Geodesic scattering on hyperboloids and Knörrer's map

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Geodesic scattering on hyperboloid

Question 1. Consider a geodesic on one-sheeted hyperboloid coming from infinity with, say, positive x_0.
Will it be reflected back, or will it pass through to infinity with negative x_0?
How many times will it rotate about the hyperboloid?

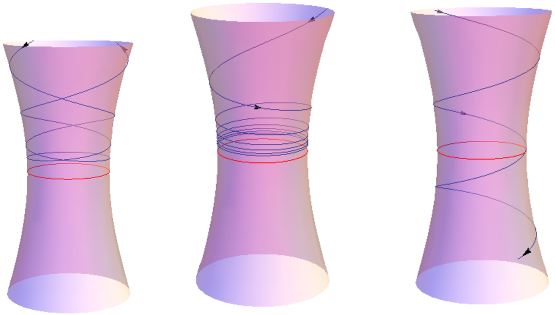

Figure: Geodesic scattering on one-sheeted hyperboloid

Question 2. What is an explicit relation between the values of asymptotic velocities at $\pm\infty$ in terms of the corresponding integrals?

Geodesics on quadrics

Consider a one-sheeted n-dimensional hyperboloid in the Euclidian space $\mathbb{R}^{n+1}$ given by

$$Q(x) := (A^{-1}x, x) = 1 \tag{1}$$

where $A = diag(a_0, a_1, \cdots, a_n)$ with $a_0 < 0 < a_1 < \cdots < a_n$.

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where $A = diag(a_0, a_1, \cdots, a_n)$ with $a_0 < 0 < a_1 < \cdots < a_n$.

The geodesics in arc length parameter s satisfy

$$x^{''} = -\lambda Bx, \quad \lambda = \frac{(Bx', x')}{(Bx, Bx)},$$

where $B = A^{-1}$ and λ is Lagrange multiplier determined by $(Bx, x) = 1$.

Joachimsthal 1843: *The function*

$$J(x, x') = (Bx, Bx)(Bx', x'),$$

is an integral of the geodesic problem.

Integrals

Let $x = u + tv$ be a straight line in $\mathbb{R}^{n+1}$ then it is tangent to the quadric $(Bx, x) = 1$ iff

$$(1 - (Bu, u))(Bv, v) + (Bu, v)^2 = 0.$$

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The condition $\Phi_z(u, v) = 0$,

$$\Phi_z(x, y) = (1 + (R_z x, x))(R_z y, y) - (R_z x, y)^2, \quad R_z = (zI - A)^{-1}$$

means that the line $x = u + tv$ is tangent to the confocal quadric

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This function can be written as $\Phi_z(x, y) = \sum_{k=0}^{n} \frac{F_k(x,y)}{z - a_k}$, where

$$F_k(x, y) = y_k^2 + \sum_{j \neq k} \frac{(x_k y_j - x_j y_k)^2}{a_k - a_j}, \quad k = 0, 1, \dots, n.$$

Uhlenbeck, Devaney 1978; Moser 1980: *The functions F_k are the involutive integrals of the geodesic flow, satisfying the relations:*

$$\sum_{k=0}^{n} a_k^{-1} F_k = 0, \quad \sum_{k=0}^{n} F_k = |y|^2.$$

Geometrically, this means that a generic line is tangent to n confocal quadrics, such that the corresponding normals are perpendicular to each other, which follow from theorem of Chasles.
The corresponding zeros $z = 0, c_1, \ldots, c_{n-1}$ of Φ_z are the confocal parameters of these quadrics:

$$\Phi_z(x, y) = \frac{z \prod_{i=1}^{n-1}(z - c_i)}{\prod_{k=0}^{n}(z - a_k)}.$$

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Jacobi (1837) showed that in the elliptic coordinates $z_1, \dots, z_n$ defined as the non-zero roots of the equation

$$((zI - A)^{-1} x, x) + 1 = 0$$

the variables in the Hamilton-Jacobi equation can be separated. Moreover, after some reparametrisation, the dynamics becomes linear at the Jacobi variety of the corresponding hyperelliptic curve

$$w^2 = -z \prod_{i=1}^{n-1} (z - c_i) \prod_{k=0}^{n} (z - a_k).$$

Answer to Question 1

Theorem

A geodesic on one-sheeted hyperboloid

$$\frac{x_0^2}{a_0} + \cdots + \frac{x_n^2}{a_n} = 1$$

in $\mathbb{R}^{n+1}$ *with* $a_0 < 0 < a_1 < \cdots < a_n$ *is crossing the neck ellipsoid with* $x_0 = 0$ *if and only if the integral*

$$F_0 = y_0^2 + \sum_{j=1}^{n} \frac{(x_0 y_j - x_j y_0)^2}{a_0 - a_j} > 0.$$

When $F_0 < 0$ *the geodesic is reflected back to infinity, while for* $F_0 = 0$ *the geodesic exponentially approaches a geodesic on the neck ellipsoid.*

