 3.$$

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The p.p.a.v. of dimension g isogenous to some Jacobian form a **countable** *union of algebraic varieties of dimension $3g-3$ in $\mathcal{A}_g$.*

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This would yield an affirmative answer to the Katz-Oort question.

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(iv) Further, recall the **Theorem of Belyi**:

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This again suggests that we should be careful before taking for granted the most obvious expectation.

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This question suggests to seek arguments not using the special nature of $\mathcal{T}_g$, but working for **any** $\mathcal{X}$.

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Proofs: Both arguments worked with appropriate sequences of a.v. of *Weyl CM type*; this concerns the ring of endomorphism, which, in particular, is in a way *maximal* for these a.v..
Also, such *CM-type* is invariant by isogeny, which was the starting point for proving (with heavy work) that eventually some member was not isogenous to any Jacobian (or to any $x \in \mathcal{X}$).

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Trivial endomorphism ring:
A $\overline{\mathbb{Q}}$-generic point $x \in \mathcal{A}_g$ has $End(x) = \mathbb{Z}$, in particular no CM.
This could already be a 'natural' condition.

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Hodge-generic: This requires that the so-called *Mumford-Tate group* of x is maximal, i.e. GSp_{2g}.
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This last property has arithmetic nature and is not directly defined in geometric terms, though believed to be equivalent to Hodge-generic (as proved in many cases by several authors).

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Theorem 1

There exists a set $\mathcal{N} \subset \mathcal{A}_g(\overline{\mathbb{Q}})$, complex-dense in $\mathcal{A}_g(\mathbb{C})$, representing pairwise non-isogenous p.p.a.v., each being ℓ-Galois-generic and not isogenous to any Jacobian.

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The method yields further information on such p.p.a.v.. For instance let us see two aspects:

Counting abelian varieties not isogenous to Jacobians

(i) Beyond the said complex-denseness, in a sense ‘the majority’ of p.p.a.v. defined over $\overline{\mathbb{Q}}$ are not isogenous to any Jacobian.

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We may then take e.g. the box $B(T)$ of integer points $(p_1, ..., p_G) \in A$ with $|p_i| \le T$ and consider the a.v. $x \in \mathcal{A}_g$ such that $\phi(x) \in B(T)$.

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Each such x is defined over a number field (of bounded degree) and their number is $\gg T^G$. On the other hand, we have:

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Theorem 2

The number of $x \in \mathcal{A}_g$ with $\phi(x) \in B(T)$, which either are not ℓ-Galois-generic or which are isogenous to some Jacobian, is $\ll T^{G-\gamma}$ for some constant $\gamma > 0$.

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- More precisely, one can add to the above theorems the uniform bound 2^{16g^4} for the degree of a (variable !) number field of definition for the a.v. in question.

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$g = 2$: We have $\dim \mathcal{A}_2 = 3$ and we may parameterize rationally $\mathcal{A}_2$ by points $(a, b, c) \in \mathbb{A}^3$, corresponding to the Jacobian of the curve $y^2 = x(x-1)(x-a)(x-b)(x-c)$ (so-called *Rosenhain coordinates*).

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Therefore, if we believe in the conjectures of Lang and Vojta, the points in $\mathcal{A}_g$ defined over any given number field should not be Zariski-dense, and we could **not hope** to answer affirmatively the general question with a **single** number field of definition.

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Theorem 3

There exist elliptic curves defined over $\mathbb{Q}(i)$ and not isogenous to any elliptic curve E with $j(E) \in \mathcal{X}$.

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Remarks.

(i) Note that in the theorem we cannot replace $\mathbb{Q}(i)$ by $\mathbb{Q}$ (take $\mathcal{X} =$ real line).

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(iii) In fact, it may be easily proved that there exists a real *analytic* curve $\mathcal{X} \subset \mathbb{C}$ such that all elliptic curves over $\overline{\mathbb{Q}}$ are isogenous to some curve with j-invariant in $\mathcal{X}$.

An analogue for $g = 1$

Remarks.

(i) Note that in the theorem we cannot replace $\mathbb{Q}(i)$ by $\mathbb{Q}$ (take $\mathcal{X} =$ real line).

(ii) Any elliptic curve over $\overline{\mathbb{Q}}$ is isogenous to some elliptic curve with j-invariant in a(ny) prescribed disk of $\mathbb{C}$.

(iii) In fact, it may be easily proved that there exists a real *analytic* curve $\mathcal{X} \subset \mathbb{C}$ such that all elliptic curves over $\overline{\mathbb{Q}}$ are isogenous to some curve with j-invariant in $\mathcal{X}$.
Hence that $\mathcal{X}$ is algebraic is crucial. (Similarly for the case of abitrary g, discussed previously.)

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In general it may be quite difficult to decide whether two given elliptic curves (say over $\overline{\mathbb{Q}}$) are or are not isogenous (deep algorithms due to Masser-Wuestholz or Faltings-Serre).

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representing that $x_i \sim x$,
where moreover the above degree estimates give estimates for their *heights*: $H(g_i) \ll m^C$.

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(In fact for our proofs Pila's more technical notion of *blocks* is needed, though the idea is the same.)

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(iii) The algebraic part of the relevant V is found to be a union of so-called *weakly special* subvarietes of $\mathcal{A}_g$, studied by authors like MOONEN, OORT, PINK,.... The theory yields that a weakly special proper subvariety containing a Hodge-generic point is itself a point, which allows to conclude.

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-The methods should allow to prove, under suitable conditions, the existence of the sought abelian varieties, restricting them to lie in a prescribed subvariety Y of $\mathcal{A}_g$.