Knörrer theorem

Consider a geodesic on the hyperboloid $(Bx, x) = 1, B = A^{-1}$ satisfying the equations

$$x'' = -\lambda Bx,\ \lambda = \frac{(Bx', x')}{(Bx, Bx)}.$$

Assuming that the Joachimsthal integral $J \neq 0$ denote its sign as

$$\varepsilon = J/|J| = \pm 1.$$

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Let us change the length parameter s to τ such that

$$\frac{d\tau}{ds} = \alpha(s), \quad \alpha^2 = |\lambda| = \frac{|(Bx', x')|}{|Bx|^2} = \frac{|J|}{|Bx|^4}.$$

Knörrer, 1982: *For any reparametrised geodesic $x(\tau)$ on the hyperboloid with $J \neq 0$ the normal vector $q = \frac{Bx}{|Bx|}$ satisfies the equations of Neumann problem on unit sphere S^n with the Hamiltonian*

$$H = \frac{1}{2}|p|^2 + \frac{1}{2}\varepsilon(Bq, q), \quad \varepsilon = J/|J|.$$

The corresponding trajectories $q(\tau)$ satisfy the relation $\Psi_0^\varepsilon(p, q) = 0$, where

$$\Psi_u^\varepsilon(p, q) = (1 + (R_u p, p))(R_u q, q) - (R_u p, q)^2, \quad R_u = (uI - \varepsilon B)^{-1}.$$

Real spectral curve

Let $d_1, \ldots, d_{n-1}$ be the non-zero roots of $\Psi_u^\varepsilon(p, q) = 0$:

$$\Psi_u^\varepsilon(p, q) = \frac{u \prod_{i=1}^{n-1}(u - d_i)}{\prod_{k=0}^{n}(u - b_k)}.$$

Then the corresponding spectral curve Γ is

$$w^2 = R(z), \quad R(z) = -z \prod_{i=1}^{n-1}(z - d_i) \prod_{k=0}^{n}(z - b_k).$$

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Its real part consists of one unbounded component and n ovals $\alpha_1, \ldots, \alpha_n$, one of them α_n containing point $(0,0)$.

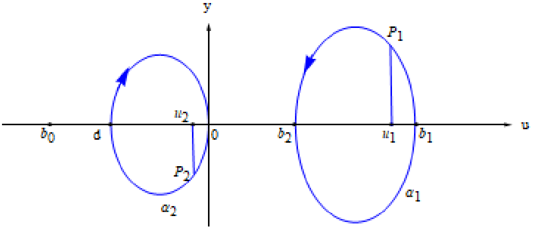
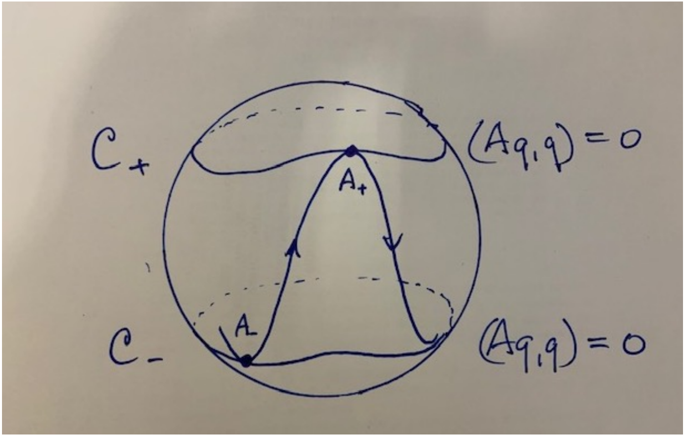


Figure: Real ovals of the spectral curve for $n = 2$

Neumann side



Let $D = (P_1, \dots, P_{n-1})$ be a set of points on the ovals $\alpha_1, \dots, \alpha_{n-1}$ and define a *real version of partial Abel map*

$$\mathbb{A} : \alpha_1 \times \dots \times \alpha_{n-1} \to \mathbb{T}^{n-1} = \mathbb{R}^{n-1}/\mathcal{L},$$

$$\mathbb{A}(D)_j := \sum_{i=1}^{n-1} \int^{P_i} \omega_j, \quad \omega_j = \frac{u^{j-1}du}{\sqrt{R(u)}}, \quad j = 1, \dots, n-1,$$

and the lattice $\mathcal{L}$ is generated by the period vectors $\Omega_1, \dots, \Omega_{n-1}$:

$$\Omega_j = (\oint_{\alpha_1} \omega_j, \dots, \oint_{\alpha_{n-1}} \omega_j), \quad j = 1, \dots, n-1.$$

Real partial Abel map

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Theorem

If all the roots of the polynomial $R(u)$ are real and distinct, then period vectors $\Omega_1, \dots, \Omega_{n-1}$ are linearly independent.
The real partial Abel map $\mathbb{A}$ is a diffeomorphism of two real tori.

Answer to Question 2

Consider the asymptotic hypersurfaces $C_\pm \subset S^n$ given by $(Aq, q) = 0, |q|^2 = 1$. In the Neumann elliptic coordinates $u_1 > u_2 > \cdots > u_n$ which are the solutions of

$$((u - \varepsilon B)^{-1} q, q) = 0,$$

they are simply the coordinate surfaces $u_n = 0$. We use the remaining $u_1, \ldots, u_{n-1}$ as the coordinates on C and consider the corresponding divisor $D = (P_1, \ldots, P_{n-1}),\ P_j = (u_j, \sqrt{R(u_j)})$.

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Theorem

The geodesic scattering on the hyperboloid is described by the relation

$$\mathbb{A}(D_+) = \mathbb{A}(D_-) - \Delta, \quad \Delta_j = \oint_{\alpha_n} \omega_j, \quad j = 1, \ldots, n-1.$$

The topological properties of the geodesic are completely determined by the position of the vector Δ with respect to the lattice $\mathcal{L}$.

Two-dimensional case

In that case the spectral curve $w^2 = R(u)$,

$$R(u) = -u(u-d)(u-b_0)(u-b_1)(u-b_2), \;\; d = b_0 b_1 b_2 J,$$

the asymptotic curve C can be identified with the double cover of the oval α_1:

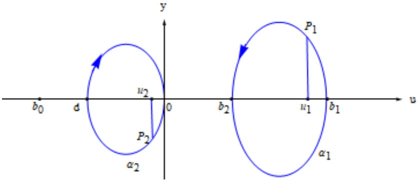

Figure: The ovals and position of roots in the reflection case

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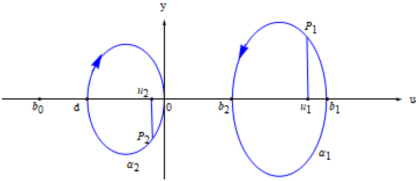

Figure: The ovals and position of roots in the reflection case

In particular, we see that the corresponding geodesic makes $N = \left[\frac{I_2}{2I_1}\right]$ full rotations about the hyperboloid before it is reflected back to infinity, where

$$I_1 = \oint_d^0 \frac{du}{\sqrt{R(u)}}, \quad I_2 = \oint_{b_1}^{b_2} \frac{du}{\sqrt{R(u)}}.$$

Cone case and flat billiard

In the case of cone given by

$$Q(x) = \frac{x_0^2}{a_0} + \frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} = 0$$

the metric is flat and the geodesics become the billiard trajectories in the corresponding corner with

$$\alpha = \frac{4}{\sqrt{k_2}} \int_0^1 \sqrt{\frac{1 - k_1 t^2}{(1 - t^2)(1 - \frac{k_1}{k_2} t^2)}} dt, \quad k_1 = \frac{a_2 - a_1}{a_2}, \; k_2 = \frac{a_2 - a_0}{a_2}.$$

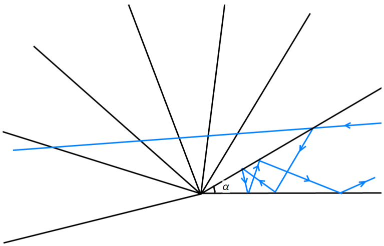

Figure: Geodesic scattering on cone and billiard in the corner

Quantum case

In the simplest symmetric case

$$\frac{x^2+y^2}{a^2} - \frac{z^2}{c^2} = 1$$

in parametrisation

$$x = a\sqrt{1+\rho^2}\cos\phi, \quad y = a\sqrt{1+\rho^2}\sin\phi, \quad z = c\rho,$$

the quantum Hamiltonian is the Laplace-Beltrami operator

$$\hat{H} = -\frac{1}{2}\left[\frac{1+\rho^2}{c^2+b^2\rho^2}\frac{\partial^2}{\partial\rho^2} + \frac{1}{a^2(1+\rho^2)}\frac{\partial^2}{\partial\phi^2}\right].$$

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Separation of variables leads to the equation

$$\mathcal{L}_k\psi_1(\rho)=E\psi_1(\rho),\quad \mathcal{L}_k=\frac{1}{2}\left[-\frac{1+\rho^2}{c^2+b^2\rho^2}\frac{d^2}{d\rho^2}+\frac{k^2}{a^2(1+\rho^2)}\right]$$

and, after the Liouville transformation, to the Schrödinger equation

$$\left(-\frac{d^2}{d\tau^2}+V(\tau)\right)\varphi=E\varphi,\quad V=\frac{k^2}{2a^2(1+\rho^2)}-\frac{a^2[6b^2\rho^4+(3b^2+c^2)\rho^2-2c^2]}{8(c^2+b^2\rho^2)^3(1+\rho^2)}$$

with

$$\tau=\int\sqrt{\frac{2(c^2+b^2\rho^2)}{1+\rho^2}}d\rho.$$

Recall that two metrics on a manifold are *projectively equivalent* if they have the same set of geodesics, possibly in different parametrisation.

Let $\mathcal{E}$ be the ellipsoid

$$b_0 x_0^2 + \cdots + b_n x_n^2 = 1, \ \ b_i > 0$$

with the standard metric ds_1^2 induced from the Euclidean metric on $\mathbb{R}^{n+1}$.

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Knörrer's change of time and the form of the Lagrange multiplier $\lambda = \frac{(Bx', x')}{(Bx, Bx)}$ suggest the following conformally flat metric given in the affine coordinates by

$$ds_2^2 = \frac{1}{|Bx|^2}(Bdx, dx) = \frac{b_0 dx_0^2 + b_1 dx_1^2 + \cdots + b_n dx_n^2}{b_0^2 x_0^2 + b_1^2 x_1^2 \cdots + b_n^2 x_n^2}.$$

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Tabachnikov, Matveev-Topalov 1998: *The restrictions of metrics ds_1^2 and ds_2^2 on ellipsoid $\mathcal{E}$ are projectively equivalent.*

Theorem

The ellipsoid $\mathcal{E}$ is totally geodesic submanifold of $\mathbb{R}^{n+1}$ with metric ds_2^2. The geodesics of the second metric coincide with the usual geodesics after Knörrer reparametrisation.

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Lemma

The involution $\sigma : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ defined by $y = \sigma(x) := \frac{x}{(Bx,x)}$ preserves the metric ds_2^2 and leaves the ellipsoid $\mathcal{E}$ as fixed point set.

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Proof.

$$dy = \frac{dx}{(Bx,x)} - \frac{2x(Bx,dx)}{(Bx,x)^2},$$

$$(Bdy,dy) = \frac{(Bdx,dx)}{(Bx,x)^2} - \frac{4(Bx,dx)^2}{(Bx,x)^3} + \frac{4(Bx,x)(Bx,dx)^2}{(Bx,x)^4} = \frac{(Bdx,dx)}{(Bx,x)^2},$$

$$\frac{(Bdy,dy)}{|By|^2} = \frac{(Bdx,dx)}{(Bx,x)^2}\frac{(Bx,x)^2}{|Bx|^2} = \frac{(Bdx,dx)}{|Bx|^2} = ds_2^2.$$

Projective closure of hyperboloids

Consider now the projective closure $\mathcal{H}$ of hyperboloid given in $\mathbb{R}P^{n+1}$ with projective coordinates $\xi_0 : \cdots : \xi_n : \xi_{n+1}$ by the equation

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$$ds_1^2 = \sum_{k=0}^{n} \frac{\xi_{n+1} d\xi_k^2 - 2\xi_k d\xi_k d\xi_{n+1}}{\xi_{n+1}^3} + \frac{|\xi|^2 d\xi_{n+1}^2}{\xi_{n+1}^4}, \ |\xi|^2 = \sum_{k=0}^{n} \xi_k^2.$$

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APV 2008: *The restriction of metric ds_2^2 to the projective closure of hyperboloids is a regular metric (which is Riemannian in two-sheeted case and pseudo-Riemannian of signature $(n, 1)$ in one-sheeted case).*

Projective Knörrer map

Define now the following projective version of Knörrer's map $\nu : \mathcal{H} \to \mathbb{R}P^n$. In the affine chart with $\xi_{n+1} \neq 0$ it is given by the formula

$$\nu(x) = [Bx], \quad x \in \mathcal{H},$$

where $[Bx] \in \mathbb{R}P^n$ is the line defined by vector Bx, and then extend it to the whole $\mathcal{H}$ by continuity.

Since Neumann system on S^n is invariant under antipodal map $\sigma : v \to -v$ it can be reduced to the quotient $\mathbb{R}P^n = S^n/\sigma$.

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APV-LW 2020: *The projective Knörrer's map maps the non-isotropic geodesics on $\mathcal{H}$ with restricted metric ds_2^2 to the solutions of the Neumann system on $\mathbb{R}P^n$, satisfying the relation $\Psi_0^\varepsilon(p, q) = 0$.*

